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More clarity on the concept of material frame-indifference in classical continuum mechanics

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Abstract There was and still is a considerable amount of confusion in the community of classical continuum mechanics on the concept of material frame-indifference. An extensive review is presented which will point out and try to resolve various misconceptions that still accompany the literature of material frame-indifference. With the tools of differential geometry a precise terminology is developed ending in a consequent mathematical framework, in which not only the concept of material frame-indifference can be formulated naturally, but showing advantages that go beyond all conventional considerations on invariance used so far in classical continuum mechanics. As an exemplification the Navier-Stokes equations and the corresponding Reynolds averaged equations are written in a general covariant form within Newtonian mechanics.

1 Introduction

The concept of material frame-indifference, also known as material objectivity, is solely a notion from continuum mechanics.

Continuum mechanics in general is driven by balance laws of mass, momentum, energy and entropy which must be supplemented by constitutive equations or response functions of the material itself. These can be modelled macroscopically in accordance with experimental findings, or as the balance laws themselves can be derived or simulated microscopically from the Boltzmann equation, which in classical mechanics rests on Newton's laws of motion.

A constitutive equation is defined as a mathematical relation between forces and motion, in the sense that if forces are applied to a body of a solid, liquid or gaseous material they will cause it to undergo a motion in which the motion itself will differ according to the nature of the body. In classical continuum mechanics the forces of interest can range from pure mechanical stresses over thermal to electromagnetic forces.

The most general Ansatz for modelling material behaviour macroscopically is to lay down a functional relationship $\mathcal{F}$ between every force that can act on the material and all relevant kinematic variables that are necessary to describe the subsequent reaction of the material, each depending on the memory as well as on the spatial non-locality of the material. The functionals themselves can either be represented explicitly by integrals or implicitly by using differential equations [1,2].

Despite the fact that there exists a large degree of freedom in modelling material laws there are three ultimate constraints that need to be obeyed to ensure physical realizability

- Determinism,
- Material symmetry,
- Entropy principle.

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The self-evident principle of determinism guarantees that any material behaviour at the present state can only depend on its past history, a remembrance of the future is not possible. Furthermore, if a material shows certain structural symmetries, constitutive equations must reflect these, while the entropy principle ensures that constitutive equations are not to be at variance with the second law of thermodynamics.

The general formulation of a material equation through a functional $\mathcal{F}$ as described above is too complex. Usually for practical applications the aim of a constitutive equation is not to encompass all the observed phenomena related to a particular material, but, rather, to define certain ideal materials. Such idealizations are very helpful in that they portray reasonably well over a definite range of forces the behaviour of real substances. Assumptions as spatial locality or fading memory offer the possibility to simplify constitutive equations for practical applications.

Based on the historic work of Cauchy, Jaumann, et al. and their success in modelling constitutive equations for an elastic spring or a viscous fluid by using the experimental fact that the responses of these materials are invariant under superposed rigid body motions, or the fact that they should not depend on any properties regarding the underlying frame of reference, Truesdell and Toupin [3] and later in more detail Truesdell and Noll [4] elevated these findings to a principle of nature: *the principle of material frame-indifference* (*MFI-principle*).

Hence, according to Truesdell et al. *every* material law has to obey, besides the three fundamental principles stated above, an additional principle, that of material frame-indifference, which then serves as a further tool to reduce possible constitutive equations during a systematic modelling process.

But the applicability of material frame-indifference within rational continuum mechanics has been debated for more than four decades now, and recent articles indicate that the discussion does not seem to settle [5,6]. The latest discussions are concerned with the correct mathematical formulations of this principle and its implications in those cases where material frame-indifference can be applied as a reasonable approximation. Hence, the aim of this article is twofold: on the one hand the aim is to construct a framework within which a principle on material frame-indifference can be formulated naturally, making the way for a formalism in classical continuum mechanics which is well suited to discuss even the general concept of *invariance*, in which material frame-indifference only plays a special part. On the other hand the article aims at clarifying still existing confusions and to help to close discussions that are still open in the literature.

To accomplish this goal, the strategy is as follows: Sect. 2 gives a detailed historic review on the MFI-principle and the state of the art as it is currently debated in rational continuum mechanics. To the best of the author's knowledge, in all its completeness and thoroughness this has never been done before. As we will see, this is not only necessary to have an extensive review at hand, but also to get a feeling of the complex diversity as how this principle can be formulated and interpreted. In the course of this only the most influential articles on this topic are considered and only their main conclusions are reflected. The review itself is presented without any personal comments, it is presented as it would appear in the literature. Any unclarities that will arise when reading this review will be discussed and solved in upcoming Sections. At the end a brief résumé is given as how the MFI-principle is currently understood in classical continuum mechanics and in which points the debate still continues.

Section 3 is to prepare and to motivate all the definitions that will be developed in Sects. 4–8. To get a deeper and thorough insight into a concept as that of material frame-indifference one needs a considerable amount of definitions beforehand. To put them on a more firm foundation several references were made towards the general theory of relativity. After all, the general theory of relativity is the basic physical theory when discussing concepts as transformations and invariance. In general it would be unwise to skip this theory with all its assumptions, results and predictions in any discussion on invariance within classical continuum physics. The references are written such that no pre-knowledge on general relativity is required.

Based on the terminology defined in the previous Sections we are then able to construct a mathematical framework in Sect. 9, in which not only the concept of material frame-indifference can be formulated naturally, but which also shows considerable advantages over the conventional mathematical framework used in classical continuum mechanics especially regarding invariant material or dynamical modelling.

Section 10 and the Appendix finally exemplify as how this new mathematical framework can be utilized. Based on precise definitions, Sect. 10 on the one hand gives an unambiguous formulation of material frame-indifference, and on the other hand it gives an outlook as how future misunderstandings can be avoided. The Appendix shows, by using the Navier-Stokes equations as a representative, that all physical laws of classical continuum mechanics, constitutive or dynamical, can be reformulated such that their structural forms do not change under arbitrary coordinate transformations, and this without changing the physical contents of the equations.

2 Historic review and state of the art on the MFI-principle

In proposing the first law of all constitutive equations for deformable materials, namely, the law of linear elasticity, Robert Hooke in 1678 [7] gave the first hint toward such a principle of material frame-indifference. He realized that by carrying a spring scale to the bottom of a deep mine or to the top of a mountain, the *response* of the spring to a given force is always the same, in other words, he realized that by fixing the physical units the value of the spring constant is not changed during such a displacement. Thus he regarded it as obvious that while a rigid displacement in altitude might alter the weight of a body, it could have no effect on the response of a given spring to a given force. Due to this invariance he could measure the change of gravity with a simple spring scale.

In 1829 Poisson and Cauchy [8,9] were inspired independently by an experiment which claimed that with a spring scale one is not only able to measure the change of gravity but also capable of measuring the centrifugal force of a rotating system. By attaching a spring with a terminal mass to the center of a horizontal, uniformly rotating table, it could be experimentally verified that the force exerted by the spring in *response* to a given elongation is independent of the observer, being the same to an observer moving with the table as to one standing upon the floor. That is, the response of the spring is not only unaffected by a rigid displacement but also by a rigid rotation. Upon this experiment Poisson and Cauchy were forced to impose the theoretical requirement that only frame-independent terms should enter into a constitutive equation of the type of material they proposed for study.

These were the earliest examples of using the MFI-principle to reduce the form of constitutive equations by demanding that only frame-independent terms should enter. For about the next 75 years the theory of continuum mechanics evolved in the spirit of Hooke, Poisson and Cauchy, leading for example to the Navier-Stokes equations which are, used up to this day in fluid dynamics, constructed from a constitutive equation which only includes the frame-independent strain rate to describe the effect of viscosity.

During the time of development towards the theory of special relativity by Einstein in 1905 [10], a new alternative perspective on the MFI-principle was given independently by Zaremba in 1903 [11] and in more detail by Jaumann in 1906 [12]. Thus, in the exact words of Truesdell and Noll [4], this principle can be stated in two forms

- in the Hooke–Poisson–Cauchy form: constitutive equations must be invariant under a superimposed rigid rotation of the material,
- in the Zaremba–Jaumann form: constitutive equations must be invariant under an arbitrary change of the observer.

The difference between the two perspectives lies in the question of exactly *what* is thought to be rotating and/or translating. The first interpretation involves relative rigid motion of the body with respect to a given observer, and the second, relative motion between different observers with respect to the *same* moving body.

It is important to note that these two statements on the MFI-principle are not the original formulations given by Hooke, Poisson & Cauchy and Zaremba & Jaumann, respectively, but are the formulations of Truesdell and Noll, which must be read in the context of Noll's special continuum mechanical space–time setting, the neoclassical space–time [13–15], where an arbitrary change of the observer only refers to Euclidean transformations.[1] Thus within Noll's neoclassical space–time a change of frame is only restricted to Euclidean transformations in order to guarantee a classical Newtonian description for physical phenomena. Now, since the relative motions between Euclidean observers are in general given by the entire Euclidean group, while superimposed rigid body motions are restricted to its proper subgroup, Truesdell and Noll [4] concluded that "a constitutive equation satisfying the Zaremba–Jaumann form satisfies also the Hooke–Poisson–Cauchy form, but not conversely".

By giving their formulation of the MFI-principle with its implication that the Zaremba–Jaumann form is more restrictive than the Hooke–Poisson–Cauchy form, Truesdell and Noll consolidated the thought in the 1960s that constitutive equations in continuum mechanics characterize materials in which the description of the material response should be independent of the observer.

To illustrate this statement mathematically let us consider a simple material [4] in a pure mechanical process fully decoupled from any thermal or electromagnetic reactions. In such a decoupled theory the material is fully

[1] Euclidean transformations are functionally equal to Galilean transformations with the only difference that both the spatial rotation matrix and the translation vector are time-dependent, thus, in contrast to Galilean transformations, they can induce a change into non-inertial reference frames.

described by the Cauchy stress tensor $\boldsymbol{\sigma}$, which for a simple material is defined by the following functional:

$$\boldsymbol{\sigma}(\mathbf{X},t) = \mathcal{F}|_{\tau=0}^{\infty}\,(\boldsymbol{\chi}(\mathbf{X},t-\tau),\mathbf{F}(\mathbf{X},t-\tau),\mathbf{X}), \tag{1}$$

in that the current state of the stress in a material point $\mathbf{X}$ depends on the past motion $\boldsymbol{\chi}$ only in the immediate neighbourhood of $\mathbf{X}$, thus showing a dependence only on the first order deformation gradient $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$. The MFI-principle under a Euclidean transformation

$$\tilde{t} = t, \quad \tilde{\boldsymbol{\chi}} = \mathbf{Q}(t)\boldsymbol{\chi} + \mathbf{c}(t), \tag{2}$$

where $\mathbf{Q}_t$ is an arbitrary time-dependent orthogonal tensor, would now demand for every material point $\mathbf{X}$ the same state of stress with the same material law, independent of the corresponding reference system

$$\mathbf{Q}_t\mathcal{F}|_{\tau=0}^{\infty}\,(\boldsymbol{\chi}(\mathbf{X},t-\tau),\mathbf{F}(\mathbf{X},t-\tau),\mathbf{X})\,\mathbf{Q}_t^T = \mathcal{F}|_{\tau=0}^{\infty}\left(\tilde{\boldsymbol{\chi}}(\mathbf{X},t-\tau),\tilde{\mathbf{F}}(\mathbf{X},t-\tau),\mathbf{X}\right), \tag{3}$$

which finally serves as a restriction for the functional $\mathcal{F}$.

However, in the following years the application of the MFI-principle itself, as well as its mathematical formulation and its consequences for reducing constitutive equations gave rise to intense controversies in the community of classical continuum mechanics—regarding the mathematical formulation it even continues up to this day.

Let us have a closer look by giving a chronological outline: The validity of this principle was first questioned by Müller in 1972 [16], according to whom material frame-indifference and the kinetic theory of gases are incompatible. In his article Müller considers the Boltzmann equation for identical Maxwellian molecules using expressions of moments up to the fourth order. From these he then derives expressions for the stress deviator and the heat flux, showing that they depend on properties of the underlying frame of reference. In his words: "It follows that the kinetic theory of gases does not support the view of continuum mechanics and thermodynamics according to which stress and heat flux are related to the fields of density, velocity and temperature in a manner dependent solely on the material".

In the following year 1973 Edelen and McLennan [17] published an article which carries the same message as Müller [16], but the conclusion is expressed more forcefully: "Constitutive relations for stress and energy flux, derived from the Boltzmann equation by the Chapman-Enskog procedure, are shown to violate the principle of material frame indifference while exhibiting invariance under Galilei transformations".

Two years later in 1975 Wang [18] claimed that the analyses of Müller [16] and Edelen & McLennan [17] were not rigorous and tried to demonstrate that frame-indifference is not contradicted by kinetic theory. His arguments, however, were not convincing, so that Müller in 1976 [19] continued his work in showing that Ohm's law of electric conduction and Fourier's law of heat conduction in a metal both contain frame-dependent terms, ending with the conclusion that for constitutive equations in a metal the "principle of material frame indifference is only approximately true", in the sense that the contributions of frame-dependent terms in a metal are relatively small compared to the rest of the terms contributing to the constitutive equation. Furthermore, he gives a suggestive interpretation in diagrams as how the Coriolis force acting upon the electrons in free flight between collisions induces frame-dependence in the constitutive laws of a metal. If the *microscopic* time scale is defined proportional to the mean free path length of the classical colliding particles being investigated, and if the *macroscopic* time scale is defined as the temporal change given by the frame of reference, the metal shows a clear-cut separation of time scales in which the ratio of microscopic to macroscopic time scale is sufficiently small in order to use the MFI-principle for a metal as a reasonable approximation.

In the same year 1976 two additional articles appear, the one article of Söderholm [20] demonstrates again that the kinetic theory contradicts the MFI-principle, while the other article of Truesdell [21] strongly states that this criticism of material frame-indifference originating from kinetic theory is wrong, claiming that in all these articles certain analytical approximations must be made in order to derive constitutive equations, but, which at the end cannot be regarded as constitutive equations in the sense of continuum mechanics, and thus no conclusions on material frame-indifference can be drawn.

In 1981 Hoover et al. [22] tried to resolve this conflict by circumventing the most criticised analytical approximations and consequently presented a molecular-dynamics simulation of the problem of heat conduction in a two-dimensional rotating disk of dense fluid. The calculation addresses the issue of whether or not the heat flux should be frame-indifferent, coming to the conclusion that "...kinetic theory and Enskog's dense-fluid modification of Boltzmann's equation correctly predicts a violation of Fourier's law."

Convinced of his findings Müller continued his work together with Heckl in 1983 [23] with the aim to restate the topic properly on which kinetic theory and common phenomenological thermodynamics are at variance, since "the confusion about material frame indifference in the kinetic theory spread among engineers and mathematicians to the extent that the issue was distorted beyond recognition." The article clearly documents that the MFI-principle is not supported by the kinetic theory of gases.

However, in the same year 1983 Murdoch [24] demonstrates that material frame-indifference is not contradicted by kinetic theory after all. By giving a new interpretation in that constitutive equations may depend on the frame, but only through an indifferent tensor called the intrinsic spin $\mathbf{W}^* := \mathbf{W} - \mathbf{\Omega}$, where $\mathbf{W}$ is the skew-symmetric part of the velocity gradient and $\mathbf{\Omega}$ the spin of the frame of the observer relative to an inertial frame, he could show that the frame-dependence found in the works of Müller [16,19], Edelen & McLennan [17] and Söderholm [20] only enters through the intrinsic spin $\mathbf{W}^*$. By introducing into the system an additional Euclidean observer, Murdoch concludes that relative to these two Euclidean observers all constitutive equations derived from kinetic theory are actually frame-indifferent, and therefore satisfy the MFI-principle.

For the first time in this whole debate Ryskin in 1985 [25] criticises in a short essay the mathematical framework within which the MFI-principle is formulated. It is the usage of Noll's neoclassical space–time structure [13–15], as well as the corresponding vague language used by Truesdell and Noll [4] to formulate this principle, which in Ryskin's discussion is at the root of the controversy. He states that "the confusion arises because the concept of general covariance of physical laws is applied in the inappropriate setting of the three-dimensional space instead of the four-dimensional space–time", and concludes "that the MFI cannot be exactly true (because the microscopic physics obeys Newton's laws...), but is a very good approximation for ordinary materials and circumstances (because the absolute accelerations due to the rigid-body motion are usually much smaller than the accelerations at the molecular scale)."

Matolcsi in 1986 [26] continues to criticise the usual mathematical framework and its formulation of the MFI-principle. In contrast to Ryskin [25], however, he states that material frame-indifference, in the sense that the response of the material is independent of the observer, is a physical principle of nature which can be used to reduce constitutive equations—it only has to be put into the correct mathematical framework, whereas the usual mathematical framework given by Noll [13–15] is fully inappropriate to express this principle correctly. In his words: "the usual formulation of this principle is insufficient from a mathematical point of view, because one describes mathematical formulae involving the notion of observes without defining mathematically the notion of the observer." Matolcsi proposes a new way in that one has to exclude the notion of the observer from the description of any physical process that evolves in nature; on the contrary to Noll's neoclassical space–time where the definition of physical processes contains an observer. He masters this by defining a new space–time setting, his so called nonrelativistic space–time model, in which all physical relations like balance equations, the Boltzmann equation and the constitutive equations are formulated in absolute terms independent of any observers.

A new perspective on the discussion of the MFI-principle is given by the article of Kempers in 1989 [27], who shows that material frame-indifference follows from the covariance principle of general relativity [28] in the nonrelativistic limit when inertia is considered to be absent. Hence, the observer-dependent MFI-principle in classical continuum mechanics cannot retain its status of a fundamental principle, since the derivation of material frame-indifference from the covariance principle only considers the limited class of physical quantities without inertia; acceleration for example does not belong to this class.

Similar to the work of Hoover et al. in 1981 [22], Sharipov and Kremer in 1995 [29] verified numerically once again, but with a higher numerical precision that constitutive equations for heat transfer show frame dependence. By performing a numerical calculation of the heat flux field in a rotating gas they could show that the radial heat flux is affected by rotation: "Thus, from the results obtained from the kinetic equation and on the basis of simulation of microprocesses it follows that Fourier's law is distorted in rotating systems or, in other words, the principle of material frame-indifference is violated."

In 1996 Matolcsi, together with Gruber [30], finally applies his previously worked-out mathematical framework from 1986 to kinetic theory. As already mentioned the great advantage of his nonrelativistic space–time model defined in [26] is the fact that "the intuitive notion of the observer (reference frame) is ruled out", which then only allows for *absolute* objects in formulating physical theories, that means, objects that are not related to observers. All constitutive equations can thus be formulated independently of the observer. Now since the properties of a given material can be solely described by absolute objects, Matolcsi concludes that within his mathematical framework "the principle of material frame indifference is satisfied automatically." However, if these absolute forms are translated into the language of observers, relative objects as inertial terms will emerge in the constitutive equations, for which the MFI-principle cannot be strictly formulated anymore. The results of that article are: (i) The kinetic theory does not contradict the MFI-principle, (ii) the kinetic theory contradicts the usual mathematical formulation as given by Truesdell and Noll [4], (iii) the usual mathematical formulation cannot express a principle on material frame-indifference correctly, in that constitutive equations relative to observers do depend on observers.

The article of Muschik in 1998 [31] shows that in formulating the observer-dependent MFI-principle unambiguously two situations are to be distinguished properly, the one where the material is described by various observes, and the other one were two identical but differently moving materials are described by the same observer. By defining three frames of references, a standard frame of reference B^*, an arbitrary frame B, and a rest frame B^0 which is fixed with the material, Muschik concludes: (i) Constitutive properties do not depend on the relative motion of frames, in other words two observers in the different frames B and B^* investigating the same material observe the same constitutive properties. Since no observer is distinguished a principle of material frame-indifference is only to be understood in this sense. (ii) Constitutive properties do depend on the motion of the material with respect to a standard frame of reference, in other words if materials perform their motion with respect to a standard frame of reference B^* its constitutive properties will depend on B^0, and thus within this situation it is senseless to formulate a MFI-principle.

The article of Svendsen and Bertram in 1999 [32] focusses in more detail on the two formulations that can be given to describe the MFI-principle, the Hooke–Poisson–Cauchy form and the Zaremba–Jaumann form. With the help of precise group-theoretic definitions they show that besides superimposed rigid body motion on the one hand and the change of an observer on the other, a "third-leg" is necessary to have a complete formulation of material objectivity in constitutive theory, that of form-invariance. They thus identify three concepts in order to fully formulate a principle of material frame-indifference: Euclidean frame-indifference (EFI), form-invariance (FI), and indifference with respect to superimposed rigid-body motions (IRBM). The concepts FI and IRBM are interpreted as being constitutive in nature, while EFI is interpreted as a general principle which hold for all materials. Using this approach they show that only EFI together with FI can represent the concept of material frame-indifference; EFI alone is insufficient to obtain restrictions on the form of constitutive equations. Also the claim of Murdoch in 1983 [24] that kinetic theory is not at variance with material frame-indifference must be reinterpreted in the sense that the sole dependence on the intrinsic spin tensor $\mathbf{W}^*$ only implies EFI, but not material frame-indifference. Finally they show that any two of the concepts EFI, FI and IRBM imply the third one. In 2001 Bertram and Svendsen [33] stress their findings once again, showing in more detail as how EFI together with FI represent the concept of material frame-indifference as stated by Truesdell and Noll [4], and in addition how material frame-indifference is equivalent to IRBM.

Regarding materials with memory, Rivlin in 2002 [34] strongly criticized the approach to material symmetry and frame-indifference by Truesdell and Noll [4]: "...Noll's version of frame indifference and material symmetry are seriously erroneous on both physical and mathematical grounds." For materials with memory, in a pure mechanical process for example, the assumption is usually made that the Cauchy stress tensor at time t in a material element is a functional of the history of the deformation gradient tensor in that element up to and including time t. Now, according to Rivlin it is essential whether the deformation gradient for a material with memory is measured with respect to a fixed "already-happened" configuration of the material, or if it is defined with respect to an arbitrarily chosen reference configuration that may, or may not, be a configuration actually adopted by the material during the deformation considered. If the latter definition is used, as Truesdell and Noll are doing, Rivlin concludes that within such a theory the MFI-principle will always imply that any material being considered is isotropic, which renders this principle "...totally unacceptable as a basic element of constitutive equation theory." Two years later, Bertram and Svendsen [35] could show that Rivlin's distinction, after all, is not essential in defining a MFI-principle: "Thus, Rivlin's and Truesdell & Noll's forms of frame-indifference are essentially equivalent." Furthermore, Bertram and Svendsen claim that frame-indifference and material symmetry are conceptually distinct, such that it is never possible to deduce any material symmetry property, as that of isotropy, from a principle of frame-indifference.

Similar to the work of Svendsen and Bertram [32,33] Murdoch too in 2003 [36] presents a more detailed investigation on the two distinct versions with which the MFI-principle can be physically interpreted. However, in stark contrast to the results given in Svendsen and Bertram, Murdoch comes to a totally different conclusion. Terming the version which is connected to the indifference of material behaviour to its observation by two or more Euclidean observers as "objectivity", and abbreviating the version which is connected to the invariance of material response under superposed rigid body motions as "isrbm", he concludes: (i) Considerations of "objectivity" alone suffices to obtain standard restrictions on response functions, (ii) the "isrbm" version of material frame-indifference is non-physical and should be discarded from classical continuum physics. He argues that demanding frame insensitivity on a material under superposed rigid motions will always impose an a priori restriction upon nature as how the material has to respond for an observer in that particular frame. Thus within the concept of superposed rigid body motions in the interpretation of "two motions one observer" it is physically senseless to formulate a MFI-principle. To recover the 'isbrm' as a principle one rather has to monitor the superposed motion of the body not only by one but by two observers in the interpretation of "two motions two observers", where one observer is attached to the superposed rigid motion. Concerning "objectivity", however, he argues that within this concept it is possible to formulate a MFI-principle, since frame sensitivity in kinetic theory as found in the works of Müller et al. [16,17,22,23] is not in conflict with objectivity if frame-dependence only enters through the intrinsic spin tensor as defined in [24].

One year later in 2004 Liu [37] tries to invalidate Murdoch's conclusions as given in [36]. Liu shows disbelief in that Murdoch was able to prove that considerations on Euclidean objectivity alone, without any requirements on form-invariance, are sufficient to obtain standard restrictions on constitutive equations, and that invariance under superposed rigid body motions is to be discarded. By presenting counter-examples Liu shows that Murdoch's proof relies "on an erroneous interpretation concerning observers and relative motion", and thus all his claims are groundless. Upon this Murdoch responded in 2005 [5] that Liu has "misinterpreted and misrepresented" his work. His response aims at answering Liu's criticisms and to show that Liu's proposed counter-examples can be even "used to amplify the tenets of Murdoch's work". Thereupon Liu immediately reacted [6] by claiming that "Murdoch failed to recognize the mathematical implication of the condition of objectivity, as pointed out by Liu", and insists on that his criticisms towards Murdoch are justifiable.

Similar to the main idea of Matolcsi [26,30] Noll in 2006 [38] presents a new intrinsic mathematical framework in which the MFI-principle is vacuously satisfied. As Matolcsi's framework can rule out observers, Noll's new framework does not have to make use of so-called external frames of references when it comes down to describe internal interactions of a physical system. This he only shows for the classical theory of elasticity. External frames of references he defines as frames that are not constructed from the system itself, and internal interactions he defines as actions which do not originate from the environment of the system and its parts. In the language of external frames of references the statement of the MFI-principle as it applies to any physical system then reads as: "The constitutive laws governing the internal interactions between the parts of the system should not depend on whatever external frame of reference is used to describe them." Noll remarks that it is important to formulate this principle only relative to internal interactions of the system. Hence, it is of no surprise that the constitutive equations in the works of Müller et al. [16,17,22,23] fail to show frame-indifference, since all formulas involve inertial actions which are external actions and not internal interactions.

2.1 MFI-principle and turbulence modelling

One important branch of continuum physics we have not mentioned so far is that of turbulence modelling. Physically one has to distinguish between modelling a turbulent flow and modelling constitutive equations of a continuous material. The closure problem in turbulence modelling is a pure dynamical problem which originates from the lack of knowledge in quantifying the complex flow behaviour of a fluid material, and not from the lack of knowledge concerning the structure and behaviour of the material itself.

Nevertheless, Lumley in 1970 [39] could show that under numerous restrictions certain turbulent flows behave like classical non-linear viscoelastic media, which means that turbulent closure relations of the Reynolds stress tensor can be identified as constitutive equations for a non-Newtonian viscous stress tensor. Although all reasoning of rational continuum mechanics can be used in these cases, Lumley concludes, which he stressed again more clearly in 1983 [40], that a principle of frame-indifference cannot be utilized for turbulence modelling, not even in an approximative sense as it can be used in rational continuum mechanics. The reason is the huge spectrum of time-scales within any turbulent flow: there nearly always exists a band of scales which are in the time-scale range of the underlying non-inertial frame of reference.

In 1981 Speziale [41] demonstrates that in analogy to the MFI-principle for reducing constitutive equations, a frame-indifference principle in the limit of two-dimensional turbulence can be formulated to reduce turbulence models. In his review article of 1998 [42] he shows in more detail how the analogy of material frame-indifference breaks down for three-dimensional turbulence, but that this analogy rigorously applies in the limit of two-dimensional turbulence under arbitrary Euclidean transformations. He further claims that the general invariance group for the fluctuating dynamics in turbulence, that is universally valid, is the extended Galilean group of transformations which includes arbitrary time-dependent translations of the spatial frame of reference, and hence "consistent with the Einstein equivalence principle"; rotational frame-dependence then enters exclusively through the intrinsic spin vector as defined in the work of Murdoch [24].

Finally the article of Sadiki and Hutter in 1996 [43] emphasizes that the MFI-principle for constitutive equations in continuum mechanics ultimately rests on two requirements under Euclidean transformations, that of form-invariance where each observer uses the constitutive equations in the same structural form, and that of frame-indifference where the constitutive equations are independent of any properties of the underlying frame of reference. By focusing on the transport equations governing second order turbulent closures they show that form-invariance does not imply frame-indifference: although the transport equations can be written form-invariantly they remain frame-dependent, and thus cannot fulfil the MFI-principle.

2.2 Résumé

Nowadays there is no serious doubt anymore that constitutive equations in non-inertial frames are different from those in inertial frames, if and only if they are formulated in an observer-dependent way. That the response of a solid or a dense fluid material may depend on the motion of the material, however, is mostly out of scope for classical continuum mechanics: the conditions under which such constitutive properties depend on the motion are too extreme. Thus, for solids and ordinary dense fluids the MFI-principle is a reasonable approximation to reduce constitutive equations.

This is no longer the case if the characteristic size of the dense fluid microstructure gets bigger as in suspensions or polymer solutions, or if the mean free path length in the microprocesses gets longer as in gases and rarefied gases. Here, material frame-indifference may not be used as a guiding modelling principle anymore. The same holds for turbulence modelling where a frame-indifference principle may not be used, not even in an approximative sense.

Altogether the community of classical continuum mechanics basically agrees on this interpretation of the MFI-principle. The controversies that still accompany this topic solely concern the fact that if material frame-indifference is applied as a modelling principle to reduce constitutive equations, as it can be done for solids and ordinary dense fluids, then what is its correct mathematical formulation and what consequences can be fully drawn from it. This is exactly where this article wants to give a contribution to properly define all elements that are necessary to formulate a MFI-principle unambiguously, and to make available an observer-dependent mathematical framework where this can be formulated naturally.

3 What is invariance?

When trying to define material frame-indifference or when trying to formulate even a principle on such an indifference it is necessary to make a step backwards and first try to define the more general notion of *invariance*.

If one consults a scientific dictionary the word “invariance” is always loosely described as being a property of a mathematical or physical object that does not change under a transformation. Now, if one starts working with this definition it immediately becomes clear that such an encyclopedia-definition is not sufficient, it rises new questions that need to be answered, like what should we really understand under “no change” or under a “transformation”. Consulting the dictionary again, one finds that the concept of invariance is intimately linked to terms like “objectivity”, “tensor”, “form-invariance”, “frame-independence”, “inertial frames of reference”, “symmetry”, and “conservation”, terms already seen in the review on material frame-indifference in Sect. 2, but for which we have not given precise definitions yet.

So, to construct a precise definition of invariance is not as simple and straightforward as one might think at first, one certainly needs more than one elementary definition in order to grasp the whole concept of invariance. Although each previously listed term can be precisely defined, it seems that still a lot of researchers, as can be easily seen in various publications and discussions, have their own independent interpretations, causing great confusion among all scientific communities who are using or in future want to use the concept of invariance as a serious working method.

The following Sections will thus build up a clear-cut terminology, which at the end serves as an essential basis to formulate a principle as that of material frame-indifference unambiguously. The Sections are arranged such that a logical flow is maintained.

4 Frames of references

Differential geometry formulates a clear distinction between a manifold and a coordinate system. A manifold of dimension N is to be seen as a set or a space of points possible to embed coordinate systems which locally assign to every point of the manifold a unique N-tuple of real values, the coordinates of the manifold. The manifold itself is defined as essential and thus immutable, whereas the embedding of a coordinate system into the manifold is not unique, it can be chosen freely by transforming the coordinates accordingly—the choice of a coordinate system is a matter of expediency, not of truth.

Our physical world can be identified as a structured four-dimensional manifold made up of particular points in space and time, the so called events. A *frame of reference* is then defined as a four-dimensional coordinate system embedded into this physical manifold, irrespective of whether the four coordinates are chosen to be Cartesian or curvilinear. Using the previous mathematical definitions we can conclude: switching to a new arbitrary frame must be equal to a four-dimensional coordinate transformation incapable of inducing any changes upon the structure of the underlying physical manifold.

Unfortunately our physical surrounding is more complicated than we would always like to have it. The statement of an unchanging space–time manifold during a coordinate transformation can only be understood as an idealization if the change of the frame is not coupled to a change in the motion of a *massive* observer. Our physical manifold is not empty, it is rather filled up with matter and various physical processes evolving in it. The theory of general relativity by Einstein [28] successfully claims that the distribution of mass in our world determines the structure of the space–time manifold, but also vice versa, that the structure of the space–time manifold determines the mass distribution in our world. So, by changing the motion of a massive object will change the structure of the manifold, which again couples back to the motion of this object as well as to the rest of the massive objects in that manifold.

First of all, we are only interested in classical Newtonian continuum mechanics, and not in the predictions of relativistic continuum mechanics. Secondly, to grasp the idea behind material frame-indifference it is fully sufficient to operate only within a classical space–time manifold in which geometry and gravitation are intrinsically decoupled. There are many mathematical possibilities how to construct such a space–time manifold, but physically the most natural one is to work with a manifold which is, conceptually as well as mathematically, the classical limit of the manifold used in the general theory of relativity: decoupling the gravitational field from the structure of the manifold and simultaneously letting the speed of light go to infinity. This manifold will be explicitly constructed in Sect. 9, which from now on will be called the *classical Newtonian space–time manifold*. In this context, we strictly define a frame of reference as a four-dimensional Cartesian or curvilinear coordinate system embedded into an unchangeable classical Newtonian space–time manifold. Thus a change in frame as a four-dimensional coordinate transformation can no longer induce a change in the geometrical structure of the manifold, irrespective of whether the frame is carrying a massive observer or not. In other words, a change in a frame of reference is of no physical significance at all, it is just a relabelling of the four coordinates: a physical system in this classical Newtonian sense is completely insensitive to a choice of the reference frame from which it is described from, any physical phenomenon thus happens independently of the motion of the observer.

Finally, it is important to note that the community of classical continuum mechanics is used to a different conception regarding reference frames. Due to the highly influential books and articles of Noll [4,13–15], the community up to this day distinguishes between the concept of a frame of reference and that of a coordinate system. On the one hand, a frame of reference in classical continuum mechanics is regarded as the synonym for a special classically structured space–time manifold, and a change of frame as a special one-to-one mapping of this space–time manifold onto itself, preserving spatial distances, time intervals, and temporal order, which can only be realized by Euclidean transformations in order to guarantee a non-relativistic description of continuum mechanics. On the other hand, a coordinate system is regarded as a *spatial* labelling system for the reference frame, and a coordinate transformation as a transformation that is not affecting the reference frame itself, but is only relabelling its spatial coordinates, irrespective of whether the three-dimensional coordinates are Cartesian or curvilinear.

Compared to the classical Newtonian space–time setting which will be constructed in Sect. 9, we will see that Noll's neoclassical space–time has the following three drawbacks: First of all, in a true four-dimensional space–time formulation the distinction between frames of references and coordinate systems is artificial. Since Noll's mathematical framework is based on a classical evolution formulation of a three-dimensional space, a so called (3+1)-formulation, and not a true four-dimensional space–time formulation, a distinction between reference frames and coordinate systems is necessary therein to separate the purely spatial coordinate transformations from the coordinate transformations that depend on time. Secondly, by construction, the space–time structure is not independent of an observer when defined as a frame of reference. Restricted to Euclidean transformations the observer is able to change the classical space–time manifold accordingly, thus unnecessarily complicating any discussions on invariance. Thirdly, Noll's concept of a frame of reference has no relativistic counterpart. In other words, Noll's classical space–time theory cannot be constructed mathematically by a classical limit out of the general theory of relativity.

These three drawbacks in Noll's classical space–time theory give rise to a mathematical framework which is not well-suited when invariance principles like the MFI-principle are to be discussed.

5 Tensors

Without loss of generality let us focus on a four-dimensional continuous and differentiable manifold $\mathcal{M}$ whose points are distinguished from each other by assigning four real values x^0, x^1, x^2, x^3 to each of them. However, this first labelling should have no prerogative over any other one

$$\begin{aligned} \tilde{x}^0 &= \tilde{x}^0(x^0, \dots, x^3), \quad \tilde{x}^1 = \tilde{x}^1(x^0, \dots, x^3), \\ \tilde{x}^2 &= \tilde{x}^2(x^0, \dots, x^3), \quad \tilde{x}^3 = \tilde{x}^3(x^0, \dots, x^3), \end{aligned} \tag{4}$$

where the $\tilde{x}^\alpha$ are four continuous and differentiable functions of the x^α, such that their functional determinant vanishes nowhere.[2]

Tensors can be defined within two different formulations, an absolute and a relative formulation. Although each formulation makes use of different mathematical objects, they both base on the same mathematical reasoning.

The *absolute* formulation, which is mostly used in mathematical textbooks, assigns to each point $\mathbf{x} = (x^0, \dots, x^3)$ in the manifold $\mathcal{M}$ an orientated linear vector space $\mathcal{V}$ having the four-dimensional basis $(\mathbf{g}_0, \dots, \mathbf{g}_3)$, together with its dual vector space $\mathcal{V}^*$ having the basis $(\mathbf{g}^0, \dots, \mathbf{g}^3)$ defined by

$$\mathbf{g}^\alpha(\mathbf{g}_\beta) = \delta^\alpha_\beta. \tag{5}$$

The vector space $\mathcal{V}$ plays the role of a tangent space at that point in the manifold $\mathcal{M}$, while $\mathcal{V}^*$ that of a cotangent space. With respect to these vector spaces one can always associate the following vector space

$$\mathcal{T}^{(r,s)} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{\text{r-times}} \otimes \underbrace{\mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^*}_{\text{s-times}}, \tag{6}$$

where all its elements $T^{(r,s)} \in \mathcal{T}^{(r,s)}$ are defined as tensors of type (r, s); the space itself is called tensor space of type (r, s), which in the special case of $(r, s) = (0, 0)$ is defined as $\mathcal{T}^{(0,0)} := \mathbb{R}$. Each tensor in this linear space can be expanded in terms of the corresponding basis[3]

$$T^{(r,s)} = t^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s} \mathbf{g}_{\alpha_1} \otimes \cdots \otimes \mathbf{g}_{\alpha_r} \otimes \mathbf{g}^{\beta_1} \otimes \cdots \otimes \mathbf{g}^{\beta_s}, \quad 0 \le \alpha_i, \beta_j \le 3, \tag{7}$$

where $t^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}$ are the expansion coefficients or the components of the tensor $T^{(r,s)}$. The index r is called the contravariant and s the covariant rank. Additionally, each tensor is to be seen as a geometrical object, which, as an absolute object, stays invariant under a coordinate transformation (4) for that particular point $\mathbf{x} \in \mathcal{M}$ in the manifold on which each tensor is defined

$$\tilde{T}^{(r,s)} = T^{(r,s)}, \quad \text{when} \;\; \tilde{\mathbf{x}} = \tilde{\mathbf{x}}(\mathbf{x}) \in \mathcal{M}. \tag{8}$$

This invariance ultimately defines the transformation property between a tensor component and its basis (7), in that each tensor component has to transform contragrediently to its corresponding basis. Since this holds for every point in the manifold the concept of a tensor field $T^{(r,s)}(\mathbf{x})$ is defined accordingly.

For more detailed information on defining differentiable manifolds and tensors within the absolute formulation see the book of Schutz [44]. As this formulation with its abstract dual spaces and transposes is sometimes difficult to read, this article will continue to define differentiable manifolds and tensors in the second alternative formulation, in the relative formulation also known as the index formulation, which is more commonly used in physics textbooks, and in general showing an easier readability than the corresponding absolute formulation. Here we will follow the excellent book of Schrödinger [45].

[2] This is necessary in order to secure a one-to-one correspondence between the two sets of labels. But sometimes local exceptions have to be made, for example, the point of origin which in the transition from a Cartesian to a polar coordinate system is to be excluded.

[3] Throughout this article the Einstein summation convention in the index notation will be used; except stated otherwise an expression is implicitly summed over all possible values for that index which occurs twice in this expression.

The *relative* formulation aims at looking for mathematical entities, numbers or sets of numbers to which a meaning can be attached to every point $\mathbf{x}$ in a given manifold $\mathcal{M}$ regarding arbitrary coordinate transformations: if we consider a coordinate transformation as given in (4), a contravariant vector A^α is defined as a quantity with 4 components which transform like the coordinate differentials dx^α in (4), that means

$$\tilde{A}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} A^\beta. \tag{9}$$

Thus the coordinate differentials themselves form a contravariant vector, but for the finite coordinate differences Δx^α this is only the case if the transformation (4) is linear, and for the coordinates x^α themselves only if it is also homogeneous. A quantity B_α is called a covariant vector if its components transform as

$$\tilde{B}_\alpha = \frac{\partial x^\beta}{\partial \tilde{x}^\alpha} B_\beta. \tag{10}$$

In general one can not unambiguously associate a covariant and a contravariant vector with each other; for this one needs more information on the inner structure of the manifold. One can now define covariant, contravariant, and mixed tensors[4] of any rank by similar expressions

$$\tilde{T}^{\alpha\beta\ldots}_{\mu\nu\ldots} = \frac{\partial \tilde{x}^\alpha}{\partial x^\kappa}\frac{\partial \tilde{x}^\beta}{\partial x^\lambda}\cdots\frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu}\cdots T^{\kappa\lambda\ldots}_{\rho\sigma\ldots}, \tag{11}$$

where a vector is a tensor of rank 1, and a scalar is an invariant tensor of rank 0.

A tensor is thus solely defined by its transformation property in that it has to transform linearly and homogeneously. This property guarantees that if a tensor is zero in one coordinate system it is zero in all coordinate systems. Without loosing the tensor character in (11) one can define an autonomous tensor algebra for each point $\mathbf{x}$ in the manifold $\mathcal{M}$, that means, referring only to a single point in the manifold one can define addition and subtraction for tensors of the same rank and multiplication and contraction for tensors of any rank.

When defining ordinary differentiation for tensors, however, the tensor character of (11) in general will be lost: Partial derivatives of tensors show no tensor transformation behaviour, since they emerge from subtraction of tensors which do not refer to a single point but which refer to different points of the manifold. That ordinary differentiation in general breaks the tensor character can be easily seen for example by differentiating the transformation rule (10) of a covariant vector,

$$\frac{\partial \tilde{B}_\alpha}{\partial \tilde{x}^\mu} = \frac{\partial x^\beta}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\frac{\partial B_\beta}{\partial x^\nu} + \frac{\partial^2 x^\beta}{\partial \tilde{x}^\mu \partial \tilde{x}^\alpha} B_\beta, \tag{12}$$

where the second inhomogeneous term, except for linear coordinate transformations, prevents the derivative of a covariant vector to be a tensor of rank $(0, 2)$.

Nevertheless, to define differentiation which maintains the tensor character of any expression in (11) one either has to specify the inner structure of the manifold, or to be in the possession of a contravariant vector field already defined in the manifold, since during differentiation different points of the manifold need to be connected. The former leads us to affinely and metrically connected manifolds, while the latter leads us to the notion of the Lie derivative.

Let V^λ be a contravariant vector field in the manifold $\mathcal{M}$. The *Lie derivative* of a tensor $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ along the vector field V^λ is a tensor of the same rank as $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ defined by

$$\mathcal{L}_V T^{\alpha\beta\ldots}_{\mu\nu\ldots} := V^\lambda \partial_\lambda T^{\alpha\beta\ldots}_{\mu\nu\ldots} - T^{\lambda\beta\ldots}_{\mu\nu\ldots}\partial_\lambda V^\alpha - T^{\alpha\lambda\ldots}_{\mu\nu\ldots}\partial_\lambda V^\beta - \cdots + T^{\alpha\beta\ldots}_{\lambda\nu\ldots}\partial_\mu V^\lambda + T^{\alpha\beta\ldots}_{\mu\lambda\ldots}\partial_\nu V^\lambda + \cdots, \tag{13}$$

with a negative term for each contravariant index, and a positive term for each covariant index. Preserving the rank of the tensor the Lie derivative $\mathcal{L}_V$ satisfies all rules of ordinary differentiation ∂_ν as the product rule and linearity. Roughly speaking one can say that the Lie derivative is the directional derivative of a tensor along a curve set by a vector field adjusted for the change in the tangent: To differentiate the tensor $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ at a point P one drags the tensor along the curve given by the vector field V^λ through P to a neighbouring point Q, the derivative then compares the dragged tensor with the tensor evaluated at Q in the limit $Q \to P$.

[4] If a coordinate transformation carries a special name like orthogonal, Lorentz, Galilei or Euclidean, all tensors within such a transformation will adopt this name in calling them orthogonal, Lorentz, Galilei or Euclidean tensors, respectively. Orthogonal tensors are also called isotropic tensors.

5.1 Affine and metric spaces

To define an arbitrary differentiation in a manifold $\mathcal{M}$ which maintains tensor character, one has to introduce an extra geometric structure on the manifold, an *affine connection* $\Gamma^{\sigma}_{\kappa\lambda}$ with the transformation law

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\sigma}} \frac{\partial x^{\kappa}}{\partial \tilde{x}^{\mu}} \frac{\partial x^{\lambda}}{\partial \tilde{x}^{\nu}} \Gamma^{\sigma}_{\kappa\lambda} + \frac{\partial \tilde{x}^{\rho}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \tilde{x}^{\mu} \partial \tilde{x}^{\nu}}. \tag{14}$$

Thus $\Gamma^{\sigma}_{\kappa\lambda}$ does not transform as a tensor of rank $(1, 2)$ except under linear coordinate transformations: an affine connection which vanishes in one coordinate system $\Gamma^{\sigma}_{\kappa\lambda} = 0$ does not necessarily vanish in any other coordinate system $\tilde{\Gamma}^{\rho}_{\mu\nu} \neq 0$. The affine connection itself can be arbitrarily assigned in one coordinate system and determines the meaning of parallel displacement in the space considered. Then one can define covariant derivatives by

$$\nabla_{\nu} T^{\kappa\lambda\dots}_{\rho\sigma\dots} := \partial_{\nu} T^{\kappa\lambda\dots}_{\rho\sigma\dots} + \Gamma^{\kappa}_{\alpha\nu} T^{\alpha\lambda\dots}_{\rho\sigma\dots} + \Gamma^{\lambda}_{\alpha\nu} T^{\kappa\alpha\dots}_{\rho\sigma\dots} + \cdots - \Gamma^{\alpha}_{\rho\nu} T^{\kappa\lambda\dots}_{\alpha\sigma\dots} - \Gamma^{\alpha}_{\sigma\nu} T^{\kappa\lambda\dots}_{\rho\alpha\dots} - \cdots, \tag{15}$$

where the covariant derivative operator ∇_{ν} satisfies all linear rules of ordinary differentiation ∂_{ν}. If $T^{\kappa\lambda\dots}_{\rho\sigma\dots}$ is a tensor of rank (r, s) then $\nabla_{\nu} T^{\kappa\lambda\dots}_{\rho\sigma\dots}$ is a tensor of $(r, s + 1)$. In the following we are only concerned with symmetric connections $\Gamma^{\sigma}_{\kappa\lambda} = \Gamma^{\sigma}_{\lambda\kappa}$, that means with affine connected manifolds which are free of torsion.

If we form the second covariant derivatives by successive applications of (15), then in general the differentiations with respect to different coordinates do not commute. This property of the affine space is called curvature and is described by the Riemann curvature tensor

$$R^{\kappa}_{\mu\lambda\nu} = \partial_{\lambda} \Gamma^{\kappa}_{\mu\nu} - \partial_{\nu} \Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\kappa}_{\rho\lambda} \Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu} \Gamma^{\rho}_{\mu\lambda}. \tag{16}$$

Since the Riemann curvature is a tensor it will be zero in all coordinate systems if it is zero in one specified coordinate system.

Up to now we have not discussed as how to measure length in a manifold. In an affine space the concept of length can only be defined along a geodesic,[5] but lengths along different geodesics cannot be compared. Such a comparison becomes possible if the space admits a symmetric tensor of rank 2 with vanishing covariant derivatives[6], the metric tensor $g_{\mu\nu}$. Then the distance ds in a manifold is determined by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{17}$$

which by suitable parametrisation agrees with the distance along geodesics.

If the determinant of the $g_{\mu\nu}$ vanishes, the metric is called singular; if it does not vanish anywhere, the space is called metric or Riemannian. In such a space the geodesics are constructed by a variation principle as the shortest as well as the straightest lines. The assumption $\nabla_{\rho} g_{\mu\nu} = 0$ give rise to equations, which can be solved for the affine connections

$$\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu}), \tag{18}$$

where the affine connections now carry the name of *Christoffel symbols*; the tensor $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ defined by

$$g_{\mu\rho} g^{\rho\nu} = \delta^{\nu}_{\mu}. \tag{19}$$

With the help of these two tensors one now can uniquely associate with each contravariant vector A^{ρ} a covariant vector $A_{\mu} = g_{\mu\rho} A^{\rho}$, and conversely $A^{\rho} = g^{\rho\mu} A_{\mu}$.

Any symmetric tensor of rank 2 such as $g_{\mu\nu}$ which can be associated with a quadratic form can be characterized by its signature, that means by the difference between the number of positive and negative terms after the quadratic form has been diagonalized. This number is an invariant by Sylvester's law of inertia of quadratic forms.

[5] A curve in an affine space is called a geodesic if its development is a straight line, in other words geodesics in affine spaces are defined to be curves whose tangent vectors remain parallel if they are transported along it.

[6] The assumption that the covariant derivative of the metric tensor should vanish guarantees the natural demand that the lengths of vectors do not change under a parallel transport.

6 Transformations

In principle one has to distinguish between two classes of variable transformations [46], the class of the so called *active* transformations, and the class of the so called *passive* transformations.

6.1 Active transformations

Active transformations are characterized by the fact that the physical state of a system is transformed, whereas the original as well as the transformed state are observed from one and the same frame of reference.

The transformations themselves are not restricted to any specific norm, they can represent any mathematical imaginable transformation, for example in that a scalar field is not restricted to transform invariantly, or in that the time coordinate does not have to behave as an absolute quantity if a classical Newtonian description of physical phenomena is desired. In other words, active transformations are more than just coordinate transformations, for example a special subset is given by all symmetry transformations which are admitted by the dynamical equation describing the physical state [47–50].

Surely, an active transformation can always be performed theoretically, by putting the transformed variables into the corresponding dynamical equation of the observed physical state. If the transformation itself is a symmetry transformation, we will get the same dynamical equation, otherwise we will get a new dynamical equation with which the new transformed state of motion can be described. The question that immediately follows, is whether such a theoretical state of motion can be constructed by an experimentalist in his laboratory? The answer depends on the complexity of the physical state being investigated. If the physical state represents only a single elementary particle where its motion is characterized by a few variables, the realization of any given theoretical active transformation within classical mechanics should be possible, if the experimentalist is creative enough. But for a physical state which needs to be described by continuum physics, the answer is not so obvious anymore. In fact, the reader may ask, how should an experimentalist in his laboratory actually perform an active transformation on a continuous physical system of finite size, is it not impossible to actively transform every single continuous mass element? It is, and for that very reason one has to distinguish in continuum mechanics between *theoretical active transformations* and *real active transformations*.

6.1.1 Active transformations in continuum mechanics

To make this distinction clear, we take an illustrative example from fluid dynamics. Let us focus for simplicity on proper time-dependent active rotations performed on an arbitrarily shaped container filled with an incompressible fluid of moderate viscosity, where the axis of rotation is given perpendicular to the bottom wall and going through the center of mass of the filled container; whether the fluid is at rest or in motion is irrelevant in the following discussion, since in both cases the Navier-Stokes equations are satisfied: Performing an active transformation on the fluid as a whole in the form that the present velocity field is collectively superimposed by a *rigid* rotation, one will arrive at a new state of motion. As already mentioned, such a kind of transformation can always be done theoretically by putting the transformed variables into the corresponding dynamical equation, in this case into the Navier-Stokes equations; surely the underlying boundary conditions must be transformed accordingly. Since a time-dependent active rotation represents no symmetry of the Navier-Stokes equations [48,51], we will get a new dynamical equation with which the new transformed state of motion can be described.

Now, the question to be answered is whether such a theoretical state of motion can be constructed in a laboratory. From the perspective of pure mechanics the answer is definitely no, since for the experimentalist there are no tools for penetrating deep into the fluid and start to actively rotate a particular mass element without disturbing the rest of the fluid motion—the only tool that would be available is the gravitational field, which, as the only volume force in pure mechanics, can act simultaneously on each fluid element, but due to its homogeneity on the scale of the container it cannot be utilized to bring all fluid elements simultaneously into a rigid rotation.

The situation is certainly different when the fluid is plasma-like and the experimentalist can use the tools of electrodynamics, since here, in contrast to the volume force of gravitation, the volume forces of electric and magnetic fields can be individually shaped such that the motion of every fluid element can be superimposed simultaneously by a rotation.

However, for the pure mechanist the only way he can superimpose a rotation onto the motion of a fluid is either to actively stir the fluid around the line going through the center of mass or to actively rotate its boundaries, in this case the outer walls of the container. He then has to wait until the viscous shear forces will transport the rotation continuously into the exterior or into interior of the fluid respectively. Now, only if the rotation is performed at a constant rate and only if the fluid motion was in stationary state before rotation, he asymptotically will get a new stationary state, which can be identified as a *rigid* rotation superimposed on the original state of motion and thus will equal the theoretical state of motion. Otherwise these experimental time-delayed rotations cannot run into a rigid form and thus cannot lead asymptotically to the same theoretical state of motion constructed by a simultaneous superimposed rigid rotation.

This example should suffice to show the necessity in continuum mechanics to distinguish between theoretical and real active transformations. The less the viscosity of the continuous material, the more crucial this distinction becomes; for a rigid body this distinction is certainly redundant, as far as we only consider macroscopic active transformations, whereas for a *pure* ideal fluid it is impossible to mechanically force it into a rotational state. On the contrary, as the Hess-Fairbank experiment [52] clearly demonstrates, it is possible to rotate a superfluid, which can be explained by Landau's two-component theory of superfluidity [53], in that a superfluid as liquid helium is not a pure ideal fluid, but that it rather consists of two fluid components which coexist simultaneously:[7] a superfluid component being inviscid and a normal component carrying viscosity which then is responsible that the superfluid as whole can be brought into a rotational state mechanically.

Finally, it is worthwhile to mention that although an experimentalist in general cannot construct any given theoretical active transformation in his laboratory, it does not automatically mean that the corresponding state of motion is not part of nature; the experimentalist is rather incapable of showing that this state of motion can exist in nature.

6.2 Passive transformations

Here, not the physical state itself but the frame of reference is transformed, so that only the perspective changes from which the very same state is observed. Since such transformations always imply a change of coordinates in space and time, they are also called *coordinate transformations*. If a classical Newtonian description of physics is desired then passive transformations are to be restricted only to those transformations where the time coordinate is absolute.

Unlike active transformations, passive transformations are nothing more than just a change of frame or likewise a change in the coordinate system, and thus, as we already have discussed in detail, without any physical significance if the change of the frame is not coupled to a change in the motion of a massive observer. This is the reason for giving these transformations the attribute "passive", otherwise a change of frame is in general no longer physically insignificant and the add-on "passive" would be fully inappropriate to characterize such transformations.

Now, it is clear that to every passive transformation one can mathematically associate a corresponding active transformation that will lead to the same consequences regarding the manifestation and description of a physical state. In other words: the appearance of a physical state is independent of whether the frame of reference of the observer is transformed or if, in the same but counter-variant way, the physical state itself is transformed. Although the actively transformed state is certainly physically different to the passively transformed state, the mathematical consequences are the same, in both cases one has to solve the same dynamical equations with the same initial and boundary conditions to describe the corresponding state of motion.

The opposite is certainly not true, that means there always exist active transformations which *cannot* be connected to passive transformations. In other words: not every imaginable transformation is also a coordinate transformation. This immediately becomes clear if one looks for example at the only scaling symmetry transformation of the Navier-Stokes equations [48,51], which certainly is not a coordinate transformation.

Finally, it is helpful to shortly repeat the essence of the two classes of transformations once again: an active transformation is one which actually transforms the physical state of a system, whereas a passive transformation is a change in the frame of reference, and thus in contrast to an active transformation without any physical significance if the frame of reference is physically decoupled from the underlying space–time manifold. In continuum mechanics it is necessary to distinguish between theoretical and real active transformations, where it is possible that a theoretical transformation cannot be realized experimentally, or where a real transformation turns out to be different from the proposed theoretical transformation. Performing a theoretical active transformation on a physical system for which one can associate a passive transformation will always lead

[7] Landau emphasizes that this terminology is largely a matter of convenience and not to be taken as an indication that a superfluid can actually be separated physically into two distinct parts.

to the same mathematical consequences as when performing the corresponding passive transformation. This subtle distinction between active and passive transformations is one which should always be kept in mind when working with variable transformations.

7 Inertial frames of reference

An inertial frame of reference is a special reference frame in which Newton's laws of motion are valid. Hence, within the inertial frame, an object or body accelerates only when a physical force is applied, while in the absence of a net force, a body at rest will remain at rest and a body in motion will continue to move uniformly in a straight line. Experiments show that inertial reference frames are those systems which move with a constant velocity in both magnitude and direction relative to a sky of fixed stars, while all reference frames different from those are called non-inertial.

According to special passive transformations[8] it is possible to perform a change from one inertial system to another[9]. Galilei was the first to realize experimentally that one inertial system can not be distinguished from any other one, which he formulated in his famous principle of relativity: "All inertial systems are equivalent". This principle is a fundamental principle of *all* physics, without any exceptions. Since the necessary interconnecting Lorentz transformations, or in the classical limit, the Galilei transformations always involve constant translations in time and space, a constant rotation in space as well as a constant velocity boost, this equivalence principle implies the following space–time symmetries for inertial systems, respectively: the homogeneity of time, the homogeneity of space, the isotropy of space and the relativity of space–time.

In practical terms, this equivalence means that scientists living inside an enclosed box within an inertial frame without any sensuous contact to the outside world cannot detect their frame motion by *any* physical experiment done exclusively inside that box (relativity of space–time), nor can they distinguish one inertial frame from any other inertial frame due to the complete insensitivity of *any* physical experiment when performing it today, tomorrow or in 100 years (homogeneity in time), at one specific location or far away from it (homogeneity of space), as the experiment is set up or rotated with a fixed angle to it (isotropy of space).

By contrast, physical objects are subject to so-called inertial or fictitious forces in non-inertial reference frames; that is, forces that result from the acceleration of the reference frame itself and not from any physical force acting on the object. Non-inertial systems are thus not equivalent, the space–time symmetries do not hold anymore as it was the case for inertial systems. Therefore, scientists living inside a box that is being rotated or otherwise accelerated *can* measure their frame motion or acceleration by observing the inertial forces on physical objects inside the box. In a rotating frame for example, one space direction is superior to all other directions, the axis of rotation which breaks the isotropy of space. In the case of our earth as a closed system, experiments like the Foucault pendulum can demonstrate the rotation of the earth, or by using gyrocompasses which can exploit the rotation of the earth to find the direction of true north during navigation.

Inertial forces definitely deserve the name of being fictitious, because by choosing proper passive transformations they can be transformed away, namely by changing the current frame of reference to an inertial one. So, two observers describing the same physical phenomenon, where one observer is embedded into an inertial frame and the other into a non-inertial frame, experience two different mathematical descriptions for the same physical phenomenon: for the observer in the non-inertial frame the inertial forces seem to be real as such as that he has to include them into the dynamics of the observing phenomenon, while for the observer in the inertial system these forces do not exist and thus finally have to be claimed as fictitious forces by both observers. Examples of such forces are the centrifugal force and the Coriolis force in rotating reference frames, but also the gravitational force is a fictitious force, since, at least locally, it can be transformed away by going for example into a free falling system which has to be seen only as a *local* inertial frame of reference[10] [54–56].

In summary: inertial frames of reference are very special frames in which their space–time structure can be characterized by four fundamental symmetries. One of them, the relativity of space–time, implies that the description of physical processes is independent of the velocity with which an inertial frame relative to a sky

[8] These transformations are given by the Lorentz transformations from the theory of special relativity, which reduce to the classical Galilei transformations in the limit of small frame velocities $\|\mathbf{v}\| \ll c$ compared to the speed of light.

[9] The theory of general relativity shows that in our material universe only local inertial reference frames can exist, it is not possible to construct a global inertial frame of reference.

[10] A free falling observer as an astronaut will notice the effect of gravity if he inspects larger space–time regions than his immediate surrounding: for example for him the planetary orbits are no straight lines. Nevertheless, our universe can exhibit local inertial systems of quite large extent relative to a human scale, in deep space it is not impossible to find reasonable local inertial systems of the spatial size as our milky way galaxy.

of fixed stars moves. This inevitably gives rise to a notion of frame-independence and form-invariance, which will be defined next.

In the end it is worthwhile to note that the distinction of inertial systems towards non-inertial systems is not yet fully understood [57–60,56]. The question remains of relative to what an inertial system is not accelerated. Only to say that it moves with a constant velocity relative to a sky of fixed stars is certainly not precise enough. This discussion was first introduced by Mach [61] and is still an open discussion today. Mach himself wanted to find a proper substitute for Newton's explanation in using an absolute space, which obviously had the flaw that the absolute space should physically act on the motion of an object but that the motion itself could not act back on it. According to Mach's maxim it is impossible to do physics, when such a quantity as Newton's absolute space is used. These ideas accumulated to *Mach's Principle*, which significantly served Einstein as a guide to his eventual discovery of general relativity. Today we understand Mach's principle as a hypothesis that all mass of the universe determines the structure and behaviour of an inertial system. In other words it asserts that physics should be defined entirely in terms of the relation of one body to another, and that the very notion of a background space should be abandoned. Later analysis of Einstein's theory showed that Mach's Principle is not incorporated by general relativity, since it is not able to remove the background space as Mach was demanding it [60]. Currently there are however newer perspectives on these issues, which are argued to support the case that Einstein's theory is "Machian" after all, see [62]. The final word in this discussion is certainly not spoken yet.

8 Form-invariance and frame-independence

The concepts of form-invariance and frame-independence are special aspects of the concept of a variable transformation, actively as well as passively.[11] To define them in a general way, we consider an arbitrary frame of reference within which a certain physical phenomenon is described by an equation symbolically given as $\mathcal{D} = 0$, where usually the functional $\mathcal{D}$ is built up of several different parts $\mathcal{D} = \sum_{i=1}^{N} P_i$. In continuum mechanics such equations are either given by fundamental balance laws of a physical quantity, or by constitutive equations of a material, where in each case all the contributions acting on the system or defining the material sum up to zero, respectively. The definitions read as follows:

- Form-invariance for an equation like $\mathcal{D}=0$ is given if after a certain variable transformation it is possible to consistently rewrite the transformed equation $\tilde{\mathcal{D}}=0$ such that in the new variables it has the same structural form as the original equation in the untransformed variables.
- Frame-independence for an equation like $\mathcal{D} = \sum_{i=1}^{N} P_i = 0$ is given if the transformed equation $\tilde{\mathcal{D}} = \sum_{i=1}^{N} \tilde{P}_i = 0$ is completely independent of all properties that are being owned by the transformation. An equation element P_i, as an own mathematical expression, is said to be frame-independent if the transformed expression $\tilde{P}_i$ does not induce frame-dependent quantities in its defining equation. Thus an equation like $\mathcal{D} = \sum_{i=1}^{N} P_i = 0$ can either be frame independent if the *sum* of all parts P_i is frame-independent, or more restrictive, if *each* part P_i is frame-independent.

These definitions clearly stress that form-invariance and frame-independence are two distinct matters which should not be confused. Although the definitions hold for all classes of variable transformations, it is necessary to discuss in what to expect when the transformation is to be chosen as an active or as a passive transformation.

8.1 Passive case

If we look at the definitions of form-invariance and frame-independence from the perspective of passive transformations, it is obvious that

- form-invariance for an equation can always be guaranteed if all its entities transform as tensors, in other words if the equation can be formulated as a tensorial equation,
- frame-independence for an equation is obviously satisfied if the transformed equation does not depend on any mathematical quantities which define the geometrical structure of the underlying space–time manifold, as the affine connection $\Gamma^{\rho}_{\mu\nu}$, or in metric spaces the essential metric tensor $g_{\mu\nu}$. Certainly, this statement

[11] In the physics community the concept of form-invariance is frequently termed as covariance, whereas in the engineering community the concept of frame-independence is frequently termed as objectivity.

can be weakened for specific physical theories, for example in Newtonian mechanics, where time is an absolute coordinate, frame-independence is already then guaranteed if the transformed quantities defining the geometry of the manifold do not depend on time,[12] otherwise the transformation will induce inertial forces into the equation under consideration.

In this sense form-invariance in general is less restrictive than frame-independence, that means an equation which is not frame-independent can still be form-invariant. This statement is easily traceable if one looks for example at the concept of a covariant derivative, which on the one hand is a tensorial object, but on the other hand depends on the inner structure of the manifold through the affine connection, or in metric spaces through the Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$.

Within passive transformations very special frames of references can be selected, the so called inertial reference frames. Since experiments show that all inertial reference frames are equivalent to each other, this principle of *equivalence of inertial reference frames* can be identically reformulated to the principle that *all* physical laws within inertial frames of references must satisfy form-invariance as well as frame-independence. In other words: all physical laws must be form-invariant *and* frame-independent under the special passive transformations given by the Lorentz transformations or in the classical limit by the Galilei transformations, otherwise it would be possible for an observer to distinguish one inertial system from another one by probing the new terms that would appear during such a transformation between inertial systems.

Changing to non-inertial frames of reference the situation is different; the discussion is more subtle.

8.1.1 Frame-independence in non-inertial systems

Since it is always possible for an observer to distinguish one non-inertial system from any other one by performing an appropriate physical experiment, it is certainly unphysical and even absolutely wrong to impose here the concept of frame-independence as a physical principle or as a physical axiom as it can be done within inertial reference frames. Again, since there does *not* exist anything like an *equivalence principle of non-inertial reference frames*, as it does exist for inertial frames, it is definitely fully inappropriate to demand frame-independence within non-inertial systems. A passively transformed equation like $\tilde{\mathcal{D}} = \sum_{i=1}^{N} \tilde{P}_i = 0$ in a non-inertial system will in general contain fictitious forces with which the equation as a whole can never be independent from the properties owned by the frame itself, irrespective of whether the equation is dynamic or constitutive. But this obviously does not mean that each part $\tilde{P}_i$ of the equation is frame dependent, only the sum of all parts $\tilde{\mathcal{D}}$ is frame-dependent; it can happen that one or even several parts $\tilde{P}_i$ are frame-independent.

In this context it is very important to note two things: First of all, by rewriting the sum $\tilde{\mathcal{D}}$ into an alternative representation $\tilde{\mathcal{D}} = \sum_{i=1}^{N} \tilde{P}_i = \sum_{i=1}^{M} \tilde{Q}_i$ it is always possible to change the amount of frame-independent parts. Secondly, if a certain part $\tilde{P}_i$ of the frame dependent equation $\tilde{\mathcal{D}} = \sum_{i=1}^{N} \tilde{P}_i = 0$ cannot be given by theory, that means if this part needs to be modelled macroscopically, only a physical experiment or a numerical simulation can decide whether the unknown part $\tilde{P}_i$ needs to be modelled frame-dependently or frame-independently.

8.1.2 Form-invariance in non-inertial systems

Now, what is to say about the less restrictive form-invariance in non-inertial systems? When Einstein formulated his general theory of relativity in 1915, he was proud to present a theory that was generally form-invariant, or as how he first called it, a theory that was generally covariant; its equations retained their form under *arbitrary* transformations of the space–time coordinate system.[13] Einstein had the following argument for general covariance: the physical content of a theory is exhausted by a catalog of events, which must be preserved under arbitrary coordinate transformations; all we do in the transformations is relabelling the space–time coordinates assigned to each event. Therefore a physical theory should be generally covariant. Einstein thus claimed that his general theory of relativity rests on three fundamental pillars: on the principle of general covariance, on the principle of constancy in the speed of light for all local inertial reference frames, and on

[12] We will discuss this in more detail in Sect. 9, together with an exemplification in the Appendix.

[13] Important to note is that the theory of general relativity is not a theory of *general* frame-independency. In general relativity it is still possible for an observer to distinguish between inertial and non-inertial systems. In this sense the name "general relativity theory" can be criticized, since this theory presents no true generalization of the relativity or equivalence principle for inertial reference frames towards an equivalence principle for non-inertial systems.

the principle of equivalence between inertial and gravitational mass; only with all three principles together the theory of general relativity arrives at a new description of gravitation in terms of a curved four-dimensional space–time.

Shortly after, Kretschmann [63] pointed out and concurred in by Einstein [64,65] that the principle of general covariance is fully devoid of any physical content. For, Kretschmann urged, it is unessential to declare general covariance as a principle of nature, since any space–time theory whatever can be formulated in a generally covariant form as long as one is prepared to put sufficient energy into the task of reformulating it; thus the theory of general relativity rests on only two pillars, on the principle of a constant speed of light for all observers in their immediate neighbourhood and on the principle of the mass equivalence. In arriving at general relativity, Einstein had used the *tensor calculus* of Ricci and Levi-Civita, where Kretschmann pointed to this calculus as a mathematical tool that made the task of finding generally covariant formulations of theories tractable. In his objection, Kretschmann agreed that the physical content of space–time theories is exhausted by a catalog of events and that they should be preserved under any coordinate transformation, but this, he argued, is no peculiarity of the new gravitational theory presented by Einstein. For this very reason *all* space–time theories can be given generally covariant formulations. For a further discussion of Kretschmann's objection, Einstein's response and of the still active debate that follows see the articles [66–68].

Nevertheless, Kretschmann's objection does seem sustainable, even to the classical laws of fluid motion and to the physics of turbulence modelling. The Appendix of this article shows how it is possible to rewrite the Navier-Stokes equations as well as the Reynolds averaged equations in a general manifest form-invariant way. For this scheme one has to change to a four-dimensional space–time formulation, but since we want to maintain the physics of classical Newtonian mechanics, the physical predictions of the general form-invariant laws of fluid motion will certainly stay identical to those in the usual formulation; the necessary mathematical framework to achieve this will be constructed in Sect. 9. In this context the word *general* form-invariance only refers to coordinate transformations which maintain the Newtonian space–time structure by always fixing the time coordinate as an absolute quantity, otherwise we would not do classical but relativistic fluid mechanics.

Fully in accord with Kretschmann's objection, we see that within the example of reformulating the existing laws of classical fluid motion into a generally form-invariant way, no new physics, which would give rise to new predictions, is implied. This insight inevitably brings about the question of what the necessity or even the advantage is when rewriting existing laws into their general form-invariant representation? The answer surely depends on what one intends to do, if for example the equations are to be solved numerically or even analytically such reformulations do not bring any advantages at all, but if for example the equations are not closed and need to be modelled as a material law or as the equations of turbulence, such form-invariant reformulations automatically bring along consistent and structured modelling arguments in the most natural way when they are based on invariant principles. Furthermore, following the recent article of Matolcsi [69] it is even necessary to operate in a four-dimensional space–time formulation when a dynamical system is to be described mathematically, as the concept of a frame-independent (objective) time derivative like the Jaumann- or the Oldroyd-derivative cannot be properly formulated in three dimensions, because the proper transformation rules for physical quantities between time-dependent or non-inertial reference frames requires the use of four-dimensional Christoffel symbols. This we will discuss in more detail in Sect. 9.

8.2 Active case

The discussion on the concepts of form-invariance and frame-independence from the perspective of active transformations is not so involved as it was in the case of passive transformations. This is due to the fact that the set of active transformations can be split up into a set of transformations which mathematically can be associated with a corresponding passive transformation, and into the remaining set of transformations which do not have to obey any specific physical norms.

Regarding the set where an active transformation is connected to a passive transformation one easily concludes that although such an actively transformed state is physically different from the corresponding passively transformed state, the mathematical consequences in each case, however, are the same in that all statements made on form-invariance and frame-independence under passive transformations equally apply to those special active transformations.

Regarding the set of *arbitrary* active transformations, our physical intuition immediately tells us that if we demand on some observing physical phenomenon the principle of frame-independence or, in contrast to passive transformation, even the weaker principle of form-invariance it will in general imply a restriction upon the nature as how this dynamical system should evolve—how severe this restriction will be depends on the structure of the transformation used, and in how far this demand is actually realized by nature must be experimentally investigated. If on the other hand a given dynamical system happens to admit form-invariance under an active transformation, such a transformation is called an *equivalence transformation* of the system [49], while if it admits frame-independence the active transformation is called a *symmetry transformation* of the system [47–50]. Since a symmetry transformation is more restrictive than an equivalence transformation it implies that not only within passive transformations but also within arbitrary active transformations frame-independence is more restrictive than form-invariance.

9 Construction of the classical Newtonian space–time manifold

In the subsequent development we consider a four-dimensional smooth manifold in which each point can be smoothly labelled by four coordinates x^α. A manifold is characterized by its geometrical structure in that it is either endowed with an affine connection or additionally endowed with a metric. To construct an affinely connected manifold it is necessary to specify the affine connection $\Gamma^\rho_{\mu\nu}$ in all its components, which then fixes the curvature of the manifold given by relation (16). To construct a Riemannian manifold it is necessary to specify a symmetric non-singular metric tensor $g_{\mu\nu}$, which then fixes the remaining geometrical properties of the manifold as the Christoffel symbols given by relation (18) and the curvature by relation (16). Our aim is to construct a physical manifold $\mathcal{N}$

- in which the four coordinates x^α are identified[14] as 3 spatial coordinates x^i and a time coordinate x^0,
- in which physics evolves on the basis of a Newtonian description emerging as a classical limit from Einstein's theory of relativity,
- and in which a change in the frame of reference of a massive observer is without physical significance.

In other words, our aim is to do Newtonian physics with a proper connection to Einsteinian physics in a true four-dimensional formulation and not in a (3+1)-formulation, that means to do Newtonian physics on a four-dimensional manifold and not on a three-dimensional spatial manifold in which time only plays the role of a parameter and not that of an independent coordinate. Furthermore, the physics evolving in the manifold should be completely insensitive to a choice of the reference frame from which it is described from, any physical phenomenon should happen independently of the motion of the observer, irrespective of whether the observer carries mass or not. To construct such a manifold we need to make a mathematical and a physical requirement.

The *mathematical* requirement is geometrical simplicity: Our manifold $\mathcal{N}$ should be such that first of all it is free of torsion, that means the affine connection is symmetric in its two covariant indices, and secondly that it is always possible to globally choose a coordinate system in which the affine connection vanishes, defining it as our *standard coordinate system*. This implies that the manifold is flat, that it is globally without any curvature, because since the curvature as given in (16) vanishes in the standard coordinate system by construction and since the curvature, in contrast to the affine connection (14), transforms as a tensor it thus vanishes in all coordinate systems.

The *physical* requirement is to measure distances: To specify the metric it is necessary to say how physics evolving in this manifold $\mathcal{N}$ should be described. Since we are only interested in a Newtonian description we have to lay down the minimal requirements for Newtonian mechanics, which we will construct from the three principles of general relativity given by Einstein:

(I) the principle of general covariance,
(II) the principle of a constant speed of light for all local inertial reference frames,
(III) the principle of equivalence between inertial and gravitational mass.

They are usually stated as: (I) If properly formulated, the laws of physics are of the same structural form in all coordinate systems; (II) General relativity asserts the existence of locally inertial frames, and with the correct choice of a time unit the speed of light can be assigned the universal numerical value of c in each of those frames. However, measuring the speed of light, with a metric fixed in a specific local inertial reference frame, in a space–time region which is not covered by this frame the numerical value can differ from c. Hence, this principle is better formulated as that light travels along null geodesics whose arc length is zero; (III) The local effects of a gravitational field are equivalent to those appearing in the description of physical phenomena

[14] Greek indices will always run from $0, \ldots, 3$, while Latin indices will only run from $1, \ldots, 3$.

relative to an accelerated frame. In other words, *any* inertial force can be *locally* replaced by a suitably chosen gravitational force, or vice versa.

As already discussed in detail in Sect. 8, Kretschmann's assertion puts principle (I) as a statement without physical content, and any theory, including Newtonian mechanics, can be fitted to it; principle (I) is not a characteristic feature of Einsteinian mechanics. The same holds true for principle (III): Newtonian mechanics can be mathematically reformulated such that it does conform with the general interpretation of principle (III) without changing the physical content and predictions of the usual classical (3+1)-formulation, but then, Newtonian mechanics has to evolve in a *curved* non-Riemannian space–time manifold as was pioneered by Cartan [70] and Friedrichs [71] and then further developed by various authors as Havas [72], Trautman [73] and Ehlers [74]. However, since we have chosen a globally flat manifold it implies that our Newtonian mechanics is not fully compatible with principle (III) in having general equivalence between gravitational and arbitrary inertial forces, but only having restricted equivalence between gravitational force and those inertial forces which emerge from constant linear accelerations [72]. This restriction certainly prevents the construction of a clear and full relationship between gravitation and geometry as the Einstein field equations do in the case of general equivalence. In this sense the gravitational field is automatically decoupled from the structure of the space–time manifold, so that at the end the mathematical requirement for simplicity of using a flat space–time manifold has the wanted physical impact that a change in a frame of reference is without any physical significance, irrespective of whether the frame carries or is carried by a massive observer.

After all, the real difference between Newtonian mechanics and Einsteinian mechanics thus only lies in principle (II). Since Einsteinian mechanics on the one hand assumes that there exists a maximum signal velocity equal to the speed of light c in an inertial system, and since on the other hand Newtonian mechanics assumes that there exist signals propagating with infinite velocity so that points in space are causally connected even if no time goes by, the whole concept of Newtonian mechanics on a space–time manifold will emerge from Einsteinian mechanics simply in the classical limit $c \to \infty$. Given that our manifold is flat the limit can be taken in the Minkowskian manifold of special relativity with the pseudo-Euclidean metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & - \end{pmatrix}, \tag{20}$$

if the standard four-dimensional coordinate system is to be chosen as a global inertial system with Cartesian coordinates. The metric can be used to either define an invariant infinitesimal line element which has the dimension of length

$$ds^2 = \eta_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu, \quad \text{with } \hat{x}^\mu = (ct, x^i), \tag{21}$$

or which has the dimension of time

$$d\tau^2 = \hat{\eta}_{\mu\nu} dx^\mu dx^\nu, \quad \text{with } \hat{\eta}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{c^2} \end{pmatrix}, \quad \text{and } x^\mu = (t, x^i). \tag{22}$$

To perform the classical limit $c \to \infty$ we will use the latter one. Together with its inverse metric $\hat{\eta}^{\mu\nu}$ the relationship

$$\hat{\eta}^{\alpha\lambda} \hat{\eta}_{\lambda\beta} = \delta^\alpha_\beta, \quad \text{with } \hat{\eta}^{\alpha\lambda} = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \end{pmatrix}, \tag{23}$$

which equivalently can be written as

$$\check{\eta}^{\alpha\lambda} \hat{\eta}_{\lambda\beta} = -\frac{1}{c^2} \delta^\alpha_\beta, \quad \text{with } \check{\eta}^{\alpha\lambda} = \begin{pmatrix} -\frac{1}{c^2} & 0 \\ 0 & \end{pmatrix}, \tag{24}$$

degenerates to the following relation as $c \to \infty$:

$$h^{\alpha\lambda} g_{\lambda\beta} = 0, \quad \text{with } h^{\alpha\lambda} = \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix}, \text{ and } g_{\lambda\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{25}$$

where the metrics $\check{\eta}^{\alpha\beta}$ and $\hat{\eta}_{\alpha\beta}$ become independent and equal to the tensors $h^{\alpha\beta}$ and $g_{\alpha\beta}$, respectively. In other words, in the classical limit $c \to \infty$ the unique non-singular Minkowski metric $\eta_{\mu\nu}$ splits up into two in space and time separate singular tensors, which can be identified as a space-like metric $h^{\alpha\beta}$ and a time-like metric $g_{\alpha\beta}$.

Thus our so constructed classical Newtonian space–time manifold $\mathcal{N}$ is a non-Riemannian manifold; its geometrical structure shows, in contrast to a Riemannian manifold, a non-unique and singular metrical connection. In such a manifold it is impossible to assign to *every* contravariant four-vector a corresponding covariant four-vector, and conversely. One rather has to distinguish between two classes of four-vectors, the class of space-like contravariant vectors $a^\alpha = (0, a^i)$ and the class of time-like covariant vectors $w_\alpha = (w_0, 0)$, where in each class one can uniquely assign a corresponding covariant or contravariant vector respectively. For our later considerations it is fundamental to construct these, but before we can do that we first have to determine the coordinate transformations that are compatible with this manifold $\mathcal{N}$.

Taking the classical limit $c \to \infty$ in the relation for the invariant infinitesimal line element of (22)

$$d\tau^2 = \hat{\eta}_{\alpha\beta} dx^\alpha dx^\beta \underset{c\to\infty}{=} g_{\alpha\beta} dx^\alpha dx^\beta = dt^2, \tag{26}$$

leads to the statement that the time coordinate $x^0 = t$ itself is an invariant. Thus the space–time coordinate transformations compatible with the classical Newtonian manifold $\mathcal{N}$ are those in which the time coordinate, up to an additive constant, transforms as an absolute quantity

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta), \text{ with } \tilde{x}^0 = x^0. \tag{27}$$

Compared to Noll's neoclassical space–time manifold [4,13–15] in which a change in frame is restricted to Euclidean transformations in order to guarantee a Newtonian description of continuum physics, the above constructed classical Newtonian space–time manifold $\mathcal{N}$ has the clear advantage in that it can allow for an arbitrary change in frame as long as time stays absolute; Euclidean transformations only form a small subset within these transformations.

9.1 Characteristic features of coordinate transformation with absolute time

For arbitrary coordinate transformations (27) in which the time coordinate transforms invariantly

$$\tilde{x}^0 = x^0, \quad \tilde{x}^i = \tilde{x}^i(x^0, x^j), \tag{28}$$

it is straightforward to show that the following two relations always hold:

$$\frac{\partial \tilde{x}^i}{\partial x^j}\frac{\partial x^j}{\partial \tilde{x}^0} = -\frac{\partial \tilde{x}^i}{\partial x^0}, \quad \text{and} \quad \frac{\partial \tilde{x}^i}{\partial x^j}\frac{\partial x^j}{\partial \tilde{x}^k} = \delta^i_k, \tag{29}$$

where the latter relation indicates that within the three-dimensional spatial subspace the chain rule of differentiation can be applied.

Since the coordinate differentials dx^α transform as a contravariant four-vector, and since the differential time coordinate transforms invariantly $d\tilde{x}^0 = dx^0$, one can construct a new fundamental contravariant four-vector

$$\frac{d\tilde{x}^\alpha}{d\tilde{x}^0} = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{dx^\beta}{dx^0} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \tilde{u}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} u^\beta, \quad \text{with } u^\alpha = (1, u^i), \tag{30}$$

the velocity vector u^α which is to be identified as a pure time-like contravariant vector, since its time component always transforms numerical invariantly $\tilde{u}^0 = u^0 = 1$ and thus can never vanish. Regarding continuum mechanics the definition (30) is to be seen as a transition going from the Lagrangian to the Eulerian description in which the velocity vector then turns into a velocity vector field $u^\alpha = u^\alpha(x^\beta)$.

From the velocity vector field one can now construct another fundamental contravariant kinematic quantity, the acceleration vector field

$$\frac{d\tilde{u}^\alpha}{d\tilde{x}^0} = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\frac{du^\beta}{dx^0} \quad \Longleftrightarrow \quad \tilde{a}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} a^\beta, \quad \text{with } a^\alpha = (0, a^i), \tag{31}$$

which is to be identified as a pure space-like contravariant vector. Within the classical Newtonian manifold $\mathcal{N}$ velocity fields u^α thus always evolve as time-like vectors while accelerations a^α and forces according to Newton's second law $F^\alpha \sim a^\alpha$ always evolve or act as space-like vectors, respectively.

If we consider an arbitrary contravariant four-vector $A^\alpha = (A^0, A^i)$ and an arbitrary covariant four-vector $B_\alpha = (B_0, B_i)$ the following three general statements can be made relative to transformations of type (28):

- The time component of a contravariant vector $A^\alpha = (A^0, A^i)$ always transforms invariantly

$$\tilde{A}^0 = \frac{\partial \tilde{x}^0}{\partial x^\beta} A^\beta = \frac{\partial \tilde{x}^0}{\partial x^0} A^0 = A^0. \tag{32}$$

- The spatial components of a covariant vector $B_\alpha = (B_0, B_i)$ always transform as a three-dimensional tensor

$$\tilde{B}_i = \frac{\partial x^\beta}{\partial \tilde{x}^i} B_\beta = \frac{\partial x^j}{\partial \tilde{x}^i} B_j. \tag{33}$$

- If the time component of the covariant vector $B_\alpha = (B_0, B_i)$ is fixed as $B_0 = -u^i B_i$, where u^i are the spatial components of a velocity four-vector $u^\alpha = (1, u^i)$, the transformation property of B_α stays unchanged, that means it still transforms as a covariant four-vector

$$\begin{aligned}
\tilde{B}_0 &:= -\tilde{u}^i \tilde{B}_i = -\left(\frac{\partial \tilde{x}^i}{\partial x^0} + \frac{\partial \tilde{x}^i}{\partial x^j} u^j \right) \frac{\partial x^k}{\partial \tilde{x}^i} B_k \\
&= \frac{\partial x^k}{\partial \tilde{x}^0} B_k - \delta^k_j u^j B_k = -u^i B_i + \frac{\partial x^i}{\partial \tilde{x}^0} B_i = B_0 + \frac{\partial x^i}{\partial \tilde{x}^0} B_i \\
&= \frac{\partial x^\beta}{\partial \tilde{x}^0} B_\beta,
\end{aligned} \tag{34}$$

where in the second line we have made use of the relations given in (29).

9.2 The relationship between contravariant and covariant tensors

Let $a^\alpha = (0, a^i)$ be an arbitrary space-like contravariant four-vector which in the standard coordinate system according to the result of (25) can be equivalently written as

$$a^\alpha = h^{\alpha\beta} a_\beta, \text{ with } a_\beta = (a_0, a^i), \tag{35}$$

where $a_\beta = (a_0, a_i)$ is a corresponding covariant four-vector with an arbitrary time component a_0, while its spatial components are fixed by $a_i = a^i$. Since the space-like metric $h^{\alpha\beta}$ is singular this relationship cannot be inverted. However, if a_β is to be addressed as the corresponding covariant vector of a space-like contravariant vector a^α, it must be possible to reconstruct a_β solely from the knowledge of a^α via a relationship

$$a_\beta = k_{\beta\alpha} a^\alpha, \tag{36}$$

where the tensor $k_{\beta\alpha}$ is then to be interpreted as the corresponding covariant space-like metric of $h^{\alpha\beta}$. But since a^α only carries space-like information such a reconstruction is only possible if we restrict the domain of the corresponding covariant vector a_β in the arbitrary time coordinate a_0. A restriction which is compatible with the covariant transformation property of a_β is to fix the time component, as proven in (34), to $a_0 = -u^i a_i$. For example if $a^\alpha = (0, a^i)$ represents an acceleration vector it can be uniquely associated with the following covariant acceleration vector $a_\alpha = (-u^i a_i, a^i)$, and conversely.

Now, it is straightforward to show that with this restriction the relation (36) can only hold if the covariant space-like metric is singular and of the following symmetric form [72]:

$$k_{\alpha\beta} = \begin{pmatrix} k_{00} & -u^j \\ -u^i & \end{pmatrix}. \tag{37}$$

Since both singular[15] space-like metrics $h^{\alpha\beta}$ and $k_{\alpha\beta}$ are related by

$$k_{\alpha\mu} h^{\mu\nu} k_{\nu\beta} = k_{\alpha\beta}, \tag{38}$$

the pure time-like component of $k_{\alpha\beta}$ can be determined as $k_{00} = \|\mathbf{u}\|^2 := k_{ij} u^i u^j$.

[15] It is important to have in mind that when operating with singular tensors conclusions are different from those when operating with invertible tensors, for example plugging (36) back into (35) one has to aware that $h^{\alpha\lambda} k_{\lambda\beta} \neq \delta^\alpha_\beta$, in contrast to invertible tensors where one had to conclude unity.

As we could associate to the given contravariant space-like metric $h^{\alpha\beta}$ a corresponding covariant space-like metric $k_{\alpha\beta}$, we can associate to the given covariant time-like metric $g_{\alpha\beta}$ a corresponding contravariant time-like metric $m^{\alpha\beta}$. Its construction is obviously based on an analogous space–time relation (25) as that between $h^{\alpha\beta}$ and $g_{\alpha\beta}$

$$k_{\alpha\lambda}m^{\lambda\beta} = 0, \tag{39}$$

which, as an equation for $m^{\alpha\beta}$, can be solved for to obtain the following symmetric and also singular tensor:

$$m^{\alpha\beta} = u^{\alpha}u^{\beta} = \begin{pmatrix} 1 & u^j \\ u^i & u^i u^j \end{pmatrix}. \tag{40}$$

The relationship between covariant and contravariant time-like four-vectors is thus as follows: Given an arbitrary covariant time-like vector in the standard coordinate system $w_\alpha = (w_0, 0)$ it can be equivalently written as

$$w_\alpha = g_{\alpha\beta}w^{\beta}, \tag{41}$$

with $w^\beta = (w^0, w^i)$ as a corresponding contravariant time-like vector. The reconstruction of w^β from w_α is given by

$$w^{\alpha} = m^{\alpha\beta}w_{\beta}, \tag{42}$$

and thus only possible if the contravariant time-like vectors are of the type $w^\alpha = w_0(1, u^i)$. Hence, the velocity vector $u^\alpha = (1, u^i)$ can be associated with the following covariant velocity vector $u_\alpha = (1, 0)$, and conversely.

Eventually we could show that our non-Riemannian manifold $\mathcal{N}$ exhibits four different singular tensors which can be interpreted as metrical tensors in the following sense: Only within the subsets of space-like contravariant four-vectors $a^\alpha = (0, a^i)$ and time-like covariant four-vectors $w_\alpha = (w_0, 0)$ one can operate with two space-like metric tensors $h^{\alpha\beta}$ and $k_{\alpha\beta}$ and with two time-like metric tensors $g_{\alpha\beta}$ and $m^{\alpha\beta}$ to establish a connection between contravariance and covariance in each case, respectively. This correspondence is not restricted to four-vectors only, but applies to all tensors $T^{\alpha\beta\ldots}_{\mu\nu\ldots}$ of any rank, if only each component can either be identified as space-like or as time-like. More physically, this interpretation states that our non-Riemannian manifold $\mathcal{N}$ allows for four different tensors, which in well defined subspaces of $\mathcal{N}$ behave as metrical tensors to measure distances and/or to measure time, showing the intended result that in Newtonian physics, unlike in Einsteinian physics, space and time measurements are uncorrelated.

For all other classes of tensors there is no unambiguous correspondence between covariance and contravariance. In these cases the four tensors $h^{\alpha\beta}$, $k_{\alpha\beta}$, $g_{\alpha\beta}$ and $m^{\alpha\beta}$ may not be seen as metric tensors, but rather as tensors which can be used to change the transformation properties of tensors to which they are applied: For example, if A^ν is an arbitrary contravariant four-vector then one can construct a covariant four-vector of the form $B_\mu = k_{\mu\nu}A^\nu$, but which in general cannot be used to reconstruct the vector A^ν back again, that means in general $A^\nu \neq h^{\nu\mu}B_\mu$ or $A^\nu \neq m^{\nu\mu}B_\mu$.

In analogy to a Riemannian manifold it is possible to express the spatial components of the affine connection $\Gamma^{\rho}_{\mu\nu}$ of the manifold $\mathcal{N}$ with its singular metric tensors. To achieve this, one first has to characterize the metric tensors in arbitrary coordinate systems. Up to now, we only characterized these singular metrics in the standard coordinate system, but with the invariant properties introduced and discussed in Sect. 5, as symmetry, signature and vanishing covariant derivatives, we are able to charaterize a metric tensor in all coordinate systems.

To determine this spatial representation one first has to recognize that on the basis of the transformation rule (14) the following components of the affine connection vanish in all coordinate systems:

$$\Gamma^{0}_{\mu\nu} = 0. \tag{43}$$

The only metric which can account for non-vanishing covariant space-like components, is the metric $k_{\mu\nu}$. Its invariant property of a vanishing covariant derivative, however, is of course only valid within the spatial subspace

$$\nabla_\lambda k_{ij} = 0, \tag{44}$$

which, when adding the cyclic permuted spatial variants to it, can be equivalently written as

$$2\Gamma^{\sigma}_{li}k_{\sigma j} = \partial_l k_{ij} + \partial_i k_{lj} - \partial_j k_{il}. \tag{45}$$

Using the numerical invariance of the following spatial expression:

$$k_{sj}h^{jr} = \delta^r_s, \tag{46}$$

together with (43), one can solve for the affine connection in (45)

$$\Gamma^r_{il} = \tfrac{1}{2}h^{rj}\left(\partial_i k_{lj} + \partial_l k_{ij} - \partial_j k_{il}\right), \tag{47}$$

as a relationship which is valid in all coordinate systems. The last expression shows that Γ^r_{il} is the three-dimensional Christoffel symbol formed from k_{il}—in this sense we will from now on call the full affine connection $\Gamma^\rho_{\mu\nu}$ of $\mathcal{N}$ a Christoffel symbol. With the relations (43) and (47) the geometrical construction of the classical Newtonian space–time manifold $\mathcal{N}$ is finally completed.

9.3 Advantage of the 4-formulation over the (3+1)-formulation

The advantage can immediately be seen in the case of the velocity vector. In the 4-formulation the velocity transforms as a tensor

$$\tilde{u}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}u^\beta, \tag{48}$$

while if the four-velocity is decomposed into its spatial components $\alpha = i$

$$\tilde{u}^i = \frac{\partial \tilde{x}^i}{\partial x^0} + \frac{\partial \tilde{x}^i}{\partial x^j}u^j, \tag{49}$$

this spatial velocity, which corresponds to the velocity field in the usual classical continuum mechanical (3+1)-formulation, does not transform as tensor anymore under the given coordinate transformation (28). Thus the clear advantage of using the four-velocity is that it contributes to form-invariance in equations manifestly, which cannot be said for the velocity field in the (3+1)-formulation. In short we can say that the four-velocity transforms form-invariantly, while this is not the case for the three-velocity.

On the other hand if we investigate the covariant material derivative $u^\alpha\nabla_\alpha u^\beta$ which transforms as a contravariant four-vector

$$\tilde{u}^\alpha\tilde{\nabla}_\alpha\tilde{u}^\beta = \frac{\partial \tilde{x}^\beta}{\partial x^\lambda}u^\alpha\nabla_\alpha u^\lambda, \tag{50}$$

and where its spatial components in the standard coordinate system correspond to the usual material time derivative in the (3+1)-formulation

$$u^\alpha\nabla_\alpha u^i = u^\alpha\partial_\alpha u^i = \partial_t u^i + u^j\partial_j u^i, \tag{51}$$

these spatial components, in contrast to the velocity field, transform as a contravariant three-vector

$$\begin{aligned}\tilde{u}^\alpha\tilde{\nabla}_\alpha\tilde{u}^i &= \frac{\partial \tilde{x}^i}{\partial x^\lambda}u^\alpha\nabla_\alpha u^\lambda \\ &= \frac{\partial \tilde{x}^i}{\partial x^0}u^\alpha\nabla_\alpha u^0 + \frac{\partial \tilde{x}^i}{\partial x^j}u^\alpha\nabla_\alpha u^j = \frac{\partial \tilde{x}^i}{\partial x^0}u^\alpha u^\sigma\Gamma^0_{\alpha\sigma} + \frac{\partial \tilde{x}^i}{\partial x^j}u^\alpha\nabla_\alpha u^j \\ &= \frac{\partial \tilde{x}^i}{\partial x^j}u^\alpha\nabla_\alpha u^j,\end{aligned} \tag{52}$$

since the components $\Gamma^0_{\alpha\beta}$ of the Christoffel symbol vanish in all coordinate systems which are linked by coordinate transformations of the type (28), as already shown in (43). Thus the material derivative is a pure space-like four-vector $u^\alpha\nabla_\alpha u^\beta = (0, u^\alpha\nabla_\alpha u^i)$ in all coordinate systems within the classical Newtonian manifold $\mathcal{N}$, and can be fully identified as the material time derivative from the (3+1)-formulation. The advantage of the 4-formulation is now that one has immediate access to the structure of the transformed material time derivative

$$\tilde{u}^\alpha\tilde{\nabla}_\alpha\tilde{u}^i = \partial_t\tilde{u}^i + \tilde{u}^j\tilde{\partial}_j\tilde{u}^i + \tilde{u}^\alpha\tilde{u}^\sigma\tilde{\Gamma}^i_{\alpha\sigma}, \tag{53}$$

which is universally valid for all coordinate transformations of the type (28). So, although the material time derivative transforms form-invariantly it does not transform frame-independently due to the appearance of the Christoffel symbol in the transformed expression (53), which ultimately will give rise to inertial forces in any physical equation where the material time derivative is part of. This quantifies the statement made in Sect. 8, that frame-independence is more restrictive than form-invariance: In the terminology of classical continuum mechanics, the material time derivative $u^\alpha \nabla_\alpha u^\beta$ does not transform objectively.

As is well known, the general construction of objective tensors in the (3+1)-formulation is rather involved and cumbersome. In the 4-formulation, however, such constructions are straightforward; for example all objective material time derivatives in the (3+1)-formulation emerge from a Lie-derivative in the 4-formulation, as it was recently discussed in detail by Matolcsi [69]. Since the definition of Lie derivatives (13) is based on a transportation along a vector field, and not on the geometrical structure of the underlying manifold, they intrinsically transform as objective tensors. The advantage of Lie-derivatives in the 4-formulation is that they include time derivatives.

Let us consider an arbitrary contravariant space-like vector $a^\alpha = (0, a^i)$; its Lie derivative (13) along the velocity vector field $u^\alpha = (1, u^i)$ is then given as the following contravariant space-like vector:

$$\begin{aligned} \mathcal{L}_u a^\alpha &= u^\lambda \partial_\lambda a^\alpha - a^\lambda \partial_\lambda u^\alpha \\ &= \Big(0, \partial_t a^i + u^j \partial_j a^i - a^j \partial_j u^i\Big) , \end{aligned} \tag{54}$$

which can be identified as the "upper convected time derivative" in the (3+1)-formulation. Thus $\mathcal{L}_u a^i$ defines an objective material time derivative for a three-dimensional vector field a^i, also known as the Oldroyd derivative. This derivative can be generalized to contravariant space-like tensors $T^{\alpha\beta\cdots}$ of any rank just by taking its Lie derivative. In particular in fluid mechanics the Oldroyd derivative is the change of rate of a tensor field along an infinitesimal fluid element which accounts for stretching during advection: a tensor field is said to be Oldroyd transported if its Oldroyd derivative vanishes.

If we consider an arbitrary covariant space-like vector $b_\alpha = (0, b_i)$ its Lie derivative along the velocity vector field $u^\alpha = (1, u^i)$ gives a covariant vector which is not space-like,

$$\begin{aligned} \mathcal{L}_u b_\alpha &= u^\lambda \partial_\lambda b_\alpha + b_\lambda \partial_\alpha u^\lambda \\ &= \Big(b_j \partial_t u^j, \partial_t b_i + u^j \partial_j b_i + b_j \partial_i u^j\Big) , \end{aligned} \tag{55}$$

but since the spatial components of a covariant four-vector always transform as covariant three-vector, as was shown in (33), the spatial components $\mathcal{L}_u b_i$ define an objective material time derivative of a three-dimensional vector field b_i, also known as the "lower convected time derivative" in the (3+1)-formulation. This derivative can be generalized to covariant space-like tensors $T_{\alpha\beta\ldots}$ of any rank.

If we now consider two contravariant space-like vectors, an arbitrary $a^\alpha = (0, a^i)$ and the constant vector $\hat{c}^\alpha = (0, \;\;)$, together with their corresponding covariant vectors $a_\beta = k_{\beta\alpha} a^\alpha$, $\hat{c}_\beta = k_{\beta\alpha} \hat{c}^\alpha$ and then construct the following mixed tensor of rank two:

$$T^\alpha_\beta := \tfrac{1}{2} \left(\hat{c}^\alpha \mathcal{L}_u a_\beta + \hat{c}_\beta \mathcal{L}_u a^\alpha \right) , \tag{56}$$

where its spatial components transform as a mixed three-tensor

$$\begin{aligned} \tilde{T}^i_j &= \frac{\partial \tilde{x}^i}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \tilde{x}^j} T^\alpha_\beta = \frac{\partial \tilde{x}^i}{\partial x^0} \frac{\partial x^l}{\partial \tilde{x}^j} T^0_l + \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} T^k_l \\ &= \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} T^k_l , \end{aligned} \tag{57}$$

the diagonal spatial components

$$T^i_i = \partial_t a^i + u^k \partial_k a^i + \tfrac{1}{2} \Big(a_k \partial_i u^k - a^k \partial_k u^i \Big) \tag{58}$$

define an objective material time derivative of a three-dimensional vector field $a^i = a_i$ which is known as the Jaumann derivative in the (3+1)-formulation. Also the Jaumann derivative can easily be generalized to contravariant space-like tensors $T^{\alpha\beta\cdots}$ of any rank. In particular in fluid mechanics the Jaumann derivative is the change of rate of a tensor field along an infinitesimal fluid element which accounts for rotations during advection: a tensor field is said to be Jaumann transported if its Jaumann derivative vanishes.

Finally, we want to mention that the Lie derivative is not only suitable to construct objective material derivatives, but that it can also be utilized to generate special objective quantities with certain constraints, as for example the following contravariant tensor of rank two:

$$
\begin{aligned}
S^{\alpha\beta} &:= -\tfrac{1}{2}\mathcal{L}_u h^{\alpha\beta} \\
&= -\tfrac{1}{2}\left(u^\lambda \partial_\lambda h^{\alpha\beta} - h^{\lambda\beta}\partial_\lambda u^\alpha - h^{\alpha\lambda}\partial_\lambda u^\beta\right).
\end{aligned} \tag{59}
$$

Here one has to be careful in stating that $S^{\alpha\beta}$ is an objective tensor, which in general is also not true, since the Lie derivative acts on a quantity $h^{\alpha\beta}$ which itself defines the geometrical structure of the underlying manifold. The Lie derivative of the space-like metric $h^{\alpha\beta}$ will in general not be frame-independent. Since time is an absolute object in Newtonian mechanics, frame-independence is thus obviously only given if the geometrical quantity defining the underlying manifold $\mathcal{N}$ transforms such $h^{\alpha\beta} \to \tilde{h}^{\alpha\beta}$ that the transformed space-like metric $\tilde{h}^{\alpha\beta}$ is independent of time. As can be easily shown, this property is guaranteed if the coordinate transformations (28) are restricted to curvilinear Euclidean transformations

$$
\tilde{x}^0 = x^0, \quad \tilde{x}^i = Q^i{}_j(x^0)\,\tilde{x}^j(x^k) + c^i(x^0), \tag{60}
$$

in which all time-dependence of the coordinate transformation (28) is extracted into an orthogonal matrix $Q^i{}_j(x^0)$ and into a translation vector $c^i(x^0)$.

Only within this restriction of rotating and linear accelerating frames the Lie derivative (59) is an objective tensor, which in the standard coordinate system has the canonical form

$$
S^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & S^{ij} \end{pmatrix}, \quad \text{with} \quad S^{ij} = \tfrac{1}{2}\left(\delta^{jk}\partial_k u^i + \delta^{ik}\partial_k u^j\right). \tag{61}
$$

The spatial quantities S^{ij} are the familiar Cartesian components of the rate of deformation tensor. Since $S^{\alpha\beta}$ is a pure space-like tensor and since it has the canonical form (61), we shall call it the rate of deformation four-tensor. Now, this process can be repeated by taking the Lie derivative of $S^{\alpha\beta}$ to give a new Euclidean objective tensor which again is space-like,

$$
\begin{aligned}
S^{\alpha\beta}_{(1)} &:= \mathcal{L}_u S^{\alpha\beta} \\
&= u^\lambda \partial_\lambda S^{\alpha\beta} - S^{\lambda\beta}\partial_\lambda u^\alpha - S^{\alpha\lambda}\partial_\lambda u^\beta,
\end{aligned} \tag{62}
$$

which in the standard coordinate system has the canonical form

$$
S^{\alpha\beta}_{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & S^{ij}_{(1)} \end{pmatrix}, \quad \text{with} \quad S^{ij}_{(1)} = \partial_t S^{ij} + u^k \partial_k S^{ij} - S^{kj}\partial_k u^i - S^{ik}\partial_k u^j. \tag{63}
$$

We see that the above process may be continued infinitely to obtain an infinite sequence of Euclidean objective four-tensors $S^{\alpha\beta}, S^{\alpha\beta}_{(1)}, \ldots, S^{\alpha\beta}_{(n)}, \ldots$. Since all of these fields are space-like and have a canonical form similar to (63) they induce a sequence of three-dimensional tensor fields $S^{ij}, S^{ij}_{(1)}, \ldots, S^{ij}_{(n)}, \ldots$, which are to be seen as differential Euclidean invariants in a classical continuum mechanical motion.

10 Towards an unambiguous formulation of material frame-indifference (MFI)

Using the elaborately worked out mathematical framework of the classical Newtonian space–time manifold $\mathcal{N}$ and its corresponding terminology defined in the previous Sections, we are finally in a position to formulate the concept of MFI unambiguously. For this we proceed as follows: we first clarify and develop this concept internally, that means in the context of this article we will reformulate it, with which we then try to clarify it externally, that means to resolve still existing confusions and misconceptions about this concept among various articles in the community of classical continuum mechanics which were introduced in the historical review of Sect. 2.

Our development will be solely based on an observer dependent formulation. We will not consider and discuss any aspects of an observer independent formulation as proposed by Matolcsi [26,30] or recently also by Noll [38]. The reason is that in such a formulation the concept of MFI on the one hand is always intrinsically satisfied, while on the other hand in quantifying physical results one always has to decompose this formulation back into an observer dependent formulation. An observer independent formulation is rather of philosophical interest, but is useless when physical quantities need to be compared with an experiment, where at least one observer is needed. Physics uses the language of mathematics, which eventually is a science of numbers. Thus physics is a science which quantifies nature by using numbers, but these numbers are relative and will only show physical significance if they refer to a given observer, so that his reasoning can be compared with experiments or with other observers; this is what physics is all about.

10.1 Internal clarification

We begin our development exactly at the root of all controversy, in the Truesdell and Noll formulation of the MFI-principle [4], which, as was already discussed in Sect. 2, they stated in two forms

(a) in the Hooke–Poisson–Cauchy form: constitutive equations must be invariant under a superimposed rigid rotation of the material,
(b) in the Zaremba–Jaumann form: constitutive equations must be invariant under an arbitrary change of the observer.

These two statements must be read in the context of Noll's neoclassical space–time structure [13–15], where an arbitrary change of the observer only refers to Euclidean transformations. As already indicated in Sect. 2 and discussed in Sect. 4, Noll's neoclassical space–time manifold is not the appropriate mathematical framework to formulate a concept as that of MFI. By construction this neoclassical manifold cannot support the following advantages which the classical Newtonian space–time manifold from Sect. 9 can offer:

- the concept of an observer as a four-dimensional coordinate system,
- the general concepts of active and true passive transformations,
- in order to maintain the decoupling from an observer to a physical phenomenon as well as to guarantee a Newtonian description of continuum mechanics, the changes of an observer are no longer restricted to Euclidean transformations, but can be arbitrary coordinate transformations which only have to respect the classical Newtonian space–time structure, that of absolute time,
- that all physical laws can be manifestly written or rewritten into a form-invariant or covariant form (Kretschmann's objection),
- the concept of frame-dependence can be localized solely in quantities which define the geometrical structure of the manifold: either as non-vanishing time-like components of the Christoffel symbol, or as a temporal change in the components of the metric tensors,
- a well-defined connection to relativistic continuum mechanics.

Now, when looking at the two statements (a) and (b) from the perspective of the classical Newtonian space–time structure, the form of Hooke, Poisson and Cauchy can be identified as a statement based on an active transformation, while the form of Zaremba and Jaumann as a statement based on a passive transformation. Since an active *rigid* rotation can always be associated with a corresponding passive rotation, and since the Zaremba–Jaumann form is based on Euclidean transformations which include rotations as a special subset, the Zaremba–Jaumann form implies the Hooke–Poisson–Cauchy form, but not conversely.[16] In other words, relative to rotations the two forms lead to the same mathematical consequences, although physically they are still two different statements.

Since a constitutive equation satisfying the Zaremba–Jaumann form also satisfies the Hooke–Poisson–Cauchy form, the MFI-principle, as stated in (a) and (b), mathematically reduces into only one form, the Zaremba–Jaumann form. Dropping the restriction of Euclidean transformations the MFI-principle, as stated in (a) and (b), now generalizes within the classical Newtonian space–time structure $\mathcal{N}$ into the following single statement:

(α) Constitutive equations must be invariant under passive transformations respecting the property of absolute time.

[16] This reduction is not a typical consequence of the classical Newtonian space–time structure, since that the Zaremba–Jaumann form is more restrictive than the Hooke–Poisson–Cauchy form can also be deduced within Noll's neoclassical space–time structure, for more details see [4].

- If the term "invariant" in statement (α) is now interpreted as form-invariant, the MFI-principle turns into a principle that will always be satisfied automatically, irrespective of the constitutive equation to be used. In this case Kretschmann's objection applies, that in a four-dimensional space–time formulation every physical law and thus also every mathematical reasonable modelled constitutive equation can be written manifestly in a form-invariant way. So, relative to form-invariance the MFI-principle turns into a physical vacuous statement, incapable of reducing constitutive equations during a modelling process. We see how easy things get, when the correct mathematical framework is used, since this information was totally hidden in the inappropriate mathematical framework of Noll.
- If the term "invariant" in statement (α) is interpreted as the more restrictive frame-independence, MFI can no longer be considered as a principle of nature anymore. It is clear that the concept of MFI can only hold for passive transformations connecting inertial frames of references, but for passive transformations which induce non-inertial frames of references, this principle can no longer hold in a general sense. As we already discussed in detail in Sects. 7 and 8, this is due to the fact, that for non-inertial systems there does not exist an equivalence principle as it does exist for inertial systems. There is no chance, therefore, that MFI can be universally valid, since the most basic ingredient of all classical constitutive equations, Newton's Second Law, as a dynamical law, will always violate frame-independency in non-inertial systems. In other words, classical constitutive equations which are statistically constructed from a classical microscopic many-particle mechanics via the classical Boltzmann equation, are always connected to Newton's laws of motion, and thus will show dependencies on the non-inertial frame used. If somewhere in the statistical averaging process or even on a macroscopical level MFI is demanded it will be a restriction upon nature; the corresponding constitutive equation can only be seen as an idealization of the physical phenomenon to be described. In how far this idealization is applicable to describe a certain physical phenomenon can only be verified by performing a physical experiment. The quality of this idealization certainly depends on how strong these dependencies will be on a particular material under consideration. There will be a wide range of dependencies depending on whether one studies solids, dense fluids or rarified gases ranging from a very weak to a very strong frame-dependence behaviour, respectively. So, relative to frame-independence the concept of MFI cannot be elevated to the status of a principle of nature; constitutive equations will in general depend on the frame properties of non-inertial systems.

However, continuum mechanical experiments reveal that for materials as solids and ordinary dense fluids the MFI-principle is a reasonable approximation to reduce constitutive equations; possible frame dependencies in these materials can be well neglected within the predictions of classical continuum mechanics. For those materials the concept of MFI can be used as an ultimate modelling technique, which in accordance with statement (α) can now be precisely formulated as

(α^*) Constitutive equations within a classical Newtonian space–time setting $\mathcal{N}$ must be *frame*-independent under passive transformations respecting the property of absolute time.[17]

Up to now we have only considered passive transformations as well as active transformations which can be associated with passive transformations. What is missing are active transformations which cannot be associated with a corresponding passive transformation. As was discussed in detail in Sect. 8, demanding MFI will imply also for those transformations a restriction upon nature, but, in contrast to passive transformations, special restrictions can already be achieved by demanding only form-invariance. However, since frame-independence is always more restrictive than form-invariance, the MFI-principle for active transformations has to be formulated as

(β^*) Constitutive equations within a classical Newtonian space–time setting $\mathcal{N}$ must be *frame*-independent under active transformations.[18]

At first sight, there may seem to be little difference in the formulations (a) and (b) given by Truesdell and Noll and its reformulations given in (α^*) and (β^*). But, by closer inspection the difference becomes noticeable: The statements (α^*) and (β^*) do not constitute a fundamental law of nature, they rather show their applicability only as an approximation for a certain class of materials. Furthermore, the statements (α^*) and (β^*) use a general and more natural terminology in which each term can be traced back to a mathematically properly defined concept worked out in this article.

[17] In how far a material law as that for a solid or an ordinary dense fluid actually can fulfill this strong principle is questionable, usually the transformations are weakened to Euclidean transformations.

[18] In the sense of the previous footnote, the MFI-principle is usually weakened here to rigid rotations.

10.2 External clarification

At last we are able to comment the historical review on MFI as given in Sect. 2.

Important to note is that in all articles to be discussed here the change in a frame of reference only refers to Euclidean transformations, since the underlying mathematical framework in all these articles is that of Noll's neoclassical space–time structure [13–15]. In the following we will thus only distinguish between inertial observers and Euclidean observers passively as well as actively, if in the latter an observer is attached to an actively transformed physical state.

The article of Murdoch from 1983 [24], where he claims that kinetic theory is not at variance with MFI as long as the intrinsic spin of the material is considered as an own constitutive variable, is not convincing at all. What Murdoch actually demonstrates is that the relevant constitutive equations can be written form-invariantly, but not frame-independently, which, as we now know, is a trivial insight in a true four-dimensional formulation. After all, he only establishes an objective communication between two Euclidean observers, in that if one observer detects frame-dependence so will the other observer, but not an objective communication between an Euclidean observer and an inertial observer as MFI in its full interpretation would demand.

The short essay of Ryskin in 1985 [25] correctly states, as we could see in the development of this article, that the root of all controversy on MFI lies in not using the appropriate mathematical framework, that of a true four-dimensional formulation. But in his reasoning Ryskin unfortunately confused Einstein's principle of general covariance with the concept of frame-independence by interpreting general covariance as a concept which automatically implies frame-independence, which is certainly not the case, as we clearly could demonstrate in this article.

The notion of form-invariance (FI) in the articles of Svendsen and Bertram [32,33] gives rise to severe confusion, since their definition of form-invariance does not conform at all to the definition used in the physics community where form-invariance is exactly defined as it is defined in Sect. 8. Actually Svendsen and Bertram's concept of FI corresponds to frame-indifference, while our defined form-invariance corresponds to their notion of Euclidean frame-indifference (EFI). To state that FI is analogous to Einstein's principle of relativity is not correct. The principle of relativity which Einstein discussed in his seminal article of 1905 [10] only refers to inertial frames of references where form-invariance and frame-independence imply each other. In non-inertial frames, however, there does not exist a *global* relativity principle as it exists for inertial frames. Thus, even within general theory of relativity, form-invariance is a much weaker principle than frame-independence.

The misleading notions of form-invariance and frame-independence in the articles of Svendsen and Bertram [32,33] form the basis of the controversy between Murdoch and Liu [5,6,36,37]. Concerning the notion of "objectivity" they both have a different conception of this term which does not match; thus in a strict sense their articles cannot be compared with each other. In our terminology Murdoch's conception of "objectivity" on one hand is that of *frame-independence* between two or more Euclidean observers, but on the other hand it is only that of *form-invariance* between an Euclidean observer and an inertial observer, while Liu's conception of "objectivity" is that of general form-invariance between *any* observers, which Liu terms as "Euclidean objectivity" according to Svendsen and Bertram's notion of Euclidean frame-indifference (EFI). Now if 'objectivity' is interpreted as passive form-invariance as defined in Sect. 8 it surely implies no restrictions upon constitutive equations, but if interpreted as passive frame-independence restrictions in general will be implied, which then can be used to reduce constitutive equations. In this sense it is clear that Murdoch's demand in "objectivity" can reduce constitutive equations relative to Euclidean observers, but this reduction does not resemble the reduction of constitutive equations when demanding the MFI-principle between *any* observers, which Liu criticizes. Thus Murdoch's principle of objectivity is only to be understood as a limited MFI-principle. Furthermore, Murdoch's claim that his notion of "isrbm" in the interpretation of 'two motions one observer' should be discarded is not convincing. Since every passive transformation can be associated with a corresponding active transformation implying the same mathematical consequences, it is clear, within the same reasoning as done before for passive transformations, that "isbrm" is not part of Murdoch's limited MFI-principle, but is part of the full MFI-principle as understood by Liu and understood in this article, and thus cannot be discarded.

The review article of Speziale in 1998 [42], where he claims that under arbitrary Euclidean transformations an analogy to the MFI-principle for reducing constitutive equations can be applied in the limit of two-dimensional turbulence, is only true as long as the *rotational part* of Euclidean transformations is restricted to pure uniform rotations. Furthermore, to claim that the extended Galilean group of transformations is the invariance group for *all* possible fluctuating dynamics in turbulence and that this result is *always* consistent with Einstein's equivalence principle, cannot be correct: first of all, when solving for fluctuating variables in specific turbulent flow configurations there is no guarantee that their solutions will always admit *full* extended Galilean invariance. Although all corresponding dynamical equations are not showing any *explicit* dependence on linear accelerations, the fluctuating solutions, however, since they are always coupled to the solutions of the averaged velocities, will eventually show *implicit* dependence originating from initial and boundary conditions which need to be implemented whenever specific physical flows are considered. In other words, implementing initial and boundary conditions in general will break invariance properties of the governing equations. Secondly, his small proof towards consistency with Einstein's equivalence principle is only valid for that specific example given in [42], for two hypothetical thermomechanical experiments, where one experiment is set in an inertial system without gravitation and the other one in a free-falling system with a time-varying gravitational field.[19] In this case the global indistinguishability between these two reference frames definitely leads to an invariance of the fluctuating dynamics under the extended Galilean group, but then it is not correct to conclude that this indistinguishability will also be valid for *all* possible reference frames which are only separated by a time-dependent linear acceleration.

Finally, an example which clearly shows the superiority of a four-dimensional space–time formulation towards the usual (3+1)-formulation in classical continuum mechanics is the article of Sadiki and Hutter [43], where after laborious calculations the authors come to the conclusion that the transport equation of the Reynolds stress tensor as well as the transport equation of the turbulent heat flux vector can be written in a form-invariant way relative to Euclidean transformations and that these equations, irrespective of their form-invariance, still show frame-dependency. After all, this insight is now of no surprise for us: If one uses right from the outset a four-dimensional space–time formulation in the background of Kretschmann's objection, this specific part of their article can be marked as redundant, since both transport equations, as being part of the space–time theory of turbulent fluid dynamics, can always be written in a manifest form-invariant way, not only relative to Euclidean transformations but also relative to any coordinate transformation which respects the Newtonian space–time structure, if surely the same physical predictions as in the usual formulation should hold; that form-invariance does not imply frame-independence can now also be marked as a redundant information in that article.

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Appendix: General covariance of the Navier-Stokes equations

In the (3+1)-formulation with Cartesian coordinates the Navier-Stokes equations have the canonical form

$$\begin{aligned} \partial_i u^i &= 0, \\ \partial_t u^i + u^j \partial_j u^i &= \partial_j \sigma^{ij}, \\ \text{with } \sigma^{ij} &= -\delta^{ij} p + 2\nu S^{ij}, \quad \text{and} \quad S^{ij} = \tfrac{1}{2}\left(\delta^{ik}\partial_k u^j + \delta^{jk}\partial_k u^i\right), \end{aligned} \tag{64}$$

where σ^{ij} is the symmetrical Cauchy stress tensor of an incompressible linearly viscous fluid with strain rate tensor S^{ij}. The transition into the 4-formulation relative to the standard coordinate system[20] of the classical Newtonian space–time manifold $\mathcal{N}$ is straightforwardly given as

$$\begin{aligned} \partial_\alpha u^\alpha &= 0, \\ u^\alpha \partial_\alpha u^\beta &= \partial_\alpha \sigma^{\alpha\beta}, \\ \text{with } \sigma^{\alpha\beta} &= -h^{\alpha\beta} p + 2\nu S^{\alpha\beta}, \quad \text{and } S^{\alpha\beta} = \tfrac{1}{2}\left(h^{\alpha\lambda}\partial_\lambda u^\beta + h^{\beta\lambda}\partial_\lambda u^\alpha - u^\lambda \partial_\lambda h^{\alpha\beta}\right), \end{aligned} \tag{65}$$

where $u^\alpha = (1, u^i)$ is the time-like four-velocity, $h^{\alpha\beta}$ the contravariant space-like metric tensor, and $S^{\alpha\beta} = -\frac{1}{2}\mathcal{L}_u h^{\alpha\beta}$ the Euclidean objective strain rate four-tensor, which are all defined and discussed in detail in Sect. 9. Since the tensor $\sigma^{\alpha\beta}$ is space-like it can be properly identified as the Cauchy stress four-tensor. It is important to have in mind that the physical content of system (65) is the same as that of system (64), it is just

[19] In relativistic mechanics Einstein's equivalence principle is a local principle in space *and* time, while in non-relativistic mechanics it is only a local principle in space, due to the absoluteness of time in Newtonian mechanics.

[20] In Sect. 9 the standard coordinate system was defined as an inertial system with spatial Cartesian coordinates.

an equivalent mathematical reformulation of the Navier-Stokes equations leading to the very same physical predictions as the usual Navier-Stokes equations in the (3+1)-formulation.
Now, since all components of the Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$ in the standard coordinate system vanish by definition, the covariant derivative in this system is equivalent to the partial derivative $\nabla_\alpha = \partial_\alpha$ so that the four-dimensional Navier-Stokes equations (65) can be written in a manifest covariant form

$$\begin{aligned} \nabla_\alpha u^\alpha &= 0, \\ u^\alpha \nabla_\alpha u^\beta &= \nabla_\alpha \sigma^{\alpha\beta}, \\ \text{with } \sigma^{\alpha\beta} &= -h^{\alpha\beta} p + 2\nu S^{\alpha\beta}, \quad \text{and } S^{\alpha\beta} = \tfrac{1}{2}\left(h^{\alpha\lambda}\nabla_\lambda u^\beta + h^{\beta\lambda}\nabla_\lambda u^\alpha - u^\lambda \nabla_\lambda h^{\alpha\beta} \right). \end{aligned} \tag{66}$$

The system (66) displays manifest form-invariance: all equations are solely built up by quantities which transform as tensors. The system (66) even displays general form-invariance in that its structural form will stay unchanged under arbitrary coordinate transformations which are compatible with the underlying classical Newtonian space–time manifold $\mathcal{N}$.
However, as was discussed in detail in Sect. 8, form-invariance does not imply frame-independence. For the Navier-Stokes equations written in the form (66) this statement can now be easily quantified: If we change the frame of reference to an arbitrary non-inertial system which is compatible with the underlying classical Newtonian space–time manifold $\mathcal{N}$, that means if we change our coordinate system according to

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta), \text{ with } \tilde{x}^0 = x^0, \tag{67}$$

in which the time coordinate is an absolute quantity, the Navier-Stokes equations in the form (66) will transform form-invariantly,

$$\begin{aligned} \tilde{\nabla}_\alpha \tilde{u}^\alpha &= 0, \\ \tilde{u}^\alpha \tilde{\nabla}_\alpha \tilde{u}^\beta &= \tilde{\nabla}_\alpha \tilde{\sigma}^{\alpha\beta}, \\ \text{with } \tilde{\sigma}^{\alpha\beta} &= -\tilde{h}^{\alpha\beta} \tilde{p} + 2\nu \tilde{S}^{\alpha\beta}, \text{ and } \tilde{S}^{\alpha\beta} = \tfrac{1}{2}\left(\tilde{h}^{\alpha\lambda}\tilde{\nabla}_\lambda \tilde{u}^\beta + \tilde{h}^{\beta\lambda}\tilde{\nabla}_\lambda \tilde{u}^\alpha - \tilde{u}^\lambda \tilde{\nabla}_\lambda \tilde{h}^{\alpha\beta} \right), \end{aligned} \tag{68}$$

but irrespective of this property, the system (68) will show frame-dependence emerging from the four-dimensional Christoffel symbol in the covariant differentiation

$$\tilde{\nabla}_\alpha \tilde{u}^\beta = \tilde{\partial}_\alpha \tilde{u}^\beta + \tilde{u}^\lambda \tilde{\Gamma}^\beta_{\alpha\lambda}, \tag{69}$$

which itself transforms as

$$\tilde{\Gamma}^\rho_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\sigma} \frac{\partial x^\kappa}{\partial \tilde{x}^\mu} \frac{\partial x^\lambda}{\partial \tilde{x}^\nu} \Gamma^\sigma_{\kappa\lambda} + \frac{\partial \tilde{x}^\rho}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu}. \tag{70}$$

Important to notice is that no frame-dependence will emerge from the three covariant derivatives in the strain rate four-tensor $\tilde{S}^{\alpha\beta}$, since this tensor is based on a Lie derivative in which all Christoffel symbols by definition will cancel

$$\begin{aligned} \tilde{S}^{\alpha\beta} &= \tfrac{1}{2}\left(\tilde{h}^{\alpha\lambda}\tilde{\nabla}_\lambda \tilde{u}^\beta + \tilde{h}^{\beta\lambda}\tilde{\nabla}_\lambda \tilde{u}^\alpha - \tilde{u}^\lambda \tilde{\nabla}_\lambda \tilde{h}^{\alpha\beta} \right) = \tfrac{1}{2}\left(\tilde{h}^{\alpha\lambda}\tilde{\partial}_\lambda \tilde{u}^\beta + \tilde{h}^{\beta\lambda}\tilde{\partial}_\lambda \tilde{u}^\alpha - \tilde{u}^\lambda \tilde{\partial}_\lambda \tilde{h}^{\alpha\beta} \right) \\ &= -\tfrac{1}{2}\mathcal{L}_{\tilde{u}} \tilde{h}^{\alpha\beta}. \end{aligned} \tag{71}$$

If, however, the coordinate transformation (67) is different from that of a curvilinear Euclidean transformation the transformed strain rate four-tensor $\tilde{S}^{\alpha\beta}$ will show frame-dependence emerging from the transformed space-like metric tensor

$$\tilde{h}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} h^{\mu\nu}, \tag{72}$$

and thus be capable of inducing inertial forces into the transformed Navier-Stokes equations. The reason is that the strain rate four-tensor is only an Euclidean objective tensor; for more details see Sect. 9. Hence, the material law for the Cauchy stress tensor $\sigma^{\alpha\beta}$ only shows the well known frame-indifference under Euclidean transformations.

Finally we want to write down the averaged Navier-Stokes equations in their general covariant form. Using the Reynolds decomposition to separate the average and the fluctuating parts in the velocity and pressure fields with vanishing average fluctuations

$$u^\alpha = \langle u^\alpha \rangle + u'^\alpha, \quad p = \langle p \rangle + p', \quad \text{with } \langle u'^\alpha \rangle = 0, \ \langle p' \rangle = 0, \tag{73}$$

by defining the average as an ensemble average

$$\langle u^\alpha \rangle := \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} (u^\alpha)^{(r)}, \quad \langle p \rangle := \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} p^{(r)}, \tag{74}$$

where N is the number of realizations, it all implies that

$$\langle u^\alpha \rangle = (1, \langle u^i \rangle), \quad u'^\alpha = (0, u'^i), \tag{75}$$

the average four-velocity $\langle u^\alpha \rangle$, like the instantaneous velocity u^α, is a pure time-like vector which can never turn space-like, and that the fluctuating four-velocity u'^α is a pure space-like vector which can never turn time-like. The 4-formulation thus shows that the average and the fluctuating four-velocities evolve differently within the classical Newtonian space–time manifold $\mathcal{N}$. The behaviour of the fluctuating velocity is not that of a velocity but rather that of an acceleration or that of a force, a property which will be useful to have in mind for turbulence modelling in the 4-formulation.

Inserting the decomposition (73) into the form-invariant Navier-Stokes equations (66) and then averaging these equations will straightforwardly lead to the general manifest form-invariant averaged Navier-Stokes equations

$$\begin{aligned} &\nabla_\alpha \langle u^\alpha \rangle = 0, \\ &\langle u^\alpha \rangle \nabla_\alpha \langle u^\beta \rangle = \nabla_\alpha \langle \sigma^{\alpha\beta} \rangle - \nabla_\alpha \tau^{\alpha\beta}, \\ &\text{with } \langle \sigma^{\alpha\beta} \rangle = -h^{\alpha\beta} \langle p \rangle + 2\nu \langle S^{\alpha\beta} \rangle, \quad \langle S^{\alpha\beta} \rangle = -\tfrac{1}{2} \mathcal{L}_{\langle u \rangle} h^{\alpha\beta}, \text{ and } \tau^{\alpha\beta} = \langle u'^\alpha u'^\beta \rangle, \end{aligned} \tag{76}$$

where $\tau^{\alpha\beta}$ is a pure space-like four-tensor which thus can be fully identified as the Reynolds stress tensor in the usual (3+1)-formulation.

References

1. Haupt, P.: Continuum mechanics and theory of materials, 2nd edn. Springer, Heidelberg (2002)
2. Hutter, K., Jöhnk, K.: Continuum methods of physical modeling. Springer, Heidelberg (2004)
3. Truesdell, C., Toupin, R.: The classical field theories. In: Flügge, S. (ed.)Prinzipien der klassischen Mechanik und Feldtheorie, vol. 2 of Handbuch der Physik, pp. 226–793, Springer, Heidelberg (1960)
4. Truesdell, C., Noll, W.: The non-linear field theories of mechanics, 3rd edn. Springer, Heidelberg (1965)
5. Murdoch, A.I.: On criticism of the nature of objectivity in classical continuum mechanics. Continuum. Mech. Thermodyn. **17**, 135 (2005)
6. Liu, I.-S.: Further remarks on Euclidean objectivity and the principle of material frame-indifference. Continuum. Mech. Thermodyn. **17**, 125 (2005)
7. Hooke, R.: Lectures de potentia restitutiva, or of spring explaining the power of springing bodies, 1678 London. In: Gunther, R.T. (ed.) Early science in Oxford, vol. VIII, pp. 331–356, Oxford University Press, New York (1931)
8. Poisson, S.-D.: Mémoire sur les équations générales de l'équilibre et du mouvement des corps élastiques et des fluides. J. École. Poly. **13**, 1 (1829)
9. Cauchy, A.-L.: Sur l'équilibre et le mouvement intérieur des corps considérés comme des masses continues. Ex. de Math. **4**, 293 (1829)
10. Einstein, A.: Zur Elektrodynamik bewegter Körper. Ann. Physik. **17**, 891 (1905)
11. Zaremba, S.: Le principe des mouvements relatifs et les équations de la mécanique physique. Bull. Int. Acad. Sci. Cracovie. (1903) 614
12. Jaumann, G.: Elektromagnetische Vorgänge in bewegten Medien. Sitzungsbericht Akad. Wiss. Wien (IIa) **15**, 337 (1906)
13. Noll, W.: Euclidean geometry and Minkowskian chronometry. Am. Math. Mon. **71**, 129 (1964)
14. Noll, W.: Space–time structures in classical mechanics. In: Bunge, M. (ed.) Delaware seminar in the foundations of physics, pp. 28–34, Springer, Heidelberg (1967)
15. Noll, W.: Lectures on the foundations of continuum mechanics and thermodynamics. Arch. Rat. Mech. Anal. **52**, 62 (1973)
16. Müller, I.: On the frame dependence of stress and heat flux. Arch. Rational Mech. Anal. **45**, 241 (1972)
17. Edelen, D.G., McLennan, J.A.: Material indifference: a principle or a convenience. Int. J. Eng. Sci. **11**, 813 (1973)
18. Wang, C.C.: On the concept of frame-indifference in continuum mechanics and in the kinetic theory of gases. Arch. Rat. Mech. Anal. **58**, 381 (1975)
19. Müller, I.: On the frame dependence of electric current and heat flux in a metal. Acta. Mech. **24**, 117 (1976)

20. Söderholm, L.H.: The principle of material frame-indifference and material equations of gases. Int. J. Eng. Sci. **14**, 523 (1976)
21. Truesdell, C.: Correction of two errors in the kinetic theory of gases which have been used to cast unfounded doubt upon the principle of frame indifference. Meccanica **11**, 196 (1976)
22. Hoover, W.G., Moran, B., More, R.M., Ladd, A.J.: Heat conduction in a rotating disk via nonequilibrium molecular dynamics. Phys. Rev. A. **24**, 2109 (1981)
23. Heckl, M., Müller, I.: Frame dependence, entropy, entropy flux, and wave speeds in mixtures of gases. Acta. Mech. **50**, 71 (1983)
24. Murdoch, A.I.: On material frame-indifference, intrinsic spin, and certain constitutive relations motivated by the kinetic theory of gases. Arch. Rat. Mech. Anal. **83**, 185 (1983)
25. Ryskin, G.: Misconception which led to the "material frame-indifference" controversy. Phys. Rev. A. **32**, 1239 (1985)
26. Matolcsi, T.: On material frame-indifference. Arch. Rat. Mech. Anal. **91**, 99 (1986)
27. Kempers, L.J.: The principle of material frame indifference and the covariance principle. Il Nuovo. Cimento B. **103**, 227 (1989)
28. Einstein, A.: Die Grundlage der allgemeinen Relativitätstheorie. Ann. Physik. **49**, 769 (1916)
29. Sharipov, F.M., Kremer, G.M.: On the frame dependence of constitutive equations. I. Heat transfer through a rarefied gas between two rotating cylinders. Cont. Mech. Thermodyn. **7**, 57 (1995)
30. Matolcsi, T., Gruber, T.: Spacetime without reference frames: an application to the kinetic theory. Int. J. Theor. Phys. **35**, 1523 (1996)
31. Muschik, W.: Objectivity and frame indifference, revisited. Arch. Mech. **50**, 541 (1998)
32. Svendsen, B., Bertram, A.: On frame-indifference and form-invariance in constitutive theory. Acta Mech. **132**, 195 (1999)
33. Bertram, A., Svendsen, B.: On material objectivity and reduced constitutive equations. Arch. Mech. **53**, 653 (2001)
34. Rivlin, R.S.: Material symmetry revisited. GAMM-Mitteilungen **25**(1/2), 109 (2002)
35. Bertram, A., Svendsen, B.: Reply to Rivlin's material symmetry revisited or much ado about nothing. GAMM-Mitteilungen **27**(1/2), 88 (2004)
36. Murdoch, A.I.: Objectivity in classical continuum physics: a rationale for discarding the 'principle of invariance under superposed rigid body motions' in favour of purely objective considerations. Continuum. Mech. Thermodyn. **15**, 309 (2003)
37. Liu, I.-S.: On Euclidean objectivity and the principle of material frame-indifference. Continuum. Mech. Thermodyn. **16**, 177 (2004)
38. Noll, W.: A frame-free formulation of elasticity. J. Elast. **83**, 291 (2006)
39. Lumley, J.L.: Toward a turbulent constitutive relation. J. Fluid. Mech. **41**, 413 (1970)
40. Lumley, J.L.: Turbulence modeling. J. Appl. Mech. **50**, 1097 (1983)
41. Speziale, C.G.: Some interesting properties of two-dimensional turbulence. Phys. Fluids. **24**, 1425 (1981)
42. Speziale, C.G.: A review of material frame-indifference in mechanics. Appl. Mech. Rev. **51**, 489 (1998)
43. Sadiki, A., Hutter, K.: On the frame dependence and form invariance of the transport equations for the Reynolds stress tensor and the turbulent heat flux vector: its consequences on closure models in turbulence modelling. Cont. Mech. Thermodyn. **8**, 341 (1996)
44. Schutz, B.: Geometrical methods of mathematical physics. Cambridge University Press, Cambridge (1980)
45. Schrödinger, E.: Space–time structure, 4th edn. Cambridge University Press, London (1985)
46. Sexl, R.U., Urbantke, H.K.: Relativity, groups, particles: special relativity and relativistic symmetry in field and particle physics, 2nd edn. Springer, Heidelberg (2001)
47. Ovsiannikov, L.V.: Group analysis of differential equations, 2nd edn. Academic Press, New York (1982)
48. Olver, P.J.: Applications of Lie groups to differential equations, 2nd edn. Springer, Heidelberg (1993)
49. Ibragimov, N.H.: Lie group analysis of differential equations, vol. I–III of CRC Handbook. CRC Press, Boca Raton (1994)
50. Bluman, G.W., Kumei, S.: Symmetries and differential equations, 2nd edn. Springer, Heidelberg (1996)
51. Cantwell, B.J.: Introduction to symmetry analysis. Cambridge University Press, London (2002)
52. Hess, G.B., Fairbank, W.M.: Measurements of angular momentum in superfluid helium. Phys. Rev. Lett. **19**, 216 (1967)
53. Landau, L.: Theory of the superfluidity of Helium II. Phys. Rev. **60**, 356 (1941)
54. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. Freeman, San Fransisco (1973)
55. Stephani, H.: General relativity: an introduction to the theory of the gravitational field, 2nd edn. Cambridge University Press, London (1990)
56. Fließbach, T.: Allgemeine Relativitätstheorie, 5th edn. Spektrum, New York (2006)
57. Friedman, M.: Foundations of space–time theories: relativistic physics and philosophy of science. Princeton University Press, New Jersey (1983)
58. Barbour, J.B., Pfister, H.: Mach's principle: from Newton's bucket to quantum gravity. Birkhäuser, Basel (1995)
59. Barbour, J.B.: The discovery of dynamics: a study from a Machian point of view of the discovery and the structure of dynamical theories. Oxford University Press, New York (2001)
60. Penrose, R.: The road to reality: a complete guide to the laws of the universe. Alfred A. Knopf Publishing, (2005)
61. Mach, E.: Die Mechanik in ihrer Entwicklung, Wissenschaftiche Buchgesellschaft Darmstadt, Reprografischer Nachdruck 1991, 9th edn (1883)
62. Barbour, J.B.: Absolute or relative motion: the deep structure of general relativity. Oxford University Press, New York (2004)
63. Kretschmann, E.: Über den physikalischen Sinn der Relativitätspostulate, A. Einsteins neue und seine ursprüngliche Relativitätstheorie. Ann. Physik **53**, 575 (1917)
64. Einstein, A.: Dialog über Einwände gegen die Relativitätstheorie. Naturwissenschaften **6**, 697 (1918a)
65. Einstein, A.: Prinzipielles zur allgemeinen Relativitätstheorie. Ann. Physik. **55**, 241 (1918b)
66. Norton, J.D.: General covariance and the foundations of general relativity: eight decades of dispute. Rep. Prog. Phys. **56**, 791 (1993)
67. Norton, J.D.: Did Einstein stumble? The debate over general covariance. Erkenntnis, Kluwer **42**, 223 (1995)
68. Dieks, D.: Another look at general covariance and the equivalence of reference frames. Stud. Hist. Philos. Modern. Phys. **37**, 174 (2006)
69. Matolcsi, T., Ván, P.: Absolute time derivatives. J. Math. Phys. **48**, 053507 (2007)

70. Cartan, E.: Sur les variétés à connexion affine et la théorie de la relativité généralisée. Annales Scientifiques de l'École Normale Supérieure **40**, 325 (1923)
71. Friedrichs, K.: Eine invariante Formulierung des Newtonschen Gravitationsgesetzes und des Grenzüberganges vom Einsteinschen zum Newtonschen Gesetz. Math. Ann. **98**, 566 (1928)
72. Havas, P.: Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity. Rev. Mod. Phys. **36**, 938 (1964)
73. Trautman, A.: Comparison of Newtonian and relativistic theories of space–time. In: Hoffmann, B. (ed.) Perspectives in geometry and relativity, Indiana University Press, Bloomington (1966)
74. Ehlers, J.: The Newtonian limit of general relativity. In: Ferrarese, G. (ed.) Classical mechanics and relativity: relationship and consistency, Bibliopolis (1991)