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DECOMPOSITION OF PRODUCTS OF SYMPLECTIC GROUP REPRESENTATIONS

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Group representations play an important role in modern physics. Recent developments require to consider group representations in a wider sense than usual.

A general definition of group representation is based on the notion of categories. A *category* consists of *objects* and *morphisms* satisfying some axioms. We are concerned, in our applications, with so called concrete categories whose objects are sets with a certain structure and the morphisms are mappings related to the structure ("preserving" it in some sense). An *isomorphism* is a bijective map whose inverse is also a morphism; an *automorphism* of an object is an isomorphism of the object onto itself. The automorphisms of an object X form a group under the composition of maps; we denote this group by $\mathrm{Aut}\,(X)$. One generally defines the notion of *subobjects* in concrete categories in a natural way and it has an intrinsic meaning that a subobject is invariant for an automorphism; hence we do not give here the precise and convenient definition of subobjects and invariant subobjects in category theory; it is done in [1]. For further details on category theory we refer to [3].

Definition. Let $\mathscr{C}$ be a concrete category. A *representation* of a group G on an object X of $\mathscr{C}$ is a group homomorphism

$$A\colon G \to \operatorname{Aut}(X), \quad g \mapsto A_g .$$

The representation is *faithful* if it is injective.

The representation is *irreducible* if there is no non-trivial subobject invariant for the representation, i.e., if a subobject is invariant under all A_g then it is invariant under all automorphisms as well.

Two representations of the same group G, A on X and B on Y, X and Y being objects of $\mathscr{C}$, are *equivalent* if there is an isomorphism $i\colon X \to Y$ such that $i \circ A_g = B_g \circ i$ for all $g \in G$.

If G is a topological group and the structure of X contains a topology, too, and the map $G \times X \to X$, $(g, x) \mapsto A_g(x)$ is continuous, we call the representation *continuous.*

If G is a Lie group and the structure of X contains a differentiable sturcture too, and the map $G \times X \to X$, $(g, x) \mapsto A_g(x)$ is differentiable, we call the representation *differentiable.*

Taking different categories we get different types of representations. For example:

1. *Linear representations:* the objects are the complex linear vector spaces and the morphisms are the linear maps.

2. *Unitary representations:* the objects are the Hilbert spaces and the morphisms are the contractions.

3. *Topological transformation group:* the objects are the topological spaces and the morphisms are the continuous maps, and we consider continuous representations of topological groups.

4. *Lie transformation groups:* the objects are differentiable manifolds and the morphisms are differentiable maps, and we consider differentiable representations of Lie groups; we shall call them *Lie representations.* Let us see more closely this last type of representations. In customary

literature one speaks about "actions" of Lie groups and one uses the notions "effective" and "transitive". Now we can see that an action in this sense is a representation, effective actions are precisely the faithful representations and transitive actions are irreducible representations; an irreducible representation is a transitive action if the group of diffeomorphisms of the manifold in question form a transitive transformation group; such differentiable manifolds can be called *homogeneous.*

Let G be a Lie group and let H be a closed subgroup of G. Then the canonical left action of G on G/H is called the *canonical Lie representation* of G induced by H and is denoted by $L^{G/H}$. The following are well-known:

1. A transitive Lie representation of a Lie group is equivalent to a canonical one.

2. Let H_1 and H_2 be closed subgroups of the Lie group G; then L^{G/H_1} and L^{G/H_2} are equivalent if and only if H_1 and H_2 are conjugate subgroups.

3. An orbit of a Lie transformation group can be equipped with a differentiable structure such that the action of the group on the orbit is a transitive Lie transformation group, in other words, a Lie representation of a Lie group can be decomposed into disjoint union of transitive Lie representations.

Let A^1 and A^2 be two Lie representations of the Lie group G on the manifolds M_1 and M_2, respectively; then the *product* $A^1 \times A^2$ of the two representations are the one given on $M_1 \times M_2$ by

$$(A^1 \times A^2)_g(x_1, x_2) := (A^1_g(x_1), A^2_g(x_2)).$$

Proposition 1. *Let* G *be a Lie group, let* H_1 *and* H_2 *be closed subgroups of* G. *There are canonical bijections among the sets*

$\{$orbits of G in $L^{G/H_1} \times L^{G/H_2}\}$,

$\{$orbits of H_2 in $L^{G/H_1}\}$,

$\{$orbits of H_1 in $L^{G/H_2}\}$.

Proof. Let P be an orbit of G in $L^{G/H_1} \times L^{G/H_2}$. Since L^{G/H_1} is transitive, we can find an element x of G/H_2 such that $(H_1, x) \in P$. The stabilizator of $H_1 \in G/H_1$ is H_1, thus if (H_1, y) is another element of P then there is an $h \in H_1$ such that $L_h^{G/H_2}(x) = y$. As a consequence, the map

$$\{\text{orbits of } G \text{ in } L^{G/H_1} \times L^{G/H_2}\} \to$$

$$\to \{\text{orbits of } H_1 \text{ in } L^{G/H_2}\}$$

$$P \to \{L_h^{G/H_2}(x) \colon h \in H_1,\ (H_1, x) \in P\}$$

is well-defined and we easily check that it is bijective.

Proposition 2. *Let P be an orbit of G in $L^{G/H_1} \times L^{G/H_2}$. Then there exists a closed subgroup H of G such that the Lie representation of G on P is equivalent to $L^{G/H}$ and $H \subset H_1, H_2$ contains a subgroup conjugate to H.*

Proof. The stabilizator of a point $(H_1, x) \in P$ is clearly a subgroup of H_1, and the stabilizator of a point (z, H_2) is a subgroup of H_2.

Now we shall consider a special type of representations; they can be used in classical mechanics.

A closed two-form with constant rank on a differentiable manifold is called *presymplectic*; it is *symplectic* if it has maximal rank. A *symplectic manifold* is a pair (M, ω) where M is a differentiable manifold and ω is a symplectic form on M. We introduce a *category* whose objects are symplectic manifolds and morphisms from (M, ω) into (M', ω') are differentiable maps $F\colon M \to M'$ such that $F^*\omega'$ is a presymplectic form, and $\omega - F^*\omega'$ is a presymplectic form as well, whose rank is the difference of the rank of ω and $F^*\omega'$. As a consequence, the isomorphisms from (M, ω) into (M^*, ω') are the *symplectic diffeomorphisms* i.e., diffeomorphisms $F\colon M \to M^*$ for which $F^*\omega' = \omega$ holds.

Differentiable representations of Lie groups on objects of the above category are called *symplectic representations.* Thus if G is a Lie group and A is a symplectic representation of G on (M, ω) then A_g is a symplectic diffeomorphism of (M, ω). An other representation A' of G on (M', ω') is equivalent to A if there is a symplectic diffeomorphism $F: M \to M'$ such that $F \circ A_g = A'_g \circ F$ for all $g \in G$. *Products of symplectic representations* are defined as products of Lie representations.

Symplectic representations are similar to unitary representations in many aspects (see [2]).

A symplectic representation is a Lie representation if we forget the symplectic form (as a unitary representation is a linear representation if we forget the scalar product), hence we can apply some results concerning the differentiable representations. The most important is that a transitive symplectic representation can be given on a coset space as well; however, on the same coset space we can give a number of inequivalent representations.

We can give a characterization of transitive symplectic representations using left actions on coset spaces; another characterization can be found in [4].

Proposition 3. *Let G be a Lie group and let H be a connected closed subgroup of G. Suppose Ω is a left invariant presymplectic form on G, i.e., $L_g^*\Omega = \Omega$ $(g \in G)$ and* Ker Ω *is the Lie algebra of H (considered as left invariant vector fields). Then one can give a symplectic form ω on G/H such that Ω is the pull-back of ω by the canonical surjection and the canonical left action of G on G/H preserves ω, i.e., it is a symplectic representation on G/H which will be denoted by $L^{G/H, \Omega}$.*

Proof. The Lie algebra of H is a foliation on G, its folia are the elements of G/H; hence there exists a symplectic form ω on G/H with the listed properties (see [4]).

Proposition 4. *$L^{G/H_1, \Omega_1}$ and $L^{G/H_2, \Omega_2}$ are equivalent if and only if H_1 and H_2 are conjugate subgroups, say $H_2 = h^{-1}H_1h$, and $R_h^*\Omega_2 = \Omega_1$ where R denotes the right action of G on itself.*

Proof. It is trivial that if the listed conditions are satisfied then the two representations are equivalent.

Now, if the two representations are equivalent, then by the above mentioned relation between symplectic representations and Lie representations we know that H_1 and H_2 are conjugate subgroups and we easily deduce the equality concerning Ω_2 and Ω_1 from the fact that the equivalence of the two representations is established by a symplectic diffeomorphism.

Proposition 5. *If* A *is a transitive symplectic representation of* G *and an* A*-isotropy subgroup* H *of* G *is connected then there is a left invariant presymplectic form* Ω *on* G *such that* A *is equivalent to* $L^{G/H,\Omega}$.

Proposition 6. *Let* P *be an orbit of a symplectic representation of* G *on* (M, ω)*. Then the pull-back of* ω *to* P *is a presymplectic form invariant under the action of* G *on* P.

Proof. The pull-back of ω to P is clearly a two-form invariant under the action of G on P. Since this action is transitive, the two-form is necessarily of constant rank.

Remark. It is very important that the pull-back of ω need not be a symplectic form; hence, in general, a symplectic representation *cannot be decomposed* into a disjoint union of transitive symplectic representations. This is an essential difference between symplectic representations and unitary representations: unitary representations can be decomposed into a direct integral (direct sum) of irreducible ones.

Examples. In physical applications one encounters the problem to decompose the "product" of irreducible representations into a "sum" of irreducible ones. The most common and well-known example is the reduction of the tensor products of irreducible representations of $SO(3)$. On the contrary, the product of transitive symplectic representations of $SO(3)$ cannot be decomposed into transitive symplectic representations. The transitive symplectic representations of $SO(3)$ can be labelled by non-negative numbers in such a way that the representation corresponding to

$\sigma \geqslant 0$ is given on $(S^2_\sigma, \omega_\sigma)$ where S^2_σ is the sphere of radius σ in $\boldsymbol{R}^3$ and ω_σ for $\sigma > 0$ is the Lebesgue volume form multiplied by $\frac{1}{\sigma}$. The orbits of the symplectic representation $A^{\sigma_1} \times A^{\sigma_2}$ can be labelled by the elements of the interval $(|\sigma_1 - \sigma_2|, \sigma_1 + \sigma_2)$. The orbits corresponding to $|\sigma_1 - \sigma_2|$ and $\sigma_1 + \sigma_2$ give symplectic representations of $SO(3)$ which are equivalent to $A^{|\sigma_1 - \sigma_2|}$ and $A^{\sigma_1 + \sigma_2}$, respectively. The other orbits are diffeomorphic to $SO(3)$, hence they are of three dimensions and cannot be symplectic manifolds: the pull-back of the symplectic form $\omega_{\sigma_1} \times \omega_{\sigma_2}$ to these orbits is only a presymplectic form.

The following simple examples exhibit different situations.

Let ω be the canonical symplectic form on $\boldsymbol{R}^2$, i.e., is given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let λ be a positive real number. Define the transitive symplectic representations A^λ and B^λ of the additive group $\boldsymbol{R}^4 = \boldsymbol{R}^2 \times \boldsymbol{R}^2$ on $(\boldsymbol{R}^2, \omega)$ by

$$A^\lambda_{(a_1, a_2)}(x) := x + \lambda a_1$$

$$B^\lambda_{(a_1, a_2)}(x) := x + \lambda a_2 \qquad (x \in \boldsymbol{R}^2, (a_1, a_2) \in \boldsymbol{R}^2 \times \boldsymbol{R}^2).$$

Then for different λ_1 and λ_2 the symplectic representations A^{λ_1} and A^{λ_2} as well as B^{λ_1} and B^{λ_2} are inequivalent and only A^0 is equivalent to B^0.

Each orbit of $A^{\lambda_1} \times A^{\lambda_2}$ establishes a transitive symplectic representation of $\boldsymbol{R}^2 \times \boldsymbol{R}^2$ which is equivalent to $A^{\lambda_1 + \lambda_2}$.

The symplectic representation $A^{\lambda_1} \times B^{\lambda_2}$ is transitive.

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