Kinematics of finite elastic and plastic deformations

Tamás Fülöp[1,3] and Péter Ván[1,2,3]

[1] KFKI Research Institute for Particle and Nuclear Physics,
Konkoly-Thege M. út 29-33, H-1121, Budapest, Hungary

[2] Dept. of Energy Engineering, Budapest Univ. of Technology and Economics,
Bertalan Lajos u. 4-6, H-1111, Budapest, Hungary

[3] Montavid Thermodynamic Research Group,
Igmándi u. 26. fsz. 4, H-1112, Budapest, Hungary

November 16, 2010

Abstract

The kinematics of finite elastic and plastic deformations is considered in an approach that does not rely on reference frame and reference configuration, and that gives account of the inertial-noninertial aspects explicitly. These features are achieved by working on nonrelativistic spacetime directly. The quantity measuring elastic deformations is introduced according to its physical role, and the definition of the quantity describing plastic changes follows as a natural consequence. The properties of both are analyzed, and their relationship to frequently used elastic and plastic kinematic quantities is discussed. One important result is that neither the elastic nor the plastic kinematic quantity can be defined from deformation gradient.

1 Introduction

Kinematics of continua is an old area, with a lot of contributions from very many scientists. Still, it is not a closed subject. A number of open questions, controversial aspects and problems stimulate further research. These problematic issues become particularly relevant when moving from small elastic and plastic deformations to large, from slow processes to fast, from essentially homogeneous bodies to considerably inhomogeneous settings — as is the case with nanostructures —, and from simple continua to ones with such microstructure that plays important role also at the macro level and where the kinematics of the micro processes is to be established and to be brought into harmony with the macro kinematics. Therefore, the open questions should be answered and kinematics must be made ready for the new and newer demands.

The approach to finite (i.e., not necessarily small) elastic and plastic deformations presented here intends to stay on the safe side — safe from the controversial problems. The methodology used is chosen according to this purpose. It has the following basic pillars.

1. The mathematical model is seriously distinguished from physical reality.
2. Nonetheless, it is constructed to express the relevant physical requirements, not less and not more.
3. Embodying this, avoided are any auxiliary elements, which could make the description *technically* more convenient but do not belong to the *physics* of the motion of the continuum.
4. As an example, nonrelativistic spacetime is considered in the reference frame free — affine space based — formulation.
5. Similarly, the continuum is considered as a smooth manifold, but it is equipped as well by some further structures as dictated by physical requirements. Meanwhile, no

convenience-motivated auxiliary elements are introduced for the elastic and plastic kinematics of the material.

The treatment of objectivity is one of the important aspects of this paper. In nonrelativistic physics, the spacetime is seemingly rather simple and transparent, therefore the need for the very concept of spacetime is not apparent. Usually, the notion of spacetime is left implicit and objectivity is formulated in an indirect way, with the help of invariance properties under changes of special (rigid) reference frames. This kind of approach can be combined with differential geometry for the spacelike part of the corresponding physical quantities, too (see e.g. [1, 2, 3]). However, restricting ourselves to the transformation of reference frames related to spacelike part of physical quantities neglecting their time dependence results in an improper (or at least insufficient) formulation of covariance as one can demonstrate by the usual rigid observers of Noll [4]. Therefore, the need for a reference frame free formulation of continuum physics is indicated by several authors [5, 6] and some realizations, slightly similar to our approach, already exist in the literature[1]. In our treatment, spacetime has an explicit mathematical structure, which clearly shows the unavoidable nontrivial intertwining of space and time, even nonrelativistically. The reference frame independence of the basic physical quantities is ensured by formulation - physical quantities are given as spacetime compatible objects. The need for such a kind of approach to nonrelativistic spacetime has been indicated some time ago [11, 12], and a fully elaborated formulation was given first in [13].

In this paper, we investigate kinematics of elastic and plastic continua, the literature of which is still far too extensive to give a fair account of it. To list at least a part of the works that are closely related to the ideas raised here, [14] is one of the definitive contributions, [15] connects the finite deformation compatibility condition with the vanishing of a Riemann curvature, Bertram [16] provides a good collection of the problematic issues of kinematics, DiCarlo [17] quotes and uses the notions of current configuration and relaxed configuration, and Epstein [18] and León [19] are two examples for sources that operate with the term reference crystal. For the methodology of mathematical models in physics, [13] is an advanced, and [20] is a more readable source. The latter is also the most comprehensive source for the frame free formulation of spacetime. As such, it can also be consulted for the mathematical treatment of the various quantities with physical dimensions: lengths, times, masses etc., for tensorial operations (identifications, dual transpose, metric adjoint), and for affine spaces.

The paper is organized as follows. First, the standard formulation of kinematics is surveyed. Then, some questionable aspects of the customary notions of finite deformation kinematics are highlighted. This is followed by a summary of the frame free formalism of nonrelativistic spacetime, a necessary ingredient for the forthcoming considerations. The fifth section defines the kinematics of a solid in spacetime. The subsequent section derives the kinematic quantities needed for elasticity, the elastic shape - a frame independent generalized deformation -, and the elastic deformedness - a frame independent generalized strain. The resulting compatibility condition is calculated, too. The last section lays the foundation of plastic kinematics. Finally, a concise summary is given.

2 A standard formulation of kinematics

So as to set the context for expressing the motivations, let us start with summarizing a usual approach to elastic and plastic kinematics.

This starts with choosing a reference frame—this step being taken only tacitly, usually. Next, the continuum is represented by a reference configuration, i.e., by its location in the space of this frame at a chosen reference instant t_0. More closely, each material point is represented by its position $\mathbf{X}$ in this space at t_0. At time t, the position of a material point is $\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X})$, and its velocity is $\mathbf{v}_t(\mathbf{X}) = \dot{\boldsymbol{\chi}}_t(\mathbf{X})$, where overdot means partial

[1] See the publications of Walter Noll about the foundations of continuum mechanics starting from [7] and more recently on his website (http://www.math.cmu.edu/ wn0g/noll/). Here, he emphasizes e.g. the need of affine spaces ([8], p24, but later on he introduces some slightly different concepts [9]). However, nonrelativistic spacetime remains implicit and time is separated in his kinematics [10].

derivative with respect to time, $\dot{} = \partial_t \big|_{\mathbf{X}}$. With the displacement since t_0,

$$\mathbf{u}_t(\mathbf{X}) := \boldsymbol{\chi}_t(\mathbf{X}) - \mathbf{X}\,, \tag{1}$$

one also has

$$\mathbf{v} = \dot{\mathbf{u}}\,. \tag{2}$$

The so-called deformation gradient is introduced as

$$\mathbf{F} := \boldsymbol{\chi} \otimes \nabla_{\mathbf{X}}\,, \tag{3}$$

with $\otimes$ denoting dyadic/tensorial product; in what follows, the partial derivative operation $\nabla_{\mathbf{X}}$ will act to the left or to the right depending on context, always to reflect the proper tensorial order ('order of tensorial indices'). The deformation gradient is assumed to be invertible. This is ensured if one requires that $\det \mathbf{F}$ (which is the Jacobian of the map $\boldsymbol{\chi}$) is nonzero, which physically means that the continuum can never be singularly compressed. Following from its definition, $\mathbf{F}$ obeys the properties

$$\dot{\mathbf{F}} = \mathbf{v} \otimes \nabla_{\mathbf{X}}, \qquad \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathbf{v} \otimes \nabla_{\mathbf{x}}, \qquad (\mathbf{F} \otimes \nabla_{\mathbf{X}})^{\mathrm{A}_{2,3}} = \mathbf{0}\,; \tag{4}$$

where in the middle formula velocity is considered in variables $t, \mathbf{x}$ [the connection with the variables $t, \mathbf{X}$ being established by $\boldsymbol{\chi}_t(\mathbf{X})$], and in the last formula antisymmetrization is carried out in the second and third 'indices'. Similarly, $^{\mathrm{S}}$ will stand for symmetric part, $^{\mathrm{T}}$ for transpose and tr for trace, and, for example, $\mathrm{tr}_{1,3}$ will denote contraction of the first and third 'indices'. Note that, according to the chain rule of differentiation of composite functions, a multiplication by $\mathbf{F}$ from the right gives the transition from the derivative $\nabla_{\mathbf{x}}$ of a quantity to the derivative $\nabla_{\mathbf{X}}$ (and multiplication by $\mathbf{F}^{-1}$ from the right gives the opposite direction).

Related to this, if one has a process from t_0 to t_2 then, for any t_1 in between,

$$\mathbf{F}_{t_2}^{(t_0)} = \mathbf{F}_{t_2}^{(t_1)}\,\mathbf{F}_{t_1}^{(t_0)} \tag{5}$$

where t_1 is also used as another reference instant, and that's why now the used reference instants are also displayed in superscript. As a special case,

$$\mathbf{F}_{t_1}^{(t_0)} = \left[\mathbf{F}_{t_0}^{(t_1)}\right]^{-1}\,. \tag{6}$$

The deformation gradient — as assumed to be invertible — admits a polar decomposition:

$$\mathbf{F} = \mathbf{U}_{\mathrm{L}}\,\mathbf{O} = \mathbf{O}\,\mathbf{U}_{\mathrm{R}}\,, \tag{7}$$

with orthogonal $\mathbf{O}$ and symmetric and positive definite $\mathbf{U}_{\mathrm{L}} = \sqrt{\mathbf{F}\mathbf{F}^{\mathrm{T}}}$, $\mathbf{U}_{\mathrm{R}} = \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$.

The various deformation tensors (Cauchy-Green, Finger, ...) are defined as various powers of $\mathbf{U}_{\mathrm{L}}$ and $\mathbf{U}_{\mathrm{R}}$:

$$\mathbf{C}_{\mathrm{L}}^{(n)} := \left(\mathbf{U}_{\mathrm{L}}\right)^n, \qquad \mathbf{C}_{\mathrm{R}}^{(n)} := \left(\mathbf{U}_{\mathrm{R}}\right)^n. \qquad (n = \ldots, -2, -1, 0, 1, 2, \ldots). \tag{8}$$

From them, the various strain tensors (Green-Lagrange, St. Venant, Biot, Almansi, Hencky, ...) are derived, all expressing in some way the deviation from the identity tensor $\mathbf{I}$:

$$\mathbf{E}_{\mathrm{L}}^{(n)} := \frac{1}{n}\left[\mathbf{C}_{\mathrm{L}}^{(n)} - \mathbf{I}\right] = \frac{1}{n}\left[\left(\mathbf{U}_{\mathrm{L}}\right)^n - \mathbf{I}\right], \qquad \mathbf{E}_{\mathrm{L}}^{(0)} := \ln \mathbf{U}_{\mathrm{L}}, \tag{9}$$

$$\mathbf{E}_{\mathrm{R}}^{(n)} := \frac{1}{n}\left[\mathbf{C}_{\mathrm{R}}^{(n)} - \mathbf{I}\right] = \frac{1}{n}\left[\left(\mathbf{U}_{\mathrm{R}}\right)^n - \mathbf{I}\right], \qquad \mathbf{E}_{\mathrm{R}}^{(0)} := \ln \mathbf{U}_{\mathrm{R}} \tag{10}$$

[the $n = 0$ cases (Hencky strains) being the l'Hôpital limits of the $n \neq 0$ series]. In addition, the Cauchy strain — with which each of the above strains coincides in the leading order of $\mathbf{F} - \mathbf{I}$, i.e., for small strains — is defined as

$$\mathbf{E}^{\mathrm{Cauchy}} = \mathbf{F}^{\mathrm{S}} - \mathbf{I} = \left(\boldsymbol{\chi} \otimes \nabla_{\mathbf{X}}\right)^{\mathrm{S}} - \mathbf{I} = \left(\mathbf{u} \otimes \nabla_{\mathbf{X}}\right)^{\mathrm{S}}\,. \tag{11}$$

The important dynamical purpose with strain is to use it as the variable on which elastic inner forces — described by the elastic stress tensor — and the corresponding elastic energy are assumed to depend.

When plastic changes also occur in a material, kinematics needs to describe what a plastic change is geometrically, and how it differs from elastic changes. One usual approach is to assume that strain (one of the definitions above, typically $\mathbf{E}^{\mathrm{Cauchy}}$) decomposes into a sum,

$$\mathbf{E}^{\mathrm{total}} = \mathbf{E}^{\mathrm{elast}} + \mathbf{E}^{\mathrm{plast}}\,, \tag{12}$$

and another conception, considered more applicable for nonsmall deformations, decomposes the deformation gradient, instead, and does it multiplicatively:

$$\mathbf{F}^{\mathrm{total}} = \mathbf{F}^{\mathrm{elast}}\,\mathbf{F}^{\mathrm{plast}}\,. \tag{13}$$

The latter may be explained that, after a subsequent complete elastic relaxation, which would bring in multiplication by $(\mathbf{F}^{\mathrm{elast}})^{-1}$ from the left [in accord with (5) and (6)], one would obtain what the plastic deformation is, and one would reach the relaxed configuration.

3 Remarks and observations

The following remarks and observations will help in giving motivations and hints for an improvement of kinematics.

3.1 So many strains

Probably the most immediate observation is that there are many, actually infinitely many definitions for deformation and for strain. In addition, it is not hard to introduce infinitely many further — and not unreasonable — versions. Then, which one to use as the variable of an elastic constitutive relation? In linear elasticity, in which strain the elastic constitutive relation is expected to be the most linear? For instance, Horgan and Murphy [21] finds that, for large deformations of hard rubber, among the $\mathbf{E}_{\mathrm{R}}^{(n)}$ strains, $n = 0$ provides the most precise linearity and in the largest regime. In parallel, in nonlinear elasticity, fitting experimental data to a nonlinear elastic constitutive relation, e.g., to the Murnaghan model, can provide unphysical values for the material coefficients, and with large uncertainties, when, for example, the $n = 2$ strain is used. When $n = 0$ is chosen, instead, then the results are much more realistic and much more reliable [22].

To summarize, there is no satisfactory amount of knowledge collected and distributed about the relevance of the various strain measures.

3.2 Volumetric properties towards infinities and in between

A less apparent, but not unimportant, aspect is that the strains (9)–(10) with positive n take finite value in the infinitely compressed singular limit ($\det \mathbf{F} \to 0$). On the other side, the cases with negative n take finite value in the infinitely expanded asymptotics ($\det \mathbf{F} \to \infty$). A geometrically really descriptive strain quantity would diverge in both these singular limits. Also, it is such a quantity from which good numerical stability could be expected. Namely, if, by numerical error, the numerical representation of the system starts to deviate towards large extension or compression, such a strain measures it sensitively, and the proportional additional elastic forces will drive the situation back towards the correct value.

In this respect, the Hencky strains do better than the others listed above, as they diverge in both asymptotics. This, therefore, provides as well a possible explanation of the findings of [22] mentioned previously.

At this point, it is worth asking the following question, too. For small deformations, the Cauchy strain is known to describe the purely volumetric changes by its spherical part (trace part), and purely torsional ones by its deviatoric part. Is there such a strain — one among those mentioned above, or some other one — that admits the same property for finite deformations as well?

Actually, one finds that the left and right Hencky strains ($n = 0$) satisfy this criterion as well. This can be shown as follows.

Let us consider any symmetric tensor $\mathbf{\Lambda}$ with positive eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Its determinant is the product of eigenvalues, for which

$$\lambda_1 \lambda_2 \lambda_3 = e^{\ln(\lambda_1 \lambda_2 \lambda_3)} = e^{\ln \lambda_1 + \ln \lambda_2 + \ln \lambda_3}. \tag{14}$$

Since the logarithm of a symmetric tensor has the logarithm of the eigenvalues and the same eigenvectors,

$$\det \mathbf{\Lambda} = e^{\operatorname{tr} \ln \mathbf{\Lambda}}, \qquad \text{in rearranged form,} \qquad \operatorname{tr} \ln \mathbf{\Lambda} = \ln \det \mathbf{\Lambda}. \tag{15}$$

The volumetric extension of the continuum is the Jacobian determinant

$$\det \mathbf{F} = \det \mathbf{U}_{\mathrm{L}} = \det \mathbf{U}_{\mathrm{R}}, \tag{16}$$

and applying (15) to $\mathbf{U}_{\mathrm{L}}$ and $\mathbf{U}_{\mathrm{R}}$ leads to

$$\operatorname{tr} \ln \mathbf{U}_{\mathrm{L}} = \operatorname{tr} \ln \mathbf{U}_{\mathrm{R}} = \ln \det \mathbf{F}. \tag{17}$$

Therefore, the trace of the Hencky strains is zero if and only if there is no volumetric expansion/compression. Furthermore, isotropic changes (meaning three identical eigenvalues for $\mathbf{U}_{\mathrm{L}}$, $\mathbf{U}_{\mathrm{R}}$) change only their spherical part but not the deviatoric part.

3.3 Elastic kinematics is about a state, not a change

Measuring strain with respect to a configuration at a t_0 means to measure a change, change that occurred during a time interval $[t_0, t]$. However, this is not what we are physically interested in. What we really have in mind is that we consider an elastic solid body, which has a distinguished state, a natural state, which the body takes when it's totally relaxed, undisturbed, with no outer surface nor volume forces. Whenever this body is not in this state then inner elastic forces rise that try to govern the body towards this relaxed state. This force somehow depends on how far the body is from the relaxed state. In accord with that nonrelativistic reversible/conservative interaction forces depend usually on the current distance between the objects in interaction, we expect that the inner elastic force (elastic stress) depends on the current pairwise distances between material points. More closely, the elastic stress would depend on how the current distances differ from the relaxed, natural distances. We wish a geometric type state quantity that compares, locally, the current pairwise distances with the relaxed ones. To express its physical role, we can call it *elastic deformedness*, for example.

At an instant t_0 chosen as an initial time, the elastic deformedness of the body is, in general, nonzero. In many laboratory testing situations, it can be considered zero—but not always: for example, the Anelastic Strain Recovery method [23, 24, 25] determines underground three dimensional *in situ* stress by measuring how rocks taken from the drill core relax from deformed to undeformed state. In parallel, in a civil engineering underground situation, initial deformedness of soil or rock is nonzero because of self-weight and other effects.

The motion of the continuum known since an initial time determines only how deformedness *evolves* from the initial condition. For the various different measures of deformedness, the rate equation expressing this evolution is different.

It is instructive to derive what rate equation the deformations and strains (8)–(11) obey. Actually, not all these rate equations can be directly calculated because a symmetric tensor does not, in general, commute with its time derivative. It is the left and the right Cauchy-Green deformations and the Cauchy strain for which the calculation can be done explicitly, finding

$$\dot{\left(\mathbf{U}_{\mathrm{L}}^2\right)} = (\mathbf{v} \otimes \nabla_{\mathbf{x}}) \mathbf{U}_{\mathrm{L}}^2 + \mathbf{U}_{\mathrm{L}}^2 (\mathbf{v} \otimes \nabla_{\mathbf{x}})^{\mathrm{T}} \tag{18}$$

$$\dot{\left(\mathbf{U}_{\mathrm{R}}^2\right)} = 2\, \mathbf{F}^{\mathrm{T}} (\mathbf{v} \otimes \nabla_{\mathbf{x}})^{\mathrm{S}}\, \mathbf{F} \tag{19}$$

$$\dot{\mathbf{E}}^{\mathrm{Cauchy}} = \left(\mathbf{v} \otimes \nabla_{\mathbf{X}}\right)^{\mathrm{S}}. \tag{20}$$

New measures for deformedness can also be defined based on a rate equation. For example, below we will make use of the 'inertial version' of the Cauchy deformedness, characterized by the differential equation

$$\dot{\mathbf{E}}^{\mathrm{in.Cauchy}} = \left(\mathbf{v} \otimes \nabla_{\mathbf{x}}\right)^{\mathrm{S}} \tag{21}$$

[with $\nabla_{\mathbf{x}}$ rather than $\nabla_{\mathbf{X}}$ used in (20)].

A consequence of the generally nonzero initial deformedness at t_0 is that

$$\mathbf{E}^{\mathrm{Cauchy}} \neq (\mathbf{u} \otimes \nabla_{\mathbf{X}})^{\mathrm{S}} \tag{22}$$

in general. Only

$$\mathbf{E}_t^{\mathrm{Cauchy}} = (\mathbf{u}_t \otimes \nabla_{\mathbf{X}})^{\mathrm{S}} + \mathbf{E}_{t_0}^{\mathrm{Cauchy}} \tag{23}$$

holds.

Nevertheless, not only $\dot{\mathbf{E}}^{\mathrm{Cauchy}}$ and — consequently — its integrals

$$\mathbf{E}_t^{\mathrm{Cauchy}} - \mathbf{E}_{t_0}^{\mathrm{Cauchy}} = \int_{t_0}^{t} \dot{\mathbf{E}}_{t'}^{\mathrm{Cauchy}} \mathrm{d}t' \tag{24}$$

satisfy the so-called Saint-Venant compatibility conditions [which, in Cartesian coordinates, is a consequence of Young's theorem of mixed partial derivatives, applied on (22)] but $\mathbf{E}_t^{\mathrm{Cauchy}}$ itself also:

$$\nabla_{\mathbf{X}} \times \mathbf{E}_t^{\mathrm{Cauchy}} \times \nabla_{\mathbf{X}} = \mathbf{0}\,. \tag{25}$$

Why (25) holds can be argued as follows. Zero deformedness does naturally fulfil the compatibility condition. Furthermore, any nonzero deformedness at a time t must be such that, by a subsequent relaxation, the body could be brought into zero deformedness, and thus being totally relaxed at a later $\bar{t}$. It is not required that the actual process of the body is such that such a later zero deformedness is reached indeed—but the possibility must hold for that. Physically, we say here that a deformed elastic body must be able to relax to undeformed state. Now, the integral

$$\mathbf{E}_{\bar{t}}^{\mathrm{Cauchy}} - \mathbf{E}_{t}^{\mathrm{Cauchy}} = \int_{t}^{\bar{t}} \dot{\mathbf{E}}_{t'}^{\mathrm{Cauchy}} \mathrm{d}t' \tag{26}$$

satisfies the compatibility condition and, for this hypothetical $\bar{t}$, $\mathbf{E}_{\bar{t}}^{\mathrm{Cauchy}} = \mathbf{0}$ so $\mathbf{E}_t^{\mathrm{Cauchy}}$ also obeys the compatibility condition. An analogous argument can be given for that $\mathbf{E}_t^{\mathrm{in.Cauchy}}$ fulfils the $\nabla_{\mathbf{x}}$ version of the compatibility condition. Note that later discussions will shed further light on this 'ability to relax'.

At this point, it is important to recall the Cesàro–Volterra formula,

$$\begin{aligned}\mathbf{u}_t^{\mathrm{Cauchy}}(\mathbf{X}) := {} & \mathbf{u}_t^{\mathrm{arb}} + \boldsymbol{\Omega}_t^{\mathrm{arb}}\left(\mathbf{X} - \mathbf{X}^{\mathrm{arb}}\right) \\ & + \int_{\mathbf{X}^{\mathrm{arb}}}^{\mathbf{X}} \left[\mathbf{E}_t^{\mathrm{Cauchy}}(\mathbf{X}') + 2(\mathbf{E}_t^{\mathrm{Cauchy}} \otimes \nabla_{\mathbf{X}})^{\mathrm{A}_{1,3}}(\mathbf{X}')\left(\mathbf{X} - \mathbf{X}'\right)\right] \mathrm{d}\mathbf{X}',\end{aligned} \tag{27}$$

where, having the auxiliary position $\mathbf{X}^{\mathrm{arb}}$ and the path of integration fixed arbitrarily, $\mathbf{u}_t^{\mathrm{Cauchy}}$ is uncertain up to the arbitrary vector function $\mathbf{u}_t^{\mathrm{arb}}$ and arbitrary antisymmetric tensor function $\boldsymbol{\Omega}_t^{\mathrm{arb}}$—*i.e.,* up to an arbitrary rigid body displacement-plus-rotation. The Cesàro–Volterra formula produces all the possible vector functions $\mathbf{u}^{\mathrm{Cauchy}}$ with which $\mathbf{E}^{\mathrm{Cauchy}}$ satisfies

$$\mathbf{E}_t^{\mathrm{Cauchy}} = \left(\mathbf{u}_t^{\mathrm{Cauchy}} \otimes \nabla_{\mathbf{X}}\right)^{\mathrm{S}}. \tag{28}$$

Such a $\mathbf{u}_t^{\mathrm{Cauchy}}$ is, however, not the displacement $\mathbf{u}_t$ that is measured since a t_0, in general. $\mathbf{u}$ can be one of the possible $\mathbf{u}^{\mathrm{Cauchy}}$'s only in the special cases when initial deformedness is zero [cf. (23)], and even then it is only one of the possible $\mathbf{u}^{\mathrm{Cauchy}}$'s.

A $\mathbf{u}^{\mathrm{Cauchy}}$ is to be considered as a vectorial type potential for a symmetric tensor that satisfies Saint-Venant's compatibility condition, the condition to admit zero left-plus-right curl. Indeed, this situation is an analogue of that a curl free vector field admits a scalar potential, which is non-unique and can be obtained from the vector field via its integral along an arbitrary curve. By this analogy, $\mathbf{u}^{\mathrm{Cauchy}}$ can be named a *Cauchy potential* of a Cauchy type tensor field (a symmetric tensor field with the Saint-Venant property). This naming explains the superscript $^{\mathrm{Cauchy}}$ in the notation $\mathbf{u}^{\mathrm{Cauchy}}$. A Cauchy potential is a mathematical auxiliary quantity, which is in general unrelated to the physical displacement quantity: the indeterminateness in $\mathbf{u}^{\mathrm{Cauchy}}$ allows displacement only as a special case of $\mathbf{u}^{\mathrm{Cauchy}}$, and even the possibility for this special choice is ruined by nonzero initial deformedness. Cauchy potential and displacement are only occasionally and weakly related quantities. These two notions should never be confused.

For later use, let us give here a rewritten form of (28),

$$\mathbf{E}_t^{\mathrm{Cauchy}} = \left[\left(\mathbf{X} + \mathbf{u}_t^{\mathrm{Cauchy}} \right) \otimes \nabla_{\mathbf{X}} - \mathbf{I} \right]^{\mathrm{S}} , \tag{29}$$

too, despite that, at the moment, not any usefulness of this version is apparent.

3.4 The aspect of the natural structure

As has already been expounded, an elastic solid body possesses a natural structure which tells all pairwise distances of material points in the relaxed state, and elastic kinematics should express the deviation of current distances from the relaxed ones. It turns out that this can be done via a tensorial quantity (this will be detailed in Sect. 5 but is also plausible from the traditional approaches to deformedness via strain). For a liquid, we intend to do something else. Namely, a liquid does not have distinguished distances and has elasticity only volumetrically. There is a distinguished volume — per moles/particle-number/mass — which is realized for relaxed liquids (for amounts large enough so that surface tension can be neglected, this volume is decided by volumetric elasticity only). Elastic force/stress in a nonrelaxed liquid depends only on how the current volume differs from the relaxed one. Therefore, for liquids a scalar quantity — which could be called *expandedness*, and could be defined as $(V_t - V_{\mathrm{relaxed}})/V_{\mathrm{relaxed}}$ or its logarithm — is needed as the kinematic state variable for a elastic constitutive relation.

In the case of a gas even less holds physically. Gases have infinite relaxed volume so there only the current volume — per moles/particle-number/mass — remains as physically relevant kinematic quantity.

Granular media, which might — under certain circumstances — also be described as a continuum, pose a further, nontrivial, question towards kinematics.

To summarize, we can see that different types of continua require different kinematic description, corresponding to the different physical properties. This should be made explicit in any approach to continuum kinematics, and the various types of media should be treated differently.

One consequence of this observation is that *the deformation gradient cannot be used for the definition of elastic deformedness.* The deformation gradient quantity exists for hydrodynamic flows, too—it could well be named "flow gradient" as well—, while elastic deformedness is physically meaningful only for solids. Elastic deformedness is an additional state variable, existing in the case of solids. In the "differential equation plus initial condition" picture shown above this is manifested in that the initial condition for deformedness cannot be calculated, cannot be derived from the deformation gradient at that initial time. The initial condition is some independent quantity, and it is just the value of the elastic deformedness state variable at that initial instant.

Hereafter, let us concentrate on solids only. There, having seen that elastic kinematics is to compare the current distances to the relaxed ones, it is plausible how plasticity is connected to this situation. Indeed, plastic change means change of the relaxed structure, the change of natural pairwise distances. Contrary to elastic kinematics, plasticity is not about a *state* but about a *change*. Neither of the relaxed structures is physically distinguished with respect to another. For plastic kinematics, such a quantity is needed that tells the rate of how the relaxed distances change.

This also implies that (12), *i.e.,* $\mathbf{E}^{\text{total}} = \mathbf{E}^{\text{elast}} + \mathbf{E}^{\text{plast}}$, is problematic: $\mathbf{E}^{\text{elast}}$ must mean elastic deformedness *at a given instant*, while $\mathbf{E}^{\text{plast}}$ can only be something like the time integral of some rate of change of natural distances, between *two given instants*. The first term is a state and the second a change, hence, their sum cannot be either a state or a change. These two terms are conceptually different and physically incompatible. [The alternate formula of plastic kinematics, (13), will be commented in the subsequent subsection.]

3.5 Arbitrary auxiliary elements in the description

The above-described approach to continuum kinematics makes use of a number of auxiliary elements, most of them being introduced for technical convenience and formal simplification. These cause that the description mixes the observed — the phenomenon we wish to speak about — with the observer—us, who speak about the phenomenon. Then we need rules what changes how when we modify the auxiliary components so that the content on the phenomenon itself remains unmodified. Experience shows that this is not so simple as we hope and generates, in fact, controversies, like those seminal two which are usually labelled as 'material frame indifference' and 'material objectivity'.

Thus it is our practical interest to reveal all the auxiliary elements, and then to devise such a description that is free from these ingredients. Let us therefore see now what these troublesome elements are in turn.

Probably the most eye-catching one is the choice of a reference instant t_0. Time is homogeneous, no instant is more distinguished than another so choosing one of them is an artificial step.

Most improved variants of continuum kinematics avoid this step and do not assume elastic undeformedness at any time either, but use only a reference configuration to describe undeformedness—which configuration may never be taken at any time during the lifetime of the medium. A reference configuration means to assign a location to each material point. Unfortunately, the reference configuration still contains arbitrary aspects. Both for elasticity and plasticity, the spatial location and orientation of the body as a whole is irrelevant. Space points are equivalent, and space directions are equivalent (space is homogeneous and isotropic). Specifying them in some way brings in arbitrary auxiliary elements in our description. Note that, for deformedness, only pairwise material distances count (the current ones and the relaxed ones).

For plastic kinematics, the not yet analyzed (13), *i.e.,* $\mathbf{F}^{\text{total}} = \mathbf{F}^{\text{elast}}\mathbf{F}^{\text{plast}}$, has been based on the following interpretation: after a complete elastic relaxation [multiplication by $(\mathbf{F}^{\text{elast}})^{-1}$] the plastic deformation — embodying the current relaxed structure — is reached. Now, unfortunately, there is no distinguished relaxed configuration as there is no distinguished spatial location. Relaxation can occur in many different ways in space, depending on that which material points of the body are kept fixed during relaxation and which move, in what directions. As an example, when we hold a sponge in our right hand and, with our left hand first squeeze it and then release, then the end configuration depends on which part of the sponge is kept fixed (or moved) by our hand and where, and in which direction we hold the sponge during its relaxing. We are also allowed to move our right hand continuously and the sponge will still relax—and move afterwards as a whole body.

A current relaxed structure includes the pairwise distances but not the individual locations of material points. Consequently, the change of relaxed structure — plasticity — does not include locations or velocities either. Both $\mathbf{F}^{\text{elast}}$ and $\mathbf{F}^{\text{plast}}$ contain arbitrary aspects, they specify more than is needed physically.

3.6 The aspect of reference frame

At last, we need to examine what 'space' is. In other words, we have to revisit the initial and usually implicit step of choosing a reference frame, mentioned at the beginning of Section 2.

A reference frame is a collection of material points with respect to which any pointlike object (phenomenon) is observed by measuring its distances from the reference material points at any time. A reference frame is inertial if it sees free — unaffected, uninfluenced — objects to move in straight line with uniform velocity (as a special case: to be at rest). Inertial reference frames are a special subclass of rigid frames, *i.e.,* of those frames where the pairwise distances of reference material points are time independent. Nonrigid frames also find applications.

In the approach to continuum kinematics considered in Section 2, it would be important to specify what reference frame is used. From a noninertial frame, freely moving bodies are seen to have possess some acceleration or rotation, while to be rest with respect to such a frame requires some appropriate forces. Therefore, using a noninertial frame brings in artifacts, the auxiliary element of using a reference frame allows the presence of disturbing features which belong to the observer and not to the phenomenon we wish to observe. Least 'harm' is done by using an inertial reference frame.

Galileo is known to point out first that inertial reference frames are physically equivalent, neither of them is distinguished from the others. His example to explain this was a ship on still water [26]: if we are in a windowless cabin within the ship, we cannot determine whether the ship is at rest with respect to the shore or moves with some uniform velocity to some direction, no matter what type of physical experiment we perform in the cabin.

Inertial reference frames move uniformly with respect to each other. When changing from one to another that moves with relative velocity $\mathbf{V}$ with respect to the former, then time and position are to be transformed as

$$t' = t, \qquad\qquad \mathbf{x}' = \mathbf{x} - \mathbf{V}t \tag{30}$$

according to the Galilean transformation rule, which is applicable in the nonrelativistic regime of relative velocities much smaller than the speed of light. From this, we can read off that, nonrelativistically, time is absolute as the transformation rule for time is of the form $t' = f(t)$. Similarly do we find that space is not absolute since $\mathbf{x}' \neq \mathbf{g}(\mathbf{x})$.

From $\mathbf{x}' = \mathbf{g}(t, \mathbf{x}) = \mathbf{x} - \mathbf{V}t$ it is also clear that, to make space absolute, it must be combined with time. Forming the four dimensional combination $\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$, the transformation rule reads

$$\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} - \mathbf{V}t \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ -\mathbf{V} & \mathbf{I} \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \tag{31}$$

so the transformation does not lead out of the set of $\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$ values, and there is an absolute, physical, four dimensional quantity (four-vector) behind whose representation in one inertial frame is $\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$ and is $\begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix}$ in another.

That space is not absolute is also clear from the fact that, for any reference frame, the reference material points are the space points of that frame, and they indicate the 'resting locations' as seen from that frame. (A side remark: in practice, for a frame we do not fill the whole world with material points but take only a restricted amount of them, and use interpolation-extrapolation rules for the other, empty, locations.) Now, if a frame moves with respect to another then the reference material points — the resting locations — do not rest with respect to the other frame. What is a space point for one frame is a process for the other one.

This actually coincides with our everyday experience as well. When travelling on a train, our reference frame is fixed to the train: the seats, the window, etc. Even when our drink spills out of our cup because of a fast breaking of the train, we view this from a viewpoint fixed to the train—although the drink wished to continue its inertial motion and it was the train that did something noninertial. In the meantime, someone seeing this little accident from the outside, standing on the earth nearby the rail, sees us, the drink and the whole train as moving.

There is not one space as 'the universal background' of motions. There is one spacetime which is the universal background of motions but spaces are relative, they are relative to frames/observers.

Certainly, to take the step of combining space with time has not been easy historically, both because of physical and mathematical aspects. Physically, time is experienced to be different from space in some senses. In parallel, mathematically, geometry originally meant a Euclidean space, and it took much time to gradually realize that geometries other than Euclidean are also conceptually acceptable—and even relevant.

The four dimensional combinations show not a four dimensional Euclidean structure but the combination of two other structures. One of them is the linear map that takes the time component t out of $\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$, and the other is a three dimensional Euclidean scalar product

$$\begin{pmatrix} 0 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{x}_1 \cdot \mathbf{x}_2 \tag{32}$$

for the spacelike combinations $\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}$ — the ones which have zero timelike component —, which combinations are actually those invariant under Galilean transformation:

$$\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}' = \begin{pmatrix} 0 \\ \mathbf{x} - \mathbf{V} \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix} . \tag{33}$$

The time component is absolute, and consequently simultaneity of spacetime points also, and spacelike four-vectors with their scalar product are also absolute.

Historically, only the advent of special relativity helped to rethink the nonrelativistic space and time ideas as well. That's why Newton rejected Galileo's result on the equivalence of inertial reference frames, and claimed the existence of an absolute space (a distinguished frame): he could not imagine any other background for his action-at-a-distance forces. He didn't know that absolute simultaneity plus absolute spacetime vectors with Euclidean structure would suffice for this purpose, and, at the same time, would allow to incorporate the equivalence of inertial reference frames, too.

Some years after the birth of special relativity, another important achievement took place: Weyl published such a formalism for spacetime — he established both the nonrelativistic and the special relativistic version — that does not rely on reference frames [11]. Spacetime points and their relationships, *i.e.,* the structures of spacetime, are defined first, then pointlike objects are modelled with world lines in spacetime, and reference frames are introduced only later, as certain extended objects moving in spacetime.

This approach has gained attention only gradually, was popularized by Arnold, for example [27] Then, the quite succinct and basic treatment by Weyl was expounded, enriched and worked out in detail for the purposes of physics by Matolcsi [13, 20].

The frame free spacetime formalism enables us to remove the reference frame, the last arbitrary auxiliary element to be listed in this Section, from continuum kinematics. Material objectivity can be viewed directly, without the disturbing technical presence of frames. Also, the question of material frame indifference can be cleanly reworded as the question of noninertial motion indifference (does some interaction within the material couple to the acceleration, or to the angular velocity of the continuum?). While the present work does not intend to contribute directly to the topics of material objectivity and noninertial motion indifference, that questions will also be possible to investigate on the grounds layed down here in the subsequent Sections: a frame free discussion of kinematics, with all the other auxiliary ingredients also avoided.

In what follows, let us follow the general methodology of mathematical models for physical phenomena [28], with all notions introduced as mathematical objects, defined fully mathematically, so as not to leave any property obscured or tacitly assumed.

4 Spacetime without reference frames

4.1 Quantities with physical dimension (length, time, etc.)

Units chosen for dimensionful physical quantities — length, time, mass, etc. — are also auxiliary elements in a description. Therefore, let us avoid them, too, by utilizing the unit free approach [MTfeher], where we introduce separate one dimensional oriented real vector spaces (in short, measure lines) $\mathbb{L}$, $\mathbb{T}$, $\mathbb{M}$ for lengths, time intervals and mass values, respectively. A length, for example, is an element $\boldsymbol{\ell}$ of $\mathbb{L}$, sums of lengths $\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2$ and real multiples of lengths $\alpha\boldsymbol{\ell}$ also being elements of $\mathbb{L}$. It will be useful to summarize here how the unit free treatment gives account of the properties of dimensionful quantities.

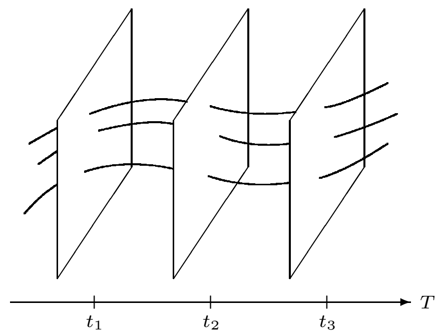

Figure 1: The structure of nonrelativistic spacetime

The product of a length $\boldsymbol{\ell} \in \mathbb{L}$ and a time $\boldsymbol{t} \in \mathbb{T}$ is their tensorial product $\boldsymbol{\ell} \otimes \boldsymbol{t} \in \mathbb{L} \otimes \mathbb{T}$ and their quotient is the tensorial quotient $\boldsymbol{\ell} /\!\!/ \boldsymbol{t} \in \mathbb{L} /\!\!/ \mathbb{T}$, with all the natural and expected properties, like

$$(\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) \otimes \boldsymbol{t} = \boldsymbol{\ell}_1 \otimes \boldsymbol{t} + \boldsymbol{\ell}_2 \otimes \boldsymbol{t}\,, \qquad (\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2) /\!\!/ \boldsymbol{t} = \boldsymbol{\ell}_1 /\!\!/ \boldsymbol{t} + \boldsymbol{\ell}_2 /\!\!/ \boldsymbol{t}\,, \tag{34}$$

$$(\alpha\boldsymbol{\ell}) \otimes (\beta\boldsymbol{t}) = (\alpha\beta)\, \boldsymbol{\ell} \otimes \boldsymbol{t}\,, \qquad (\alpha\boldsymbol{\ell}) /\!\!/ (\beta\boldsymbol{t}) = (\alpha/\beta)\, \boldsymbol{\ell} /\!\!/ \boldsymbol{t}\,. \tag{35}$$

The tensorial power of a measure line is defined for any real power and is also a measure line, *e.g.,* the pth power of $\mathbb{L}$ is $\mathbb{L}^{(\!(p)\!)}$. For $\boldsymbol{\ell} \in \mathbb{L}$, $\boldsymbol{\ell}^{(\!(p)\!)} \in \mathbb{L}^{(\!(p)\!)}$, and properties like $(|\alpha|\boldsymbol{\ell})^{(\!(p)\!)} = |\alpha|^p \boldsymbol{\ell}^{(\!(p)\!)}$ are satisfied. For positive integer powers, tensorial power coincides with what is straightforwardly expected, *i.e.,* $\mathbb{L}^{(\!(2)\!)} = \mathbb{L} \otimes \mathbb{L}$, $\mathbb{L}^{(\!(3)\!)} = \mathbb{L} \otimes \mathbb{L} \otimes \mathbb{L}$, etc. Furthermore, $\mathbb{L}^{(\!(0)\!)}$ is the vector space of real numbers, $\mathbb{R}$, and $\mathbb{L}^{(\!(-1)\!)}$ turns out to equal $\mathbb{R} /\!\!/ \mathbb{L} \equiv \mathbb{L}^*$, where * denotes the dual vector space.

If $\boldsymbol{A}$ is a linear map between vector spaces, $\boldsymbol{A} : \boldsymbol{U} \to \boldsymbol{V}$, then (taking again $\mathbb{L}$ as an example) $\boldsymbol{A}$ also determines a linear map $\boldsymbol{U} \otimes \mathbb{L} \to \boldsymbol{V} \otimes \mathbb{L}$ (if $\boldsymbol{A} : \boldsymbol{u} \mapsto \boldsymbol{v}$ then the corresponding map does $\boldsymbol{u} \otimes \boldsymbol{\ell} \mapsto \boldsymbol{v} \otimes \boldsymbol{\ell}$ for all $\boldsymbol{\ell} \in \mathbb{L}$). It will not cause confusion to denote this corresponding map also by $\boldsymbol{A}$.

Continuity, derivative, integral etc. of measure line valued functions, and of functions of measure line variable, is analogous to that of $\mathbb{R} \to \mathbb{R}$ functions, with the expected properties. For example, the derivative of a function $\mathbb{T} \to \mathbb{L}$ is $\mathbb{L} /\!\!/ \mathbb{T}$ valued, *i.e.,* has the dimension of velocity.

Units are not needed when formulating physical theories. Nevertheless, for completeness, it is worth closing this brief summary with how units are included in this approach. Choosing a unit for length is choosing a basis vector $\boldsymbol{\ell}_0$ in the one dimensional vector space $\mathbb{L}$, with which any length $\boldsymbol{\ell}$ can be given in a form $\boldsymbol{\ell} = \lambda \boldsymbol{\ell}_0$ and, consequently, can be characterized by a real number, the uniquely determined λ. Change to use a new unit means change to a new basis, to a new single basis vector.

4.2 The structure of nonrelativistic spacetime

The notations for the reference frame free treatment of nonrelativistic spacetime will be as follows. See Figures 1 and 2 for corresponding illustrations.

The set of spacetime points, M, is a four dimensional real oriented affine space over $\boldsymbol{M}$, the set of spacetime vectors (also called four-vectors). To any two spacetime points $x_1, x_2 \in M$ belongs a spacetime vector $\mathbf{x} \in \boldsymbol{M}$, in notation, $x_2 - x_1 = \mathbf{x}$, $x_2 = x_1 + \mathbf{x}$. Absolute time structure of spacetime vectors is given by a nondegenerate linear map $\boldsymbol{\tau} : \boldsymbol{M} \to \mathbb{T}$, assigning a time interval to any spacetime vector. On Fig. 2, the 'slices' depict equal-$\boldsymbol{\tau}$ surfaces. The elements of the three dimensional null space of $\boldsymbol{\tau}$,

$$\boldsymbol{S} := \operatorname{Ker} \boldsymbol{\tau} = \{ \mathbf{x} \in \boldsymbol{M} \,:\, \boldsymbol{\tau}\mathbf{x} = \mathbf{0} \} \tag{36}$$

are named spacelike vectors since their 'timelike aspect', the value of $\boldsymbol{\tau}$, is zero. They are also called three-vectors. $\boldsymbol{S}$ plays two important roles. First, the equivalence relation

$$x_1 \sim x_2 \iff x_2 - x_1 \in \boldsymbol{S} \tag{37}$$

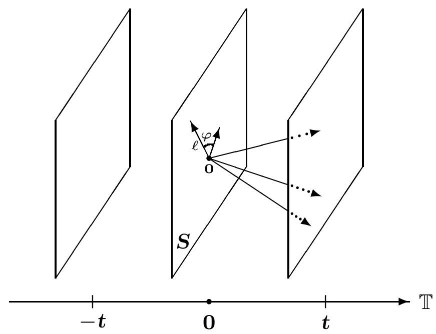

Figure 2: The structure of nonrelativistic spacetime vectors

defines the time instants of spacetime as the equivalence classes in $\boldsymbol{M}$—the 'slices' in Fig. 1. The set of time instants is a one dimensional real oriented affine space T. The time interval evaluation of spacetime vectors, $\boldsymbol{\tau}$, induces a time instant evaluation of spacetime points, which is an affine surjection $\tau : M \to T$. Therefore, expressing in terms of τ, the slices in Fig. 1 are the equal-τ-surfaces. To emphasize again: an instant is a collection of simultaneous spacetime points.

The other important aspect of $\boldsymbol{S}$ is regarding the spatial structure on nonrelativistic spacetime, which is a Euclidean scalar product, *i.e.,* a positive definite symmetric bilinear map $\boldsymbol{h} : \boldsymbol{S} \times \boldsymbol{S} \to \mathbb{L}^{(\!(2)\!)}$. Note that, nonrelativistically, only a three dimensional subset of spacetime vectors possesses a — Euclidean — metric structure, as opposed to the four dimensional — pseudo-Euclidean — metric structure of special relativistic spacetime vectors. Lengths and angles (see Fig. 2) have meaning only within $\boldsymbol{S}$.

Summarizing, nonrelativistic spacetime is a quintuplet $(M, \mathbb{T}, \boldsymbol{\tau}, \mathbb{L}, \boldsymbol{h})$, with the explanations above.

After the defining properties, let us also consider those consequences and physically important notions which will be utilized for the forthcoming discussion of continuum kinematics.

The dual vector space of $\boldsymbol{M}$, $\boldsymbol{M}^*$, consists of the linear $\boldsymbol{M} \to \mathbb{R}$ maps. Unlike special relativistically, in the nonrelativistic case four-covectors cannot be identified with four-vectors. (See, *e.g.,* [20] for tensorial identifications and other various notions and notations related to tensors that are utilized hereafter.) Let $\boldsymbol{\eta} : \boldsymbol{M}^* \to \boldsymbol{S}^*$ denote the linear map that restricts any four-covector to the subspace of spacelike vectors:

$$\boldsymbol{\eta} : \boldsymbol{k} \mapsto \boldsymbol{k}\big|_{\boldsymbol{S}} \tag{38}$$

Its dual transpose, $\boldsymbol{\eta}^* : \boldsymbol{S} \to \boldsymbol{M}$ is nothing but $\boldsymbol{I_M}\big|_{\boldsymbol{S}}$, the restriction of the identity map of $\boldsymbol{M}$ on spacelike vectors.

The motion — or, so to say, the existence — of a pointlike object in spacetime is described by a world line, a curve with futurelike tangent vectors, where futurelike means vectors that have positive $\boldsymbol{\tau}$ value. (Fig. 1 depicts three world lines.) A world line function is a world line with its natural parametrization by time:

$$r : T \to M\,, \qquad \tau(r_t) = t \quad (\forall t \in T) \tag{39}$$

(let's keep writing time variable into subscript), fulfilling the property that its derivative, the four-velocity at any time t, satisfies

$$\boldsymbol{\tau}\dot{r}_t = 1\,. \tag{40}$$

For this reason, it is useful to give a notation to the set of all possible four-velocity values:

$$V(1) := \{\boldsymbol{v} \in \boldsymbol{M}/\!\!/\mathbb{T} : \boldsymbol{\tau}\boldsymbol{v} = 1\}\,, \tag{41}$$

which is a three dimensional Euclidean affine space over $\boldsymbol{S}/\!\!/\mathbb{T}$.

The motion (spacetime existence, 'fate') of an extended and continuous material object is a continuous collection of world lines, which provides a foliation of spacetime. Then, at any spacetime point, we have a four-velocity value, which is the derivative there of the world line function that goes through that spacetime point. Thus a four-velocity field on spacetime can be read off. The opposite way is also possible for giving a motion of a continuum, when we start with a — smooth enough — four-velocity field on spacetime and determine the maximal integral curves of that four-vector field, which integral curves will be the world lines of the material body.

Any reference frame or spacetime observer is also such a possible continuum motion in spacetime. The space points of an observer are the world lines. Especially, an inertial observer admits world lines that are parallel straight lines with a single prescribed four-velocity value as their tangent vector that is constant along each world line and is the same value for all world lines.

A more general and also important family of observers is the rigid observers, where the distance of any two space points of the observer are time independent. Note that an instant is a spacelike three dimensional affine subspace in spacetime which intersects each world line, each futurelike curve at exactly one spacetime point, and the distance of two space points of an observer at a time is the Euclidean length of the two such intersection points. When this distance is the same for all times for this given pair of world lines, and if all such pairwise distances are time independent then the observer is called rigid.

The space of a rigid observer proves to be a three dimensional Euclidean affine space. For nonrigid observers, not even a time independent Riemann metric structure can be found, in general, so we are left with only a smooth manifold.

Calculus of functions defined on spacetime is similar to that of defined on $\mathbb{R}^4$, though some differences emerge because $\mathbb{R}^4$ is a Euclidean vector space while, on $\boldsymbol{M}$, which lies under M, the structures $\boldsymbol{\tau}$ and $\boldsymbol{h}$ are more tricky. The spacetime derivative of a scalar field $f : M \to \mathbb{R}$, denoted by $f \otimes \boldsymbol{D}$ (to be systematic, derivatives will always be written indicating the correct tensorial order) takes an $\boldsymbol{M}^*$ value at each spacetime point. It expresses how fast f changes in a spacetime direction. The restriction $\boldsymbol{\eta}(f \otimes \boldsymbol{D}) = (f \otimes \boldsymbol{D})\boldsymbol{\eta}^*$, denoted hereafter by $f \otimes \nabla$, tells how fast f changes in spacelike directions. Hence, ∇ is the spatial derivative, which proves thus to be an absolute (frame independent) operation, although absolute space does not exist but spaces depend on observers. The explanation is that spacelikeness is absolute and this is enough for spatial derivative.

The derivative of a four-vector field $\boldsymbol{f} : M \to \boldsymbol{M}$ is $\boldsymbol{M} \otimes \boldsymbol{M}^*$ valued at every point, the trace of which is the four-divergence. In parallel, the spatial derivative $\boldsymbol{q} \otimes \nabla$ of a three-vector field or spacelike vector field $\boldsymbol{q} : M \to \boldsymbol{S}$ is $\boldsymbol{S} \otimes \boldsymbol{S}^*$ valued, the trace of which is the three-divergence. On the other side, the spatial derivative of an $M \to \boldsymbol{S}^*$ three-covector field is $\boldsymbol{S}^* \otimes \boldsymbol{S}^*$, whose the antisymmetric part leads to the curl, $\nabla \times$, of the three-covector field.

An important special case is the derivative of a four-velocity field $\boldsymbol{v} : M \to V(1) \subset \boldsymbol{M}/\!\!/\mathbb{T}$. Since the difference of any two elements of $V(1)$ is spacelike, differences of four-velocities are always spacelike. Consequently, $\boldsymbol{v} \otimes \boldsymbol{D}$ and $\boldsymbol{v} \otimes \nabla$ are not only $\boldsymbol{M}/\!\!/\mathbb{T} \otimes \boldsymbol{M}^*$ and $\boldsymbol{M}/\!\!/\mathbb{T} \otimes \boldsymbol{S}^*$ valued, respectively, but $\boldsymbol{S}/\!\!/\mathbb{T} \otimes \boldsymbol{M}^*$, resp. $\boldsymbol{S}/\!\!/\mathbb{T} \otimes \boldsymbol{S}^*$ valued. Therefore, the three-divergence of a four-velocity field is automatically meaningful, while the curl only after utilizing the identification $\boldsymbol{S} \equiv \boldsymbol{S}^* \otimes \mathbb{L}^{(\!(2)\!)}$ provided by $\boldsymbol{h}$.

5 The material manifold of solids

Let the mathematical model for a continuum be a three dimensional simply connected complete smooth manifold $\mathcal{C}$. Its tangent space at a material point, an element P of $\mathcal{C}$, is $T_P(\mathcal{C})$. The differential map of a smooth mapping ϕ from $\mathcal{C}$ to an arbitrary smooth manifold $\mathcal{M}$ at a point P is a linear map $T_P(\mathcal{C}) \to T_{\phi(P)}(\mathcal{M})$, denoted by $(\phi \otimes \widetilde{\nabla})(P)$. Henceforth, overtilde will indicate material vectors, covectors, tensors etc., *i.e.,* objects related to the tangent spaces of $\mathcal{C}$ (*i.e.,* referring to material directions) to make a distinction to vectors etc. related to $\boldsymbol{S}$ of spacelike spacetime vectors — spatial directions — (as well as to quantities related to $\boldsymbol{M}$, though these latter will occur only infrequently in what follows).

A motion or spacetime process of such a body is given by assigning a world line function r to each material point P in a smooth enough way. Thence, at a time t, point P is at spacetime point $r_t(P)$. For a fixed t, $P \mapsto r_t(P)$ is a mapping from $\mathcal{C}$ to t — let's recall that a t is a Euclidean affine space, a three dimensional 'slice' within spacetime —, with its differential map or Jacobian map at P

$$\boldsymbol{J}_t(P) = (r_t \otimes \widetilde{\nabla})(P) . \tag{42}$$

This Jacobian is the frame free & reference time free & reference configuration free generalization of the deformation gradient.

The four-velocity of a material point P at a time t is

$$\mathbf{v}_t(P) = \dot{r}_t(P) . \tag{43}$$

For convenience, hereafter let us not distinguish in notation functions defined on $\mathcal{C}$ (Lagrangian description; *e.g.,* $\mathbf{v}_t$) from their Eulerian-like spacetime counterpart (their composition with r_t^{-1} for each t; *e.g.,* $\mathbf{v}_t \circ r_t^{-1}$). It is easy to check that, for a function f defined on spacetime, the comoving derivative $\dot{f}$ is to be introduced as $(f \otimes \boldsymbol{D})\boldsymbol{v}$ so that it be in accord with the overdot of Lagrangian variabled functions.

Using the chain rule of differentiation of composite functions,

$$\boldsymbol{L}_t := \mathbf{v}_t \otimes \nabla \tag{44}$$

can be expressed with $\boldsymbol{J}_t$ as

$$\boldsymbol{L}_t = \dot{\boldsymbol{J}}_t \boldsymbol{J}_t^{-1} . \tag{45}$$

At any instant t, the current distance of two material points P, Q is the distance of the two spacetime points $r_t(P), r_t(Q)$ where the two material points stay at t,

$$d_t(P, Q) = \| r_t(Q) - r_t(P) \|_{\boldsymbol{h}} . \tag{46}$$

This induces a Riemann metric tensor on $\mathcal{C}$, which proves to be

$$\widetilde{\boldsymbol{h}}_t = \boldsymbol{J}_t^* \boldsymbol{h} \boldsymbol{J}_t , \qquad \widetilde{\boldsymbol{h}}_t(P) : T_P(\mathcal{C}) \times T_P(\mathcal{C}) \to \mathbb{L}^{(\!(2)\!)} . \tag{47}$$

Actually, the following three characterizations of distance relationships are equivalent on a Riemann manifold: 1) lengths of curves, 2) distances of pairs of points (defined as the length of the shortest, geodesic, curve between the two points), 3) the metric tensor. The third choice uses a local quantity while the other two are nonlocal characterizations so it is favourable to work in terms of the metric tensor.

So far we haven't restricted our discussion to elastic solids. To make this step, let us realize the physical ideas put forward in Sects. 3.3 and 3.6 by adding to our model of the continuum a mathematical object that expresses the relaxed/natural distances of pairs of material points. This means that we furnish our material manifold with a natural/relaxed Riemann metric tensor $\widetilde{\boldsymbol{g}}$. In general, this metric is different from the currently realized metric $\widetilde{\boldsymbol{h}}_t$. In other words, in general the body is not relaxed, it is not undeformed, not even within some local neighbourhood. It might be considered as a 'blessed moment' when the two metrics coincide.

Nevertheless, we require something from the relaxed metric: that, under appropriate circumstances, the current metric *could* evolve into the relaxed metric: that it is possible to bring the body into relaxed state, even within a finite time interval, by some finite later time $\bar{t}$, via some motion $\bar{r}$ that would connect smoothly to the current motion r_t of the material and would continue in such a way that the distances realized at $\bar{t}$ coincide with the relaxed ones for the whole body. (In short: The relaxed metric must be realizable as a current metric at some time $\bar{t}$. A solid must have the ability to relax.) In the language of metrics, this is expressed as $\widetilde{\boldsymbol{h}}_{\bar{t}} = \widetilde{\boldsymbol{g}}$. Writing this in terms of $\bar{r}_{\bar{t}}$ and its gradient $\bar{\boldsymbol{J}}_{\bar{t}} = \bar{r}_{\bar{t}} \otimes \nabla$ reads

$$\bar{\boldsymbol{J}}_{\bar{t}}^* \boldsymbol{h} \bar{\boldsymbol{J}}_{\bar{t}} = \widetilde{\boldsymbol{g}} , \tag{48}$$

which tells that $\bar{r}_{\bar{t}}$ establishes an isometry, a metric preserving diffeomorphism, between $\mathcal{C}$ as a Riemann manifold with metric $\widetilde{\boldsymbol{g}}$ and t as a Riemann manifold with metric $\boldsymbol{h}$.

Now, we know that t is actually a Euclidean affine space and, hence, a flat Riemann manifold (*i.e.,* having zero Riemann curvature tensor). Due to a theorem (see eg. [29]), a

Riemann manifold is isometric to a flat one iff it is flat. Therefore, the relaxed structure on $\mathcal{C}$ is required to be a flat Riemann metric.

This condition can be further reformulated in a technically more convenient way. Namely, $\mathcal{C}$ being three dimensional, not only its Riemann tensor determines its Ricci tensor but the Ricci tensor also determines the Riemann tensor (one proof of this can be based on p88 of [30]). Consequently, the requirement of flatness is equivalent to the zeroness of the Ricci tensor. To summarize, the physical expectations dictate the relaxed structure of a solid body to be a Ricci-flat Riemann metric.

6 Elastic kinematics

6.1 Elastic shape

In the light of Sect. 3.3, elastic kinematics needs a quantity that expresses how the current distances — on $\mathcal{C}$, the current metric $\widetilde{\boldsymbol{h}}_t$, — differ from the relaxed distances — on $\mathcal{C}$, from the relaxed metric $\widetilde{\boldsymbol{g}}$. Or, in terms of spatial spacetime tensors, the current distances — on t, $\boldsymbol{h}$ — are to be compared to the relaxed distances — on t, to

$$\boldsymbol{g}_t := \left(\boldsymbol{J}_t^{-1}\right)^* \widetilde{\boldsymbol{g}}\, \boldsymbol{J}_t^{-1} . \tag{49}$$

Both $\widetilde{\boldsymbol{h}}_t$ and $\widetilde{\boldsymbol{g}}$ are $T_P(\mathcal{C}) \to \left[T_P(\mathcal{C})/\mathbb{L}^{(\!(2)\!)}\right]^*$ type linear maps, and both $\boldsymbol{h}$ and $\boldsymbol{g}_t$ are $\boldsymbol{S} \to \left[\boldsymbol{S}/\mathbb{L}^{(\!(2)\!)}\right]^*$ type maps.

It is also instructive to remark that it would be advantageous if the elastic kinematic quantity agreed with the Cauchy strain in the regime of small deformedness. We can formulate its definition (21) in the frame free approach via its evolution equation as

$$\dot{\boldsymbol{E}}^{\text{in. Cauchy}} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{h}^{-1}\boldsymbol{L}^*\boldsymbol{h}\right) = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^+\right) = \boldsymbol{L}^{\mathrm{S}} , \tag{50}$$

where the appearance of $\boldsymbol{h}$ is unavoidable and the $\boldsymbol{h}$-adjoint

$$\boldsymbol{L}^+ := \boldsymbol{h}^{-1}\boldsymbol{L}^*\boldsymbol{h} \tag{51}$$

of $\boldsymbol{L} = \boldsymbol{v} \otimes \nabla$ must be used because $\boldsymbol{L}$ is an $\boldsymbol{S} \to \boldsymbol{S} /\!\!/ \mathbb{T}$ tensor, and its dual transpose, a $\boldsymbol{S}^* \to \boldsymbol{S}^* /\!\!/ \mathbb{T}$ tensor cannot be added to it, except when $\boldsymbol{h} : \boldsymbol{S} \to \left(\boldsymbol{S} /\!\!/ \mathbb{L}^{(\!(2)\!)}\right)^*$ is also inserted as appropriate. The resulting $\boldsymbol{E}^{\text{in. Cauchy}}$ is $\boldsymbol{S} \to \boldsymbol{S}$ valued, and is $\boldsymbol{h}$-symmetric.

Incidentally, the reinterpretation of the traditional Cauchy strain would require to involve the Jacobian $\boldsymbol{J}$ as well, since the material gradient of velocity maps from $T_P(\mathcal{C})$ to $\boldsymbol{S}$ so it can be combined with its dual transpose only when an appropriate map between $T_P(\mathcal{C})$ and $\boldsymbol{S}$ is also made use of. Note that the Jacobian tensor $\boldsymbol{J}$ (the spacetime consistent generalization of the deformation gradient) can never be 'close to the identity tensor' since it connects *different* vector spaces so the traditional Cauchy strain would become a rather complicated object when seen in the frame free & reference time free & reference configuration free way. In what follows, let us use (for comparison purposes and for checking the small deformedness regime) only the inertial version, which is a simple and clear object.

Now, having seen that from what vector space and to what vector space the current metric tensor and the relaxed metric tensor map, mathematics allows to compose these two linear maps only in the combination

$$\boldsymbol{A}_t := \boldsymbol{g}_t^{-1}\boldsymbol{h} \tag{52}$$

as the spacelike spacetime tensorial version, and as

$$\widetilde{\boldsymbol{A}}_t = \boldsymbol{J}_t^{-1}\boldsymbol{A}_t\boldsymbol{J}_t = \widetilde{\boldsymbol{g}}^{-1}\widetilde{\boldsymbol{h}}_t , \tag{53}$$

the material equivalent of $\boldsymbol{A}_t$. It is easy to check that $\boldsymbol{A}_t$ is an $\boldsymbol{h}$-symmetric $\boldsymbol{S} \to \boldsymbol{S}$ tensor, while $\widetilde{\boldsymbol{A}}_t$ is a $\widetilde{\boldsymbol{g}}$-symmetric $T_P(\mathcal{C}) \to T_P(\mathcal{C})$ type tensor.

If the material is in its relaxed state, $\widetilde{\boldsymbol{h}}_t = \widetilde{\boldsymbol{g}}$, $\boldsymbol{g}_t = \boldsymbol{h}$, then $\widetilde{\boldsymbol{A}}_t(P) = \boldsymbol{I}_{T_P(\mathcal{C})}$, $\boldsymbol{A}_t = \boldsymbol{I}_{\boldsymbol{S}}$. This seems to indicate that we have introduced not a deformedness (a reinterpreted strain) but a reinterpreted deformation. We can call it *elastic shape tensor*.

It is possible to derive the time evolution equation of the elastic shape tensor. With (45) and (51), it is in fact straightforward to find

$$\dot{\boldsymbol{A}} = \boldsymbol{L}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{L}^{+} \,. \tag{54}$$

Comparing this with (18) shows that what we have at hand is actually the frame free & reference time free & reference configuration free generalization of the left Cauchy-Green deformation tensor. Its material version is, therefore, the reformed version of the right Cauchy-Green tensor.

Based on the definition of $\boldsymbol{A}$, one can show that

$$\det \sqrt{\boldsymbol{A}_t} = \sqrt{\det \boldsymbol{A}_t} = \frac{\sqrt{|\boldsymbol{h}|}}{\sqrt{|\boldsymbol{g}_t|}} = \frac{\sqrt{|\widetilde{\boldsymbol{h}}_t|}}{\sqrt{|\widetilde{\boldsymbol{g}}|}} \,, \tag{55}$$

where $|\sqrt{\cdot}|$ denotes the volume form that a Riemann metric defines. This quantity is, therefore, the ratio of the current volume and the relaxed volume of a small amount of material. This formula is in accord with (the square of) equation (16).

For small deformedness, which can already be formulated as $\boldsymbol{A}_t \approx \boldsymbol{I}_{\boldsymbol{S}}$, *i.e.,* their difference being much smaller than $\boldsymbol{I}_{\boldsymbol{S}}$ in tensorial norm, the rate equation (54) becomes, in leading order of $\boldsymbol{A}_t - \boldsymbol{I}_{\boldsymbol{S}}$,

$$\dot{\overline{(\boldsymbol{A}_t - \boldsymbol{I}_{\boldsymbol{S}})}} \approx 2\boldsymbol{L}^{\mathrm{S}} \,. \tag{56}$$

Comparing this with (50) shows that we have found the link

$$\tfrac{1}{2}\left(\boldsymbol{A}_t - \boldsymbol{I}_{\boldsymbol{S}}\right) \approx \boldsymbol{E}^{\mathrm{in.\,Cauchy}} \qquad \left(\boldsymbol{A}_t - \boldsymbol{I}_{\boldsymbol{S}} \ll \boldsymbol{I}_{\boldsymbol{S}}\right) \tag{57}$$

with the Cauchy deformedness (*i.e.,* from any given small initial deformedness, the two evolve the same way in leading order).

6.2 Deformedness

For deformedness itself, we are interested in such a function of $\boldsymbol{A}$ that, for $\boldsymbol{A}_t \approx \boldsymbol{I}_{\boldsymbol{S}}$, is approximately $\frac{1}{2}\left(\boldsymbol{A}_t - \boldsymbol{I}_{\boldsymbol{S}}\right)$. This is actually a large amount of freedom but we already have a geometrically distinguished desire as well: that the trace of the deformedness tensor should express the volumetric aspect of elastic expandedness and its deviatoric part the constant–volume deformednesses. In the light of Sect. 3.2 and formula (55), it is not surprising to find that this can be accomplished by the Hencky type choice:

$$\boldsymbol{D} := \ln \sqrt{\boldsymbol{A}} = \frac{1}{2} \ln \boldsymbol{A} \,, \qquad \Longrightarrow \qquad \mathrm{tr}\boldsymbol{D} = \ln \sqrt{\det \boldsymbol{A}} \,. \tag{58}$$

As such, $\boldsymbol{D}$ also satisfies the wish raised in Sect. 3.2 that our deformedness quantity should diverge both for infinite compression and for infinite extension.

In parallel, it is easy to see that

$$\boldsymbol{D} \approx \tfrac{1}{2}\left(\boldsymbol{A} - \boldsymbol{I}_{\boldsymbol{S}}\right) \approx \boldsymbol{E}^{\mathrm{in.\,Cauchy}} \qquad (\text{whenever } \boldsymbol{D} \ll \boldsymbol{I}_{\boldsymbol{S}}) \,. \tag{59}$$

Although the present work concentrates on the kinematic issues, some arguments can already be added on the advantage of the Hencky deformedness in constitutive relations. On the theoretical side, one can show — the simple details of the calculation omitted here — that, if the Cauchy stress $\boldsymbol{\sigma}$ can be derived from a quadratic elastic potential $U(\boldsymbol{D})$ of an isotropic solid as

$$\boldsymbol{\sigma} = \rho \frac{\mathrm{d}U}{\mathrm{d}\boldsymbol{D}} \tag{60}$$

(ρ denoting density) then its mechanical power, $\mathrm{tr}\,\boldsymbol{\sigma}\boldsymbol{L}$, is found to be simply

$$\mathrm{tr}\,\boldsymbol{\sigma}\boldsymbol{L} = \rho \dot{U} \,. \tag{61}$$

This is exactly the expression that is the natural one in connection with a continuity equation for energy. Hence, this simple example suggests that the variable $\boldsymbol{D}$ behaves distinguishedly from energetic and, consequently, thermodynamic point of view.

In parallel, it has already been mentioned in Sect. 3.1 that the Hencky choice is favoured by experiments in the linear regime of elastic stress-deformedness relation, as well as for the role of the variable of nonlinear elastic constitutive relations. Naturally, these aspects require extensive further investigation to collect a satisfactory amount of knowledge on the subject.

6.3 The compatibility condition

We have seen, in Sect. 5, that the relaxed metric must be Ricci-flat. This imposes a restriction on $\boldsymbol{A}$. Working with the spacelike spacetime tensorial version $\boldsymbol{g}_t$, its Ricci tensor can be expressed in a coordinate free version of the usual coordinate formula [30] since the manifold, t, on which $\boldsymbol{g}_t$ is given is a Euclidean affine space:

$$\boldsymbol{R}_t^{\mathrm{Ricci}} = \nabla \cdot \boldsymbol{\Gamma}_t - (\mathrm{tr}_{1,3}\boldsymbol{\Gamma}_t) \otimes \nabla + (\mathrm{tr}_{1,3}\boldsymbol{\Gamma}_t)\boldsymbol{\Gamma}_t - \mathrm{tr}_{1,3}(\boldsymbol{\Gamma}_t\boldsymbol{\Gamma}_t)\,, \tag{62}$$

where the Christoffel tensor is

$$\boldsymbol{\Gamma}_t = \tfrac{1}{2}\,\boldsymbol{g}_t^{-1}\left[\boldsymbol{g}_t \otimes \nabla + (\boldsymbol{g}_t \otimes \nabla)^{\mathrm{T}_{2,3}} - \nabla \otimes \boldsymbol{g}_t\right]. \tag{63}$$

As $\boldsymbol{g}_t = \boldsymbol{h}\boldsymbol{A}_t^{-1}$ [cf. (52)], the requirement $\boldsymbol{R}_t^{\mathrm{Ricci}} = \boldsymbol{0}$ can be reformulated as a condition on $\boldsymbol{A}_t$. Omitting the rather lengthy details of the calculation [31], one can obtain

$$\begin{aligned}
&\mathrm{tr}_{1,5;3,4}\left[\boldsymbol{A}^{-1} \otimes \boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)(\boldsymbol{A}\otimes\nabla)\right] + 2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{1,2}\left[\boldsymbol{A}\boldsymbol{h}^{-1}(\nabla\otimes\nabla\otimes\boldsymbol{A})\right]\boldsymbol{A}^{-1} + \\
&+2\boldsymbol{h}\,\mathrm{tr}_{2,3}\left[\boldsymbol{A}^{-1}\otimes(\nabla\cdot\boldsymbol{A})\,\boldsymbol{h}^{-1}(\nabla\otimes\boldsymbol{A})\right]\boldsymbol{A}^{-1} - 2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{2,4}\left[(\boldsymbol{A}\otimes\nabla\otimes\nabla)\,\boldsymbol{h}^{-1}\right] + \\
&+2\mathrm{tr}_{1,4;3,5}\left[\boldsymbol{A}^{-1}\otimes\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)(\boldsymbol{A}\otimes\nabla)\right] + \mathrm{tr}_{1,2;3,5}\left[\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)(\boldsymbol{A}\otimes\nabla)\otimes\boldsymbol{A}^{-1}\right] - \\
&-2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{2,5;3,6}\left[(\boldsymbol{A}\otimes\nabla)\,\boldsymbol{A}\boldsymbol{h}^{-1}(\nabla\otimes\boldsymbol{A})\otimes\boldsymbol{A}^{-1}\right]\boldsymbol{A}^{-1} + \\
&+2\mathrm{tr}_{2,4}\left[(\nabla\otimes\boldsymbol{A})\,\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)\right]\boldsymbol{A}^{-1} - 3\mathrm{tr}_{2,6;3,5}\left[(\nabla\otimes\boldsymbol{A})\,\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)\otimes\boldsymbol{A}^{-1}\right] - \\
&-2\left[\nabla\otimes(\nabla\cdot\boldsymbol{A})\right]\boldsymbol{A}^{-1} - \boldsymbol{h}\,\mathrm{tr}_{3,4;2,5}\left[\boldsymbol{A}^{-1}\otimes\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla)\,\boldsymbol{A}\boldsymbol{h}^{-1}(\nabla\otimes\boldsymbol{A})\right]\boldsymbol{A}^{-1} + \\
&+2\mathrm{tr}_{1,2}\left[\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla\otimes\nabla)\right] - 2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{2,4;3,5}\left[(\boldsymbol{A}\otimes\nabla)\otimes\boldsymbol{h}^{-1}(\nabla\otimes\boldsymbol{A})\right]\boldsymbol{A}^{-1} = \boldsymbol{0}\,.
\end{aligned} \tag{64}$$

This complicated nonlinear equation is to be satisfied by $\boldsymbol{A}$. 'Fortunately', it is enough to satisfy it at a given time and then the time evolution equation will ensure that at later times it still holds true.

In the small deformedness regime, the leading order term of this condition turns out to be extremely simple:

$$\nabla \times \boldsymbol{D} \times \nabla + (\text{higher order terms}) = \boldsymbol{0}\,. \tag{65}$$

In fact, this is what we would expect even without calculation, as being the spacetime version of the Saint-Venant compatibility condition (25). Formula (64) is, hence, the compatibility condition for finite deformedness. Nevertheless, the derivation of (65) from (64) proceeds as follows. In (64), there are two types of terms: either containing a second derivative of $\boldsymbol{A}$ or containing the product of two first derivatives of $\boldsymbol{A}$. Assuming that $\boldsymbol{A} - \boldsymbol{I_S}$ is small even in the sense that the product of two first derivatives is much smaller than a second derivative, we can keep in (64) only the terms with a second derivative:

$$\begin{aligned}
&2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{1,2}\left[\boldsymbol{A}\boldsymbol{h}^{-1}(\nabla\otimes\nabla\otimes\boldsymbol{A})\right]\boldsymbol{A}^{-1} - 2\boldsymbol{h}\boldsymbol{A}^{-1}\mathrm{tr}_{2,4}\left[(\boldsymbol{A}\otimes\nabla\otimes\nabla)\,\boldsymbol{h}^{-1}\right] - \\
&\qquad -2\left[\nabla\otimes(\nabla\cdot\boldsymbol{A})\right]\boldsymbol{A}^{-1} + 2\mathrm{tr}_{1,2}\left[\boldsymbol{A}^{-1}(\boldsymbol{A}\otimes\nabla\otimes\nabla)\right] \approx \boldsymbol{0}\,.
\end{aligned} \tag{66}$$

Inserting $\boldsymbol{A} \approx \boldsymbol{I_S}$ directly as well yields

$$\begin{aligned}
&2\boldsymbol{h}\,\mathrm{tr}_{1,2}\left[\boldsymbol{h}^{-1}(\nabla\otimes\nabla\otimes\boldsymbol{A})\right] - 2\boldsymbol{h}\,\mathrm{tr}_{2,4}\left[(\boldsymbol{A}\otimes\nabla\otimes\nabla)\,\boldsymbol{h}^{-1}\right] - \\
&\qquad -2\left[\nabla\otimes(\nabla\cdot\boldsymbol{A})\right] + 2\mathrm{tr}_{1,2}\left[(\boldsymbol{A}\otimes\nabla\otimes\nabla)\right] \approx \boldsymbol{0}\,.
\end{aligned} \tag{67}$$

This formula is already linear in $\boldsymbol{A}$. Substituting (59) into it, and after simple technical steps, we arrive at (65).

For technical purposes, it would be advantageous to be able to characterize those $\boldsymbol{A}$ which satisfy the compatibility condition in some alternative way that is simpler than the condition itself. An idea to do this is as follows. The relaxed metric $\widetilde{\boldsymbol{g}}$ is flat so, for any t, it can be brought into isometry with $\boldsymbol{h}$, the metric of any t. Let us choose such an isometry

$$\hat{r}_t : \mathcal{C} \to t \qquad (\forall t)\,. \tag{68}$$

It looks like a motion of the continuum in spacetime but its is not the true motion but only an auxiliary 'pseudo-motion'. It can be chosen fairly arbitrarily: actually, it is arbitrary up to the isometries of an Euclidean affine space, which are the translations and rotations. In formula, any other such isometry $\hat{r}'_t$ is related to $\hat{r}_t$ as

$$\hat{r}'_t = \boldsymbol{R}_t\,(\hat{r}_t - o_t) + o_t \tag{69}$$

with an arbitrary—world line function-like—function o_t (carrying the translation freedom) and an arbitrary $\boldsymbol{h}$-orthogonal tensor valued function $\boldsymbol{R}_t$, which embodies the rotation degree of freedom, and rotates around o_t. For the Jacobian of the auxiliary pseudo-motion,

$$\hat{\boldsymbol{J}}_t := \hat{r}_t \otimes \nabla\,, \tag{70}$$

a part of the freedom disappears and only the rotation remains:

$$\hat{\boldsymbol{J}}'_t = \boldsymbol{R}_t\,\hat{\boldsymbol{J}}_t\,. \tag{71}$$

All in all, this auxiliary pseudo-motion is introduced not for any principal role but as a technical aid.

The property that it is an isometry between $\mathcal{C}$ with $\widetilde{\boldsymbol{g}}$ and t with $\boldsymbol{h}$ provides the relationship

$$\widetilde{\boldsymbol{g}} = \hat{\boldsymbol{J}}_t^{*}\,\boldsymbol{h}\,\hat{\boldsymbol{J}}_t \tag{72}$$

between the two metrics, and, similarly, substituting this into (52) using (53) yields

$$\boldsymbol{A}_t = \boldsymbol{J}_t\,\hat{\boldsymbol{J}}_t^{-1}\,\boldsymbol{h}^{-1}\big(\hat{\boldsymbol{J}}_t^{-1}\big)^{*}\boldsymbol{J}_t^{*}\,\boldsymbol{h}\,. \tag{73}$$

Let us observe that

$$\boldsymbol{J}_t\,\hat{\boldsymbol{J}}_t^{-1} = \big(r_t \otimes \widetilde{\nabla}\big)\big(\hat{r}_t \otimes \widetilde{\nabla}\big)^{-1} = \big(r_t \circ \hat{r}_t^{\,-1}\big) \otimes \nabla = \hat{\boldsymbol{q}}_t \otimes \nabla \tag{74}$$

with

$$\hat{\boldsymbol{q}}_t := r_t \circ \hat{r}_t^{\,-1} - I_t\,, \tag{75}$$

a spacelike four-vector field on t, where I_t is the identity affine map of the affine space t. In terms of this vector field,

$$\boldsymbol{A}_t = (\hat{\boldsymbol{q}}_t \otimes \nabla)\,(\hat{\boldsymbol{q}}_t \otimes \nabla)^{+}\,. \tag{76}$$

Therefore, any $\boldsymbol{A}_t$ allowed by the compatibility condition can be given in terms of a spacelike vector field $\hat{\boldsymbol{q}}_t$ on t. This $\hat{\boldsymbol{q}}_t$ is not unique but carries the arbitrarinesses that hold for $\hat{r}_t$. A $\hat{\boldsymbol{q}}_t$ can be called a vector potential of elastic shape.

We are actually more familiar with this $\hat{\boldsymbol{q}}_t$ than it seems at first sight. Indeed, when

$$\hat{\boldsymbol{q}}_t \otimes \nabla \approx \boldsymbol{I}_{\boldsymbol{S}} \tag{77}$$

(in the sense as for the above approximations) then

$$\boldsymbol{A}_t \approx \boldsymbol{I}_{\boldsymbol{S}} \tag{78}$$

so we are in the small deformedness regime, and, keeping one more order,

$$\boldsymbol{A}_t \approx \boldsymbol{I}_{\boldsymbol{S}} + 2\,(\hat{\boldsymbol{q}}_t \otimes \nabla - \boldsymbol{I}_{\boldsymbol{S}})^{\mathrm{S}}\,, \qquad \text{or} \qquad \boldsymbol{D}_t \approx (\hat{\boldsymbol{q}}_t \otimes \nabla - \boldsymbol{I}_{\boldsymbol{S}})^{\mathrm{S}}\,. \tag{79}$$

Let us now take a look at (29) to realize that $\hat{\boldsymbol{q}}_t$ is nothing but the finite deformedness generalization and frame & etc. free generalization of the Cauchy potential.

7 Plastic kinematics

As analyzed in Sect. 3.4, plasticity is a situation when the natural, relaxed, structure of a solid becomes time dependent. Accordingly, plastic kinematics must be described by such a quantity that measures the rate of change of this time dependence.

On the material manifold, this is expressed by $\dot{\widetilde{\boldsymbol{g}}}_t$ —or, equivalently, by $\left(\widetilde{\boldsymbol{g}}_t^{-1}\right)^{\bullet}$. On spacetime, the corresponding quantity, $\boldsymbol{J}_t \left(\widetilde{\boldsymbol{g}}_t^{-1}\right)^{\bullet} \boldsymbol{J}_t^*$, provides the additional term that appears when (54) is generalized to time dependent $\widetilde{\boldsymbol{g}}$:

$$\dot{\boldsymbol{A}}_t = \left(\boldsymbol{J}_t\, \widetilde{\boldsymbol{g}}_t^{-1} \boldsymbol{J}_t^* \boldsymbol{h}\right)^{\bullet} = \boldsymbol{L}_t \boldsymbol{A}_t + \boldsymbol{A}_t \boldsymbol{L}_t^+ - \boldsymbol{W}_t \tag{80}$$

with

$$\boldsymbol{W}_t := -\boldsymbol{J}_t \left(\widetilde{\boldsymbol{g}}_t^{-1}\right)^{\bullet} \boldsymbol{J}_t^* \boldsymbol{h} = \boldsymbol{J}_t\, \widetilde{\boldsymbol{g}}_t^{-1} \dot{\widetilde{\boldsymbol{g}}}_t \widetilde{\boldsymbol{g}}_t^{-1} \boldsymbol{J}_t^* \boldsymbol{h}\,. \tag{81}$$

$\boldsymbol{W}$ is a state variable that expresses plastic change rate, and can be named the *metric change tensor*. Therefore, in a continuum theory involving plastic phenomena, the complete set of dynamical and constitutive equations must contain an equation that determines the current value of $\boldsymbol{W}$ as a function of other quantities of the continuum. For example, $\boldsymbol{W}$ may be given as a function of the stress tensor. This function may be nonzero only when a stress value, *e.g.,* the shear stress, exceeds a critical value.

The form of this $\boldsymbol{W}$-determining function is limited by the requirement that $\boldsymbol{W}$ must be such that the natural metric structure $\widetilde{\boldsymbol{g}}_t$ remains Ricci-flat at any time t. This follows from the physical expectation that a sudden unloading should stop plastic changes, it should stop the change of the natural metric and— in case of zero volume forces —bring the material to an elastically nondeformed state, bring the current metric to the natural one.

Bearing in mind the complicated form of the compatibility condition (64), to ensure that the time dependent natural metric stays Ricci-flat is a rather strong restriction on $\mathbf{W}$ mathematically. On the other side, physically it is very valuable since many candidates for the $\boldsymbol{W}$-determining function that seem reasonable for small deformations may fail to admit a satisfying finite deformation generalization. The strong requirement helps to find physically admissible models by allowing only a rather limited range of choices.

The two widely used formulas for plastic kinematics, $\mathbf{F}^{\rm total} = \mathbf{F}^{\rm elast}\mathbf{F}^{\rm plast}$ and $\mathbf{E}^{\rm total} = \mathbf{E}^{\rm elast} + \mathbf{E}^{\rm plast}$ [appeared in Sect. 2 under (13) resp. (12), and already discussed in Sects. 3.4–3.5] can be viewed from the kinematic picture presented here as follows.

Let us rewrite (74) as

$$\boldsymbol{J} = r \otimes \widetilde{\nabla} = (\hat{\boldsymbol{q}} \otimes \nabla)\left(\hat{r} \otimes \widetilde{\nabla}\right) = (\hat{\boldsymbol{q}} \otimes \nabla)\,\hat{\boldsymbol{J}} \tag{82}$$

The Jacobian tensor $\boldsymbol{J}$ has already been found to be the spacetime compatible generalization of the deformation gradient $\mathbf{F} = \mathbf{F}^{\rm total}$. Furthermore, the pseudo-motion $\hat{r}$ is a mapping of the relaxed metric structure into spacetime at any instant so it can be 'interpreted' (pseudo-interpreted) as a 'motion of relaxed positions'. In this sense, $\hat{r} \otimes \widetilde{\nabla} = \hat{\boldsymbol{J}}$ is the incarnation of $\mathbf{F}^{\rm plast}$. Finally, $\hat{\boldsymbol{q}} \otimes \nabla$ has been seen to be related to the elastic shape tensor $\boldsymbol{A}$ — compare (76) with the usual formula $\mathbf{U}_{\rm L}^2 = \mathbf{F}^{\rm elast}(\mathbf{F}^{\rm elast})^{\rm T}$ of Sect. 2 and recall that $\mathbf{U}_{\rm L}^2$ is the traditional counterpart of $\boldsymbol{A}$. Hence, $\hat{\boldsymbol{q}} \otimes \nabla$ embodies $\mathbf{F}^{\rm elast}$, and thus the relationship between $\mathbf{F}^{\rm total} = \mathbf{F}^{\rm elast}\mathbf{F}^{\rm plast}$ and (82) becomes clear.

It is also apparent that the unphysical arbitrarinesses of $\mathbf{F}^{\rm elast}$ and $\mathbf{F}^{\rm plast}$, pointed out in Sect. 3.5, appear here as the unphysical arbitrarinesses in the pseudo-motion $\hat{r}$ and in the related quantities ($\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{q}}$).

Next, let us investigate (80) in the regime of small *elastic* deformedness. (Note that it has no meaning to consider 'small plastic deformedness' because there is no plastic deformedness/deformation/strain quantity as a state variable.) In the leading order of small $\boldsymbol{A} - \boldsymbol{I_S}$, (80) simplifies to

$$2\dot{\boldsymbol{D}}_t \approx \boldsymbol{L}_t + \boldsymbol{L}_t^+ + \boldsymbol{J}_t \left(\widetilde{\boldsymbol{g}}_t^{-1}\right)^{\bullet} \boldsymbol{J}_t^* \boldsymbol{h} = (\mathbf{v}_t \otimes \nabla) + \boldsymbol{h}^{-1}(\nabla \otimes \mathbf{v}_t)\,\boldsymbol{h} + \boldsymbol{J}_t \left(\widetilde{\boldsymbol{g}}_t^{-1}\right)^{\bullet} \boldsymbol{J}_t^* \boldsymbol{h}\,, \tag{83}$$

which can be rearranged as

$$\frac{1}{2}\Big[\big(\mathbf{v}_t\otimes\widetilde{\nabla}\big)\boldsymbol{J}_t^{-1}+\boldsymbol{h}^{-1}(\boldsymbol{J}_t^*)^{-1}\big(\widetilde{\nabla}\otimes\mathbf{v}_t\big)\boldsymbol{h}\Big]\approx\dot{\boldsymbol{D}}_t-\boldsymbol{J}_t\,\dot{\big(\widetilde{\boldsymbol{g}}_t^{-1}\big)}\,\boldsymbol{J}_t^*\boldsymbol{h}\,. \tag{84}$$

Then, let $[t_1,t_2]$ be a time interval during which $\boldsymbol{J}$ changes only a little:

$$\boldsymbol{J}(t')\boldsymbol{J}(t)^{-1}\approx\boldsymbol{I}_{\boldsymbol{S}}\qquad(\forall t,t'\in[t_1,t_2])\,. \tag{85}$$

Then we can replace $\boldsymbol{J}$ with $\boldsymbol{J}_{t_1}$, for example. Integrating in time over $[t_1,t_2]$ at a fixed material point, and noting that

$$\int_{t_1}^{t_2}\mathbf{v}=\int_{t_1}^{t_2}\dot{\boldsymbol{r}}=r_{t_2}-r_{t_1}=:\Delta r\,, \tag{86}$$

we obtain

$$\frac{1}{2}\Big[\big(\Delta r\otimes\widetilde{\nabla}\big)\boldsymbol{J}_{t_1}^{-1}+\boldsymbol{h}^{-1}\big(\boldsymbol{J}_{t_1}^*\big)^{-1}\big(\widetilde{\nabla}\otimes\Delta r\big)\boldsymbol{h}\Big]\approx\Delta\boldsymbol{D}-\boldsymbol{J}_{t_1}\left[\int_{t_1}^{t_2}\dot{\big(\widetilde{\boldsymbol{g}}_t^{-1}\big)}\right]\boldsymbol{J}_{t_1}^*\,\boldsymbol{h}$$
$$\approx\Delta\boldsymbol{D}-\boldsymbol{J}_{t_1}\left[\widetilde{\boldsymbol{g}}_{t_2}^{-1}-\widetilde{\boldsymbol{g}}_{t_1}^{-1}\right]\boldsymbol{J}_{t_1}^*\,\boldsymbol{h}\approx\Delta\boldsymbol{D}+\Delta\Big(-\boldsymbol{J}\,\widetilde{\boldsymbol{g}}^{-1}\boldsymbol{J}^*\,\boldsymbol{h}\Big) \tag{87}$$

(where, again, Δ means changes between t_1 and t_2). On the rightmost hand side, $\Delta\boldsymbol{D}$ is the change of elastic deformedness and the second term expresses purely plastic changes. Therefore, we have reached a formula that embodies $\mathbf{E}^{\text{total}}=\mathbf{E}^{\text{elast}}+\mathbf{E}^{\text{plast}}$.

Now, the formalism in Sect. 2 suggests $\mathbf{E}^{\text{total}}$ be some realization of

$$\frac{1}{2}\Big[\big(\Delta\mathbf{u}\otimes\nabla_{\mathbf{X}}\big)\text{ “+” }\big(\nabla_{\mathbf{X}}\otimes\Delta\mathbf{u}\big)\Big] \tag{88}$$

on the leftmost hand side of (87). However, it has no meaning that $\boldsymbol{J}_{t_1}$ is approximately identity because $\boldsymbol{J}_{t_1}(P):T_P(\mathcal{C})\to\boldsymbol{S}$ connects different vector spaces, between which there is no identity map. Only taking t_1 as a reference time — and thus choosing an identification between $T_P(\mathcal{C})$ and $\boldsymbol{S}$ — enables one to transform $\boldsymbol{J}_{t_1}$ into the quantities which measure changes with respect to the reference time. The same step is needed to give a meaning to the addition in (88), too, since an $\boldsymbol{S}\otimes T_P(\mathcal{C})^*$ tensor and a $T_P(\mathcal{C})^*\otimes\boldsymbol{S}$ tensor cannot be added—that's why the + in (88) is put within quotation marks. Furthermore, $\Delta\mathbf{u}$ is defined only with respect to a reference frame. Without reference elements and artificial identification, (88) is not meaningful and, for a $\mathbf{E}^{\text{total}}$ quantity, one has to be content with the leftmost side of (87).

8 Summary

The paper has emphasized and elaborated the following points.

1. Continuum kinematics must avoid the use of auxiliary elements like reference frame, reference time and reference configuration, which cause that the observed gets mixed with the observer and *unphysical artefacts* emerge in the description.

2. If working solely with material manifold based quantities, one cannot give account of the inertial-noninertial aspects of the motion/flow of the continuum.

3. By *working on spacetime*, one is able to define continuum kinematics without any auxiliary reference elements, and the inertial-noninertial aspects are properly included. The question of material frame indifference is clarified as the question of *noninertial motion indifference*.

4. The spacetime generalization of the deformation gradient is the gradient of the system of material world lines, which exists for solids, liquids and gases each.

5. Restricting ourselves to solids, *the kinematic quantity for elasticity* is not to be a two-point function which expresses a change but to be a one-point function, *i.e.,* a *state variable*, to express a state: the deviation of the *currently realized metric* in spacetime from the *natural metric* of the solid.

   The natural metric is to which the solid relaxes when unloaded. It follows that the natural metric must have zero Riemann curvature tensor and zero Ricci tensor. This requirement is the ‘finite deformation compatibility condition’.

6. This elastic deformedness quantity evolves in time via a *rate equation*. The solution of this differential equation is fixed by, for example, an initial condition, which, in many cases, cannot be chosen zero.

   (A side remark: because of the generally nonzero initial condition, the Cesàro–Volterra formula for small deformations cannot give the displacement since a reference time in a reference frame but rather a vectorial potential-like solution of the Saint-Venant compatibility condition.)

7. More closely, the natural quantity comparing the current metric and the natural metric turns out to be, in some respect, the spacetime generalization of the left Cauchy-Green deformation tensor. From this elastic shape tensor, the elastic deformedness tensor is defined as a spacetime analogue of the left Hencky strain, which is kinematically distinguished by that its trace expresses volumetric expansion and its deviatoric part the shear degrees of freedom.

8. *Plasticity* is a phenomenon when the *natural metric* structure of the solid *changes*. In the rate equation for the elastic shape tensor, the term emerging for nonzero change rate of the natural metric is to be provided by a plastic constitutive equation, in such a way that the compatibility condition is preserved by the rate equation.

9. No "elastic deformation gradient" and no "plastic deformation gradient" can be defined. The typically used additive or multiplicative rules $\mathbf{E}^{\mathrm{total}} = \mathbf{E}^{\mathrm{elast}} + \mathbf{E}^{\mathrm{plast}}$ and $\mathbf{F}^{\mathrm{total}} = \mathbf{F}^{\mathrm{elast}}\,\mathbf{F}^{\mathrm{plast}}$ for 'elastic and plastic deformation', resp. 'elastic and plastic deformation gradient' fail to carry an objective, *i.e.,* free-of-auxiliary-reference-elements meaning.

9 Acknowledgement

The authors thank J. M. Martín-García for his help in using the *xAct* software packages. The work was supported by the grant OTKA K81161.

References

[1] Chiara de Fabritiis and P. M. Mariano. Geometry of ineractions in complex bodies. *Journal of Geometry and Physics*, 54:300–323, 2005.

[2] P. M. Mariano. Geometry and balance of hyperstresses. *Rendiconti dei Lincei Matematica Applicata*, 18:311–331, 2007.

[3] A. Yavari, J. E. Marsden, and M. Ortiz. On spatial and material covariant balance laws in elasticity. *Journal of mathematical physics*, 47:042903, 2006.

[4] T. Matolcsi and P. Ván. Can material time derivative be objective? *Physics Letters A*, 353:109–112, 2006. math-ph/0510037.

[5] A. I. Murdoch. On criticism of the nature of objectivity in classical continuum physics. *Continuum Mechanics and Thermodynamics*, 17:135–148, 2005.

[6] M. Frewer. More clarity on the concept of material frame-indifference on classical continuum mechanics. *Acta Mechanica*, 202:213–246, 2009.

[7] W. Noll. Space-time structures in classical mechanics. In *The foundations of mechanics and thermodynamics (Selected papers by Walter Noll)*, pages 204–210. Springer Verlag, Berlin-Heidelberg-New York, 1974. originally: pp28-34, Delaware Seminar in the Foudnations of Physics, Berlin-Heidelberg-New York, Springer, 1967.

[8] W. Noll. Five contributions to natural philosophy. 2004. www.math.cmu.edu/∼wn0g/noll/FC.pdf.

[9] W. Noll and B. Seguin. Basic concepts of thermomechanics. *Journal of Elasticity*, 101:791–858121–151, 2010.

[10] W. Noll. A frame free formulation of elasticity. *Journal of Elasticity*, 83:291–307, 2006.

[11] H. Weyl. *Raum-Zeit-Matterie*. Julius Springer, Berlin, 1918. in German.

[12] P. Havas. Four-dimensional formulations of Newtonian mechanics and their relation to the special and the general theory of relativity. *Reviews of Modern Physics*, pages 938–965, 1964.

[13] T. Matolcsi. *A Concept of Mathematical Physics: Models for SpaceTime*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1984.

[14] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics*. Springer Verlag, Berlin-Heidelberg-New York, 1965. Handbuch der Physik, III/3.

[15] P. Haupt. *Continuum Mechanics and Theory of Materials*. Springer Verlag, Berlin-Heidelberg-New York, 2nd edition, 2000.

[16] A. Bertram. *Elasticity and plasticity of large deformations (An introduction)*. Springer, 2005.

[17] A. DiCarlo. Surface and bulk growth unified. In P. Steinmann and G. A. Maugin, editors, *Mechanics of material forces*, pages 53–64. Springer, 2005.

[18] M. Epstein. On material evolution laws. In G.A. Maugin, editor, *Geometry, continua and microstructure*, pages 1–9. Hermann, Paris, 1999.

[19] M. de Leon. A geometrical description of media with microstructure: uniformity and homogeneity. pages 11–20. Hermann, Amsterdam-etc., 2005.

[20] T. Matolcsi. *Spacetime Without Reference Frames*. Akadémiai Kiadó Publishing House of the Hungarian Academy of Sciences), Budapest, 1993.

[21] C.O. Horgan and J.G. Murphy. A generalization of Hencky's strain-energy density to model the large deformations of slightly compressible solid rubbers. *Mechanics of Materials*, 79:943–950, 2009.

[22] J. Plešek and A. Kruisová. Formulation, validation and numerical procedures for Hencky's elasticity model. *Computers and Structures*, 84:1141–1150, 2006.

[23] K. Matsuki and K. Takeuchi. Three-dimensional in situ stress determination by anelastic strain recovery of a rock core. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 30:1019–1022, 1993.

[24] K. Matsuki. Anelastic strain recovery compliance of rocks and its application to in situ stress measurement. *Int. J. Rock Mech. Min. Sci.*, 45:952–965, 2008.

[25] W. Lin et al. A case study of 3D stress orientation determination in Shikoku Island and Kii Peninsula, Japan. In I. Vrkljan, editor, *Rock Engineering in Difficult Ground Conditions – Soft Rocks and Karst*, pages 277–282. Taylor & Francis Group, London, 2010.

[26] G. Galilei. *Dialogo sopra i due Massimi Sistemi del Mondo*. Gio. Battista Londini, Firenze, 1632. In Italian.

[27] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer Verlag, 1978.

[28] T. Matolcsi. *A Concept of Mathematical Physics: Models in Mechanics*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 1986.

[29] J. Dieudonné. *Foundations of Modern Analysis 4*. Academic, New York, 1960.

[30] B. O'Neill. *Semi-Riemannian geometry with applications to relativity*, volume 103 of *Pure and Applied Mathematics*. Academic Press, London, 1983.

[31] For deriving formula (64), the *xAct* packages for *Mathematica*^{TM} (`http://www.xact.es`), written by J. M. Martín-García, have also been used.