GPS revisited: the relation of proper time and coordinate time

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Abstract

The widely accepted approximate formula for the relation between proper time rate s and coordinate time rate t used in the GPS is given by $ds/dt = 1+\frac{V}{c^2}-\frac{v^2}{2c^2}$, where V is the gravitational potential at the position of the satellite and v is the velocity. In this note we derive, without approximation, that the precise formula reads as $ds/dt = \sqrt{(1+\frac{2V}{c^2})(1-\frac{v^2}{c^2})}$.

1 Introduction

The most significant application of the theory of General Relativity in everyday life, arguably, is the Global Positioning System. The GPS uses accurate, stable atomic clocks in satellites and on the ground to provide world-wide position and time determination. These clocks have relativistic frequency shifts which need to be carefully accounted for, in order to achieve synchronization in an underlying Earth-centered inertial frame, upon which the whole system is based.

To determine the time rate s of the clocks carried by satellites and t of ideal clocks measuring the time of the underlying Earth-centered inertial frame the customary approach is to use (a slightly modified form of) Schwarzschild spacetime, and arrive at the formula

$$ds/dt = 1+\frac{V}{c^2}-\frac{v^2}{2c^2}, \tag{1}$$

after several first-order approximations in the calculations. Here V denotes the gravitational potential at the position of the satellite, and v is the velocity of the satellite measured in the underlying non-rotating Earth-centered inertial system. A detailed description of the GPS, including the calculations of relativistic effects leading to formula (1) is available in [1]. Formula (1) is the

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internationally accepted standard relating the clock frequencies, as described in [1, 2] and references therein. To make this formula applicable in practice one needs to change the ideal time t to the time t_E measured by Earth based clocks, which is achieved by the relation

$$\frac{dt}{dt_E} = 1 - \frac{\Phi_0}{c^2}, \tag{2}$$

where Φ_0 is a constant corresponding to the Earth's geoid. This, also, is an approximate formula.

In this note we derive precise formulae for the clock frequency rates in question *without approximation*. Our calculations are based on an abstract treatment of Schwarzschild spacetime, rather than the customary coordinate transformations. In fact, we extend the notions of an abstract model of special relativity (see [3]) to describe Schwarzschild spacetime, and this new point of view leads to our precise formulae. This description of Schwarzschild spacetime is *abstract* enough to give a mathematically *clear view of the appearing concepts* (much like in [6]) but also *concrete* enough to make *calculations easy to carry out* (much like calculations with coordinates, but not needing approximations). Much of the paper is devoted to the description of the formalism to make it self-contained. The actual calculations are fairly brief and contained in Sections 4 and 5.

As of now, our results are only of theoretical interest as the existing formulae provide good approximations to the desired precision in the GPS (see [2]).

2 The formalism

Throughout the paper we normalize the universal constants for simplicity so that light speed as well as the gravitational constant are the unity, $c = 1$, $\gamma = 1$.

We shall use some notions and results of the special relativistic spacetime model whose mathematical structure was expounded in [3, 4, 5]. This coordinate-free formalism differs from the usual textbook treatments of special relativity, and is close to that of [6] whose terminology we mostly adopt.

Special relativistic spacetime is an oriented four dimensional affine space M over the vector space $\mathbf{M}$. A time oriented Lorentz form $\mathbf{M} \times \mathbf{M} \to \mathbb{R}$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ is given.

A world vector $\mathbf{x}$ is *spacelike* if $\mathbf{x} \cdot \mathbf{x} > 0$, *timelike* if $\mathbf{x} \cdot \mathbf{x} < 0$, and *lightlike* if $\mathbf{x} \cdot \mathbf{x} = 0$, and $\mathbf{x} \neq 0$. For a spacelike vector $\mathbf{x}$ we put $|\mathbf{x}| := \sqrt{|\mathbf{x} \cdot \mathbf{x}|}$.

For two elements $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{M}$ we define $\mathbf{a} \otimes \mathbf{b}$ to be the bilinear map $\mathbf{M} \times \mathbf{M} \to \mathbb{R}$, $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{y})$.

The history of a classical material point is a curve in spacetime whose tangents are timelike. Such a curve is called a *world line*. The *time passing on a world line* or the *proper time of the world line* is

$$\int \sqrt{-\dot{p}(a) \cdot \dot{p}(a)}\, da \tag{3}$$

where p is an arbitrary parametrization of the world line in question. The world line can be parameterized by its proper time; such a parametrization is called a *world line function*. Thus, if $r : \mathbb{R} \to M$ is a world line function, then $\dot{r}(\mathbf{t}) \cdot \dot{r}(\mathbf{t}) = -1$ for all proper time values $\mathbf{t}$.

Accordingly, a futurelike vector $\mathbf{u}$ in $\mathbf{M}$ for which $\mathbf{u} \cdot \mathbf{u} = -1$ holds is called a *four-velocity*. For a four-velocity $\mathbf{u}$, we define the three dimensional spacelike linear subspace

$$\mathbf{E_u} := \{\mathbf{x} \in \mathbf{M} \mid \mathbf{u} \cdot \mathbf{x} = 0\}; \tag{4}$$

and

$$\pi_{\mathbf{u}} := \mathbf{M} \to \mathbf{E_u}, \quad \mathbf{x} \mapsto \mathbf{x} + \mathbf{u}(\mathbf{u} \cdot \mathbf{x}) \tag{5}$$

is the projection onto $\mathbf{E_u}$ along $\mathbf{u}$. The restriction of the Lorentz form onto $\mathbf{E_u}$ is positive definite, so $\mathbf{E_u}$ is a Euclidean vector space.

A *reference frame* $\mathbf{U}$ is a four-velocity valued smooth map defined in a connected open subset of M (cf. [6]). A maximal integral curve of $\mathbf{U}$ – a world line – is a *space point* of the reference frame, briefly a $\mathbf{U}$-*space point*. The set of maximal integral curves of $\mathbf{U}$ is the *space* of the reference frame, briefly the $\mathbf{U}$-*space*.

In reality, any synchronization procedure aims at establishing when two world points should be considered simultaneous. Thus in the spacetime model a *synchronization* or *simultaneity* is a smooth equivalence relation on a connected open subset of M such that the equivalence classes are connected three-dimensional smooth submanifolds (hypersurfaces) whose tangent spaces are spacelike. Given a synchronization S, an equivalence class is called an S-*instant*; the set I_S of S-instants is called S-*time*.

A *reference frame with synchronization* is a pair $(\mathbf{U}, S)$, where $\mathbf{U}$ is an reference frame and S is a synchronization.

A reference frame having constant value is called *inertial*. The space points – the integral curves – of an inertial frame with four-velocity $\mathbf{u}$ are straight lines directed by $\mathbf{u}$. The $\mathbf{u}$-space point containing the world point x is the straight line $x + \mathbb{R}\mathbf{u}$, where $\mathbb{R}\mathbf{u} := \{\mathbf{t}\mathbf{u} \mid \mathbf{t} \in \mathbb{R}\}$.

The *standard synchronization* according to the inertial frame having four-velocity $\mathbf{u}$ is defined in such a way that two world points x and y are $\mathbf{u}$-simultaneous if and only if $\mathbf{u} \cdot (x - y) = 0$, i.e. $x - y \in \mathbf{E_u}$. Thus, $\mathbf{u}$-simultaneous world points form a hyperplane directed by $\mathbf{E_u}$. A $\mathbf{u}$-*instant* is the collection of simultaneous world points; so a $\mathbf{u}$-instant is a hyperplane in spacetime directed by $\mathbf{E_u}$. The $\mathbf{u}$-instant corresponding to a world point x is $x + \mathbf{E_u}$. The set of $\mathbf{u}$-instants is $\mathbf{u}$-time.

The time passed between two $\mathbf{u}$-instants (hyperplanes in spacetime) s and t is the time passed between them on an arbitrary straight line directed by $\mathbf{u}$; in formula,

$$t - s := \sqrt{-(x - y) \cdot (x - y)} \tag{6}$$

where x is an arbitrary world point in t and y is an arbitrary world point in s such that $x - y \in \mathbb{R}\mathbf{u}$; alternatively,

$$(x + \mathbf{E_u}) - (y + \mathbf{E_u}) := -\mathbf{u} \cdot (x - y) \tag{7}$$

for arbitrary world points x and y.

The $\mathbf{u}$-space vector between two $\mathbf{u}$-space points (straight lines in spacetime) p and q is the world vector between $\mathbf{u}$-simultaneous world points of the straight lines in question; in formula,

$$p - q := x - y \tag{8}$$

where x is an arbitrary world point in p and y is an arbitrary world point in q such that $x - y \in \mathbf{E_u}$; alternatively,

$$(x + \mathbb{R}\mathbf{u}) - (y + \mathbb{R}\mathbf{u}) := \pi_{\mathbf{u}}(x - y) \tag{9}$$

for arbitrary world points x and y.

3 Schwarzschild spacetime

In this section we describe Schwarzschild spacetime with the notions introduced earlier. This description is *equivalent* to the customary one, but gives a new point of view.

We conceive that Schwarzschild's spacetime describes the gravitational field of a pointlike inertial mass m. Thus, we accept that the point mass has an inertial world line L (a straight line) in a special relativistic spacetime M. Let the four-velocity $\mathbf{u}$ be the direction vector of this straight line.

For a world point x let $r(x)$ denote the $\mathbf{u}$-distance of x from L, i.e. $r(x) := |x - y|$ for $y \in L$, y being $\mathbf{u}$-simultaneous with x. In other words,

$$r(x) = |\pi_{\mathbf{u}}(x - o)| \tag{10}$$

where o is an arbitrary element of L. Furthermore, let us introduce the 'radial normal vector'

$$\mathbf{n}(x) := \frac{\pi_{\mathbf{u}}(x - o)}{|\pi_{\mathbf{u}}(x - o)|} \tag{11}$$

for x not in L.

It is not hard to see that introducing

$$V(x) := -\frac{m}{r(x)} \qquad (x \notin L), \tag{12}$$

we obtain Schwarzschild's metric – i.e. a Lorentz form $g(x)$ depending on spacetime points x – in our terms as follows:

$$g(x) = -(1 + 2V(x))\mathbf{u} \otimes \mathbf{u} + \frac{1}{1 + 2V(x)}\mathbf{n}(x) \otimes \mathbf{n}(x) + \big(\mathbf{1} + \mathbf{u} \otimes \mathbf{u} - \mathbf{n}(x) \otimes \mathbf{n}(x)\big), \tag{13}$$

where $\mathbf{1}$ the Lorentz form of the special relativistic spacetime in question.

Indeed, the above form is transformed into the usual one by taking an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in $\mathbf{E_u}$, and establishing the coordinates $t := -\mathbf{u} \cdot (x - o)$, $r := |\pi_{\mathbf{u}}(x - o)|$, $\cos\theta := \mathbf{e}_3 \cdot \mathbf{n}(x)$, $\tan\varphi := \mathbf{e}_1 \cdot \mathbf{n}(x)/\mathbf{e}_2 \cdot \mathbf{n}(x)$. Then

$$x = o + \mathbf{u}t + (\mathbf{e}_1 \sin\theta\cos\varphi + \mathbf{e}_2 \sin\theta\sin\varphi + \mathbf{e}_3 \cos\theta)r, \tag{14}$$

and the usual form is obtained by composing g by the derivative of x as a function of (t, r, θ, φ). In other words, formally

$$dx = \mathbf{u}dt + (\mathbf{e}_1 \sin\theta\cos\varphi + \mathbf{e}_2 \sin\theta\sin\varphi + \mathbf{e}_3\cos\theta)dr + \tag{15}$$

$$+ (\mathbf{e}_1 \cos\theta\cos\varphi + \mathbf{e}_2\cos\theta\sin\varphi - \mathbf{e}_3\sin\theta)d\theta + \tag{16}$$

$$+ (-\mathbf{e}_1\sin\theta\sin\varphi + \mathbf{e}_2\cos\theta\cos\varphi)d\varphi, \tag{17}$$

and one has to take $g(x)(dx, dx)$ as a function of the coordinates.

Schwarzschild's metric can be rewritten in the form

$$g(x) = \mathbf{1} - 2V(x)\mathbf{u}\otimes\mathbf{u} - \frac{2V(x)}{1+2V(x)}\mathbf{n}(x)\otimes\mathbf{n}(x). \tag{18}$$

In what follows, m and L are called the mass and the (world line of the) center of the Earth.

To describe the actual gravitational field of the Earth, Schwarzschild's metric is often modified so that

$$V(x) := -\frac{m}{r(x)}\big(1 + h(x)\big), \tag{19}$$

where $h(x)$ is much less than 1; in particular, one frequently takes (see [1])

$$V(x) := -\frac{m}{r(x)}\left(1 - k\left(\frac{R}{r(x)}\right)^2\left(3\left(\frac{\mathbf{e}\cdot\mathbf{n}(x)}{R}\right)^2 - 1\right)\right), \tag{20}$$

where k is Earth's quadrupole moment coefficient, R is Earth's equatorial radius and $\mathbf{e} \in \mathbf{E_u}$ is the direction vector of Earth's rotational axis. We emphasize, however, that the rest of our calculations would just as well be valid for other modifications of V, i.e. if one includes higher multipole moment contributions too.

Now we have the following important relations for $\mathbf{x} \in \mathbf{M}$ and $\mathbf{q} \in \mathbf{E_u}$:

$$g(x)(\mathbf{u}, \mathbf{x}) = (1 + 2V(x))\mathbf{u}\cdot\mathbf{x}, \tag{21}$$

in particular

$$g(x)(\mathbf{u}, \mathbf{u}) = -(1 + 2V(x)), \qquad g(x)(\mathbf{u}, \mathbf{q}) = 0, \tag{22}$$

$$g(x)(\mathbf{n}(x), \mathbf{x}) = \frac{\mathbf{n}(x)\cdot\mathbf{x}}{1+2V(x)}, \tag{23}$$

$$g(x)(\mathbf{q}, \mathbf{q}) = |\mathbf{q}|^2 - \frac{2V(x)}{1+2V(x)}(\mathbf{n}(x)\cdot\mathbf{q})^2. \tag{24}$$

We introduce notions analogous to those in the underlying special relativistic spacetime. Since in this framework the objects of the special relativistic spacetime and those of the general relativistic spacetime appear together, for a clear distinction, we shall mark the general objects by a prefix 'Sch': Sch-spacelike, Sch-world line, Sch-four-velocity, etc.

A world vector $\mathbf{x}$ at the world point x is *Sch-spacelike* if $g(x)(\mathbf{x},\mathbf{x}) > 0$, *Sch-timelike* if $g(x)(\mathbf{x},\mathbf{x}) < 0$, and *Sch-lightlike* if $g(x)(\mathbf{x},\mathbf{x}) = 0$, and $\mathbf{x} \neq 0$.

In the sequel we restrict our consideration to world points x for which $1 + 2V(x) > 0$ (i.e. for world points outside the Schwarzschild radius). Then it follows that $\mathbf{u}$ is Sch-timelike at every x in question. A timelike vector $\mathbf{x}$ at x is *Sch-futurelike* if $g(x)(\mathbf{u},\mathbf{x}) < 0$.

A Sch-world line is a curve whose tangents are Sch-timelike. The Sch-proper time of a Sch-world line, parametrized by $p : \mathbb{R} \to M$, is

$$\int \sqrt{-g(p(a))(\dot{p}(a),\dot{p}(a))}da. \tag{25}$$

Then the Sch-world line can be parameterized by its proper time; such a parametrization is called an *Sch-world line function*. The derivative of Sch-world line functions results in Sch-four-velocities. An Sch-four-velocity at the world point x is an Sch-futurelike vector $\mathbf{w}$ in $\mathbf{M}$ for which $g(x)(\mathbf{w},\mathbf{w}) = -1$ holds, i.e.

$$\mathbf{w}\cdot\mathbf{w} - 2V(x)(\mathbf{u}\cdot\mathbf{w})^2 - \frac{2V(x)}{1+2V(x)}(\mathbf{n}(x)\cdot\mathbf{w})^2 = -1. \tag{26}$$

In particular, $\frac{\mathbf{u}}{\sqrt{1+2V(x)}}$ is an Sch-four-velocity at x (outside the Schwarzschild radius).

An Sch-frame is a smooth Sch-four-velocity field defined in a connected domain of spacetime. According to the above formula,

$$\mathbf{U}(x) := \frac{\mathbf{u}}{\sqrt{1+2V(x)}} \qquad (x \in M, 1 + 2V(x) > 0) \tag{27}$$

is an Sch-frame (outside the Schwarzschild radius). The $\mathbf{U}$-space points (maximal integral curves of $\mathbf{U}$) are straight lines directed by $\mathbf{u}$.

The special relativistic inertial frame having four-velocity $\mathbf{u}$ together with its standard synchronization will be called the Earth Centered Reference Frame (ECRF). This is an ideal frame with space points at rest with respect to the center of the Earth, and synchronization corresponding to 'clocks at infinity'. The main task in the GPS is to achieve this synchronization in practice. According to this ideal frame the time elapsed between world points is given by (7).

The space points of the Sch-frame (27) coincide with the ECRF-space points. This Sch-frame, together with the ECRF-synchronization will be called Schwarzschild Earth Centered Reference Frame (Sch-ECRF). Thus, the Sch-ECFR-instants are just the ECRF-instants. Accordingly, the vector between two $\mathbf{U}$-space points is defined by (9) (difference between Sch-simultaneous world points). On the other hand, formula (7) cannot be applied now because the Sch-proper time passed in a $\mathbf{U}$-space point between two synchronization instants depends on the location of the $\mathbf{U}$-space point. Explicitly, the Sch-proper time passed between two instants s and t (hyperplanes directed by $\mathbf{E_u}$) in the $\mathbf{U}$-space point

q (a world line directed by $\mathbf{u}$) is obtained by (25). A parametrization of q is $p(a) := z + a\mathbf{u}$ where z is an arbitrary world point of q. Then $g(z+a\mathbf{u}) = g(z)$ for all a and the Sch-proper time interval in question is the integral of $\sqrt{1+2V(z)}$ from a_s to a_t where $p(a_s) \in s$ i.e. $a_s = -\mathbf{u} \cdot (x - z)$ for an arbitrary world point y of s and similarly, $a_t = -\mathbf{u} \cdot (y - z)$ for an arbitrary world point x of t. Thus, we get that the Sch-proper time interval in question is

$$\sqrt{-g(z)(x-y, x-y)} = \sqrt{1+2V(z)}(-\mathbf{u} \cdot (x-y)). \tag{28}$$

In other words, if $\mathbf{t}$ denotes the ideal special relativistic time interval between two ECRF-instants, then the Sch-time interval between them is

$$\mathbf{t}_{Sch}(z) = \sqrt{1+2V(z)}\mathbf{t}. \tag{29}$$

Thus, we can conceive that the ideal time interval $\mathbf{t}$ is measured by a clock 'infinitely far' from the Earth's center.

4 Uniformly rotating frames

The rotating Earth is modeled in Schwarzschild's spacetime as a uniformly rotating frame around the world line of the gravitating point mass. Such a frame is given similarly to that in special relativity ([1]). Namely, we assume a Lorentz antisymmetric linear map $\mathbf{\Omega} : \mathbf{E_u} \to \mathbf{E_u}$, the angular velocity of the rotation, and the frame is

$$U_{rot}(x) := \alpha(x-o)\big(\mathbf{u} + \mathbf{\Omega}\pi_{\mathbf{u}}(x-o)\big), \tag{30}$$

where o is an arbitrary world point of the straight line L, and

$$\alpha(x-o) := \frac{1}{\sqrt{1+2V(x) - |\mathbf{\Omega}\pi_{\mathbf{u}}(x-o)|^2}} \tag{31}$$

and x is a world point for which the expression under the square root is positive.

We easily get that the solutions of the differential equation $\dot{x} = \mathbf{U}_{rot}(x)$ are of the form

$$\mathbb{R} \to M, \quad \mathbf{t} \mapsto o + \alpha(\mathbf{q})\mathbf{t}\mathbf{u} + e^{\alpha(\mathbf{q})\mathbf{t}\mathbf{\Omega}}\mathbf{q}, \tag{32}$$

where $\mathbf{q}$ is an arbitrary element of $\mathbf{E_u}$ such that $o + \mathbf{q}$ is in the domain of $\mathbf{U}_{rot}$. This solution is an Sch-world line function, $\mathbf{t}$ is its proper time. Recall that such an Sch-world line function represents a space point of the frame $\mathbf{U}_{rot}$ (a space point of the rotating Earth). The Sch-four-velocity of the above world line,

$$\mathbf{t} \mapsto \alpha(\mathbf{q})\mathbf{u} + e^{\alpha(\mathbf{q})\mathbf{t}\mathbf{\Omega}}\mathbf{\Omega}\mathbf{q} \tag{33}$$

is periodic, the proper time period is $\frac{2\pi}{\alpha(\mathbf{q})\omega}$ where $\omega := |\mathbf{\Omega}| := -\frac{1}{2}\mathrm{Tr}\mathbf{\Omega}^2$ is the magnitude of the angular velocity.

Let $\mathbf{q}_0$ be an equatorial radius vector of the Earth. The proper time period of a world line function given by $\mathbf{q}$ (a space point of the Earth) equals the equatorial period of rotation if $\alpha(\mathbf{q}) = \alpha(\mathbf{q}_0)$ which is equivalent to

$$\Phi(\mathbf{q}) := V(o+\mathbf{q}) - \frac{|\mathbf{\Omega q}|^2}{2} = V(o+\mathbf{q}_0) - \frac{|\mathbf{\Omega q}_0|^2}{2} =: \Phi_0. \tag{34}$$

For the original Schwarzschild metric this holds if and only if $|\mathbf{q}| = |\mathbf{q}_0| =: R$.

Φ represents an effective gravitational potential which includes the gravitational potential of the Earth and a centripetal term. The modified Schwarzschild metric corresponding to (20) is defined exactly in such a way that the Earth surface be equipotential with respect to Φ. Thus we obtained that the proper time periods of all points on the Earth surface are equal. This means that the clocks at rest on the Earth surface all beat at the same rate. Exploiting this fact, we find it convenient to measure time intervals between **u**-instants (ECRF-instants) on the surface of the Earth.

Since the **u**-time period of the rotation is (see (7))

$$-\mathbf{u}\cdot\left(\left(o+\frac{2\pi}{\omega}\mathbf{u}+\mathbf{q}\right)-(o+\mathbf{q})\right) = \frac{2\pi}{\omega} \tag{35}$$

(which corresponds to clocks at rest at infinity), we see that the rate between the time periods is $\alpha(\mathbf{q}_0) = \frac{1}{\sqrt{1+2\Phi_0}}$. More explicitly, if $t(\mathbf{t})$ denotes the **u**-time point (ECRF-instant) corresponding to the proper time value $\mathbf{t}$ on the Earth surface, then

$$\frac{dt(\mathbf{t})}{d\mathbf{t}} = \frac{1}{\sqrt{1+2\Phi_0}}. \tag{36}$$

This is the precise formula relating Earth based clocks to ideal clocks at infinity. The customary formula (2) is a very good *approximation*.

Considering the inverse function, $\mathbf{t}(t)$ being the **u**-time point (ECRF-instant) corresponding to the proper time value $\mathbf{t}$ on the surface of the Earth, we have

$$\frac{d\mathbf{t}(t)}{dt} = \sqrt{1+2\Phi_0}. \tag{37}$$

5 A satellite in the Earth's gravitational field

Now let us consider a material point – a satellite – in Schwarzschild's spacetime, described by the Sch-world line function $p:\mathbb{R}\to M$; then $\dot{p}(\mathbf{s})$ is an Sch-four-velocity for all proper time values $\mathbf{s}$. The ECRF-instant $t(\mathbf{s})$ corresponding to the proper time value $\mathbf{s}$ is the hyperplane $p(\mathbf{s})+\mathbf{E_u}$. Then, according to (7),

$$\frac{dt(\mathbf{s})}{d\mathbf{s}} = \lim_{\mathbf{h}\to 0}\frac{(p(\mathbf{s}+\mathbf{h})+\mathbf{E_u})-(p(\mathbf{s})+\mathbf{E_u})}{\mathbf{h}} = -\mathbf{u}\cdot\dot{p}(\mathbf{s}). \tag{38}$$

Of course, we can give the inverse of this function, $\mathbf{s}(t)$ being the proper time value of the satellite corresponding to the ECRF-instant t; then

$$\frac{d\mathbf{s}(t)}{dt} = \frac{1}{-\mathbf{u}\cdot\dot{p}(\mathbf{s}(t))}. \tag{39}$$

According to (9) and (28), the Sch-relative velocity of the satellite with respect to Sch-ECRF is

$$\mathbf{v}(t) = \lim_{\mathbf{h}\to 0} \frac{\pi_{\mathbf{u}}\big(p(\mathbf{s}(t+\mathbf{h})) - p(\mathbf{s}(t))\big)}{\sqrt{1+2V(p(\mathbf{s}(t))}\,\big(-\mathbf{u}\cdot\big(p(\mathbf{s}(t+\mathbf{h})) - p(\mathbf{s}(t))\big)\big)} = \\ = \frac{1}{\sqrt{1+2V(p(\mathbf{s}(t)))}} \left(\frac{\dot{p}(\mathbf{s}(t))}{-\mathbf{u}\cdot\dot{p}(\mathbf{s}(t))} - \mathbf{u} \right). \quad (40)$$

Then we find for the Schwarzschild magnitude of the relative velocity

$$v(t)^2 := g(p(\mathbf{s}(t)))(\mathbf{v}(t), \mathbf{v}(t)) = 1 - \frac{1}{(1+2V(p(\mathbf{s}(t)))(-\mathbf{u}\cdot\dot{p}(\mathbf{s}(t)))^2}. \quad (41)$$

Thus, we infer from (39) that

$$\frac{d\mathbf{s}(t)}{dt} = \sqrt{(1+2V(p(\mathbf{s}(t)))(1-v(t)^2)}. \quad (42)$$

This is the precise formula relating the proper time s of the satellite to the ideal time t of the underlying inertial frame. The customary formula (1) is a very good *approximation*. We remark that instead of the Sch-velocity $\mathbf{v}(t)$ it is customary to use the ECRF-velocity of the satellites; we will discuss this below.

Next, let us measure time intervals by Earth-based clocks, i.e. let us replace t with $t(\mathbf{t})$ and, for the sake of brevity, let us write simply $\mathbf{t}$ instead of $t(\mathbf{t})$; then, in view of (37), we have

$$\frac{d\mathbf{s}(\mathbf{t})}{d\mathbf{t}} = \sqrt{\frac{(1+2V(p(\mathbf{s}(\mathbf{t})))(1-v(\mathbf{t})^2)}{1+2\Phi_0}}. \quad (43)$$

Of course, we can consider the inverse function $\mathbf{t}(\mathbf{s})$, the time measured by clocks on the Earth surface as a function of the proper time of the satellite; then

$$\frac{d\mathbf{t}(\mathbf{s})}{d\mathbf{s}} = \sqrt{\frac{1+2\Phi_0}{(1+2V(p(\mathbf{s}))(1-v(\mathbf{t}(\mathbf{s}))^2)}}. \quad (44)$$

We repeat the meaning of our symbols: $\mathbf{s}$ denotes the Sch-proper time of the satellite (denoted by τ in [1] and by T_{sv} in [2]), and $\mathbf{t}$ denotes the Sch-proper time on the surface of the Earth (denoted by t'' in [1] and by t in [2]).

Lastly, it is customary to use the Newtonian approximation for the velocity of the satellite. This means that in formula (1) the symbol v is meant to be the magnitude of the ECRF-velocity for which a good Newtonian approximation is available. Therefore, in the formulae above we replace the Sch-ECRF-velocity $\mathbf{v}(t)$ by the ECRF-velocity $\mathbf{v}_N(t)$ as follows:

In equation (40) we need to measure time by ideal clocks, i.e. we get

$$\mathbf{v}_N(t) = \sqrt{1+2V(p(\mathbf{s}(t)))}\,\mathbf{v}(t). \quad (45)$$

Combining (24), (41) and (45) we get

$$v(t)^2 = \frac{1}{1+2V(p(\mathbf{s}(t)))}\left(|\mathbf{v}_N|^2 - \frac{2V(p(\mathbf{s}(t)))}{1+2V(p(\mathbf{s}(t)))}(\mathbf{n}(p(\mathbf{s}(t)))\cdot\mathbf{v}_N(t))^2\right), \quad (46)$$

which is to be substituted to (43). As a result we obtain

$$\frac{d\mathbf{s}(\mathbf{t})}{d\mathbf{t}} = \sqrt{\frac{1+2V(p(\mathbf{s}(\mathbf{t}))) - |\mathbf{v}_N(\mathbf{t})|^2 - \frac{2V(p(\mathbf{s}(t)))}{1+2V(p(\mathbf{s}(t)))}(\mathbf{n}(p(\mathbf{s}(t)))\cdot\mathbf{v}_N(\mathbf{t}))^2}{1+2\Phi_0}}. \quad (47)$$

Using a series expansion, assuming that all the terms on the right-hand side of (47) are much less than 1, we get back formula (1)

$$\frac{d\mathbf{s}(\mathbf{t})}{d\mathbf{t}} \approx 1 + V(p(\mathbf{s}(\mathbf{t}))) - \Phi_0 - \frac{|\mathbf{v}_N(\mathbf{t})|^2}{2}. \quad (48)$$

Similarly,

$$\frac{d\mathbf{t}(\mathbf{s})}{d\mathbf{s}} \approx 1 - V(p(\mathbf{s})) + \Phi_0 + \frac{|\mathbf{v}_N(\mathbf{s})|^2}{2}. \quad (49)$$

These expressions coincide with formulae (27) and (28) in [1] and (6) in [2]. Assuming that the satellite moves along a Newtonian orbit with semimajor axis a – which is a good approximation for orbits far enough from the Earth surface –, one can evaluate $\mathbf{v}_N(\mathbf{t})$ as

$$|\mathbf{v}_N(\mathbf{t})|^2 = -2V(p(\mathbf{s}(\mathbf{t}))) - \frac{m}{2a}, \quad (50)$$

and then proceed with the calculations as in [1] to achieve the desired synchronization in the underlying ideal inertial frame.

References

[1] N. Ashby: Relativity in the global positioning system, in *100 years of relativity*, 257–289, World Sci. Publ., Hackensack, NJ, (2005). (also at Livingreviews in Relativity, http://relativity.livingreviews.org/Articles/lrr-2003-1/)

[2] J. Kouba: Improved relativistic transformations in GPS, *GPS Solutions* (2004), **8**, 170180.

[3] T. Matolcsi: *Spacetime without Reference Frames*, Akadémiai Kiadó Budapest, 1993

[4] T. Matolcsi: Spacetime without reference frames: an application to synchronizations on a rotating disk, *Found. Phys.* (1998), **28**, no. 11, 1685–1701.

[5] T. Matolcsi and A. Goher: Spacetime without reference frames: An application to the velocity addition paradox, *Stud. Hist. Phil. Mod. Phys.* (2001), **32**, 83-99.

[6] R. K. Sachs, H. Wu: *General Relativity for Mathematicians*, Springer, New York, (1977).