Group Representations in Mechanics

by

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I. Introduction

Group representations play an important role in modern physics. It seems, however, that the usual definition of group representations hinders this theory from being a more general, all-embracing branch of mathematics and from being a more powerful tool in physics.

Elementary particles are classified in quantum mechanics by representations of the space-time symmetry groups. Recently a classification of elementary particles appeared also in classical mechanics which seems similar to the quantal one ([1], [2], [3], [4]). However, the analogy is not perfectly clear and "it would be desirable to formulate a correspondence principle to relate the classifications" ([4]). To establish the desired correspondence, one must first recognize that some sort of group representations should be used also in classical mechanics.

I defined a general notion of group representations ([6]) which includes the usual representations, that is, the linear and unitary representations, ray representations and actions of transformation groups as well. By the aid of this general notion of group representations we can state a perfect parallelism between classical and quantal elementary particles.

After recalling the general notion of group representations, here I shall examine the special feature of representations needed in quantum mechanics and in classical mechanics. The investigations are based on the formulation of mechanics using the algebraic structure of events, the so called logic of a physical system ([7], [8]).

II. Representations of groups

A category consists of objects and morphisms satisfying some axioms. We are concerned, in our applications, with so called concrete categories, the objects of which are sets with a certain structure and the morphisms are maps related to the structure e.g. preserving it . An isomorphism is a one-to-one and onto morphism whose inverse is also a morphism; an automorphism of an object is an isomorphism of the object onto itself. The automorphisms of an object X form a group under the composition of maps; we denote this group by $\mathrm{Aut}(X)$. One generally defines the notion of subobjects in concrete categories in a natural way and it has an intrinsic meaning that a subobject is invariant for an automorphism; hence, we do not give here the precise and convenient definition of subobjects and invariant subobjects in category theory. These definitions are given for an arbitrary (not necessary concrete) category in ([6]). For further details on categories we refer to [9].

<u>Definition 1.</u> Let G be a group and $\mathcal{C}$ a concrete category. A <u>representation</u> A of G on the object X of $\mathcal{C}$ is a group homomorphism $G \to \mathrm{Aut}(X)$, $g \mapsto A_g$.

The representation is <u>faithful</u> if it is one-to-one.

The representation is <u>irreducible</u> if there is no subobject of X invariant for all A_g and not invariant for all automorphisms of X.

The representation is <u>weakly irreducible</u> if there is no element of X invariant for all A_g and not invariant for all automorphisms of X.

Two representations of the same group G, A on X and B on Y, X and Y being objects of $\mathcal{C}$, are <u>equivalent</u> if there is an isomorphism $i : X \to Y$ such that $i \circ A_g = B_g \circ i$ for all $g \in G$.

If G is a topological group and the structure of X contains a topology too, we require that the map $G \times X \to X$, $(g,x) \mapsto A_g(x)$ be continuous. Such representations are called <u>continuous.</u>

If G is a Lie group and X has a differentiable structure, we require that the map $G \times X \to X$, $(g,x) \mapsto A_g(x)$ be differentiable. Such a representation is called <u>differentiable.</u>

As an example, take the category whose objects are Hilbert spaces and morphisms are linear contractions. Then continuous representations on such objects are exactly the strongly continuous unitary representations.

As a second example, let us consider the category of topological spaces and continuous maps. Then we obtain topological transformation groups in the customary sense ([10], p.110). Effective actions are nothing else than faithful representations. Transitive actions are irreducible representations; the converse is only true, in general, on those spaces whose homeomorphisms form a transitive transformation group; such spaces are called <u>homogeneous.</u>

In the theory of topological transformation groups one examines a given single action. In representation theory one is looking for different that is inequivalent representations. The two theories meet in the theorem that every continuous irreducible representation A of a locally compact second countable group G on a locally compact homogeneous Hausdorff space X is equivalent to the representation L on G/H where H is a closed subgroup of G and L is the canonical or standard action left translation of G ([10], p.111). Remind that equivalence means here that there is a homeomorphism $f: G/H \to X$ such that $f \circ L_g = A_g \circ f$ for all $g \in G$. Quite the same is true for differentiable representations of a Lie group on a differentiable manifold ([10], p.114). However, if X has a further additional structure then representations and equivalences are given by maps related to this structure too and there can be much more inequivalent representations of G on X.

III. Representations in quantum mechanics

The logic of a physical system in quantum mechanics is the orthocomplemented σ-lattice $\mathcal{P}(\mathcal{H})$ of projections on a separable complex Hilbert space $\mathcal{H}$, called the structure space (of the physical system). The lattice structure, that is, the ordering of projections is defined by the ordering of closed linear subspaces. Now we introduce a category in which objects are such lattices and morphisms are orthocomplementation preserving lattice σ-homomorphisms.

We assume from now on that the dimension of $\mathcal{H}$ is greater than two. By a theorem of Wigner ([7], vol I, p.169), every isomorphism $S : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}')$ is induced by a unitary or antiunitary map $U : \mathcal{H} \to \mathcal{H}'$, determined uniquely up to a unit factor, such that $S(E) = U E U^{-1}$ for all $E \in \mathcal{P}(\mathcal{H})$. Consequently, as it is known, a representation of a group on $\mathcal{P}(\mathcal{H})$ gives rise to a unitary-antiunitary <u>ray representation</u> on $\mathcal{H}$ ([11]). Conversely, a ray representation on $\mathcal{H}$ determines a representation on $\mathcal{P}(\mathcal{H})$. Representations on projection lattices will be called <u>projective representations</u>. Let us see more closely the connection between ray representations and projective representations.

Let A be a projective representation of a group G on $\mathcal{P}(\mathcal{H})$ and (U,τ) its realisation as a ray representation on $\mathcal{H}$. It means that we have a map $g \mapsto U_g$, U_g being a unitary or antiunitary operator on $\mathcal{H}$, and we have a function $\tau: G \times G \to \mathbb{T}$ ($\mathbb{T}$ is the complex unit circle) such that

$$A_g(E) = U_g E U_g^{-1}, \tag{1}$$

$$U_g U_h = \tau(g,h)\, U_{gh}, \tag{2}$$

$$\tau(e,e) = 1 \quad (e \text{ is the unit element of } G) \tag{3}$$

$$\tau(g,h)\,\tau(gh,f) = \tau(h,f)\,\tau(g,hf) \tag{4}$$

for all $E \in \mathcal{P}(\mathcal{H})$ and for all $g, h, f \in G$. In the sequel, a function $\tau: G \times G \to \mathbb{T}$ satisfying (3) and (4) will be called a <u>unitary cocycle</u> of G. Two unitary cocycles τ and τ' will be said <u>cohomologous</u> if there is a function $\rho: G \to \mathbb{T}$ such that

$$\tau'(g,h) = \tau(g,h)\,\frac{\rho(g)\,\rho(h)}{\rho(gh)} \quad \text{for all } g,h \in G.$$

τ and τ' are <u>weakly cohomologous</u> if either τ and τ' or $\bar{\tau}$ (the complex conjugate of the function τ) and τ' are cohomologous.

Now, first we mention the known fact that if G is a connected topological group then U_g is unitary for all $g \in G$ ([11]). Then we state

<u>Proposition 1.</u> <u>The projective representation</u> A <u>of a connected topological group</u> G <u>is weakly irreducible if and only if</u> (U,τ) <u>is irreducible in the usual sense</u>.

Proof. We have $U_{g^{-1}} = \tau(g,g^{-1})\, U_g^{-1}$ from (2) and thus we can repeat the familiar arguments for unitary representations to get that a closed linear subspace is invariant for all U_g if and only if the corresponding projection commutes with all U_g. Consequently, we conclude from (1) that a projection is invariant for all A_g if and only if the corresponding subspace is invariant for all U_g. Recall that A is weakly irreducible if the only elements of the projection lattice, which are invariant for all A_g, are the identity and the zero.

Recall now that it follows from our general definition concerning group representations that two projective representations of the same group, A on $\mathcal{P}(\mathcal{H})$ and A′ on $\mathcal{P}(\mathcal{H}')$, are equivalent if and only if there is an isomorphism $S : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}')$ such that

$$S \circ A_g = A'_g \circ S \qquad \text{for all} \quad g \in G . \tag{5}$$

Proposition 2. Let A and A′ be two projective representations of the connected group G and let (U,τ) and (U',τ') be their realizations as ray representations. A and A′ are equivalent if and only if there is a unitary or antiunitary map V and a function $\rho : G \to \mathbf{T}$ such that

$$VU_g = \rho(g)\, U'_g V \qquad \text{for all} \quad g \in G . \tag{6}$$

Consequently, τ and τ' or $\bar{\tau}$ and τ', depending on whether V is unitary or antiunitary, are cohomologous by ϱ.

Proof. (5) and (6) are simple consequences of each other. Furthermore, multiplying the equality (2) from the left and from the right by V and by V^{-1} respectively and using (6) we get that, according to the nature of V, τ and τ' or $\bar{\tau}$ and τ' are cohomologous by ϱ.

As a consequence of Proposition 2, if (U,τ) and (U',τ') are ray representations such that τ and τ' are not weakly cohomologous, the projective representations determined by the ray representations are not equivalent. In other words, the weak cohomology classes of unitary cocycles of a connected group can be used for certain labelling of the equivalence classes of projective representations of the group.

Now we want to consider projective representations continuous in the sense of our definition. For this purpose recall that there is a well working notion of continuity of ray representations. A ray representation (U,τ) of a topological group G is continuous in the sense of Bargmann if there exists a neighbourhood of the identity of G so that U is strongly continuous and τ is continuous in this neighbourhood. The continuity of ray representations can be related to projective representations as follows. A distance d is defined on the set of one-dimensional projections (on the rays) by

$$d(E,F) = \inf \{\|x - y\| ,\ x \in E\mathcal{H},\ \ y \in F\mathcal{H},\ \ \|x\| = \|y\| = 1\}.$$

Let A be a projective representation induced by a continuous ray representation of G . Then the map $g \mapsto A_g(E)$ is continuous in the distance d for every one-dimensional projection E . Conversely, if A is a projective representation for which the map $g \mapsto A_g(E)$ is continuous for all one-dimensional E , then one can find a continuous ray representation inducing A .

We would like to define a topology on a projection lattice $\mathcal{P}(\mathcal{H})$ with the following conditions:

1. The topology is compatible with the structure of $\mathcal{P}(\mathcal{H})$ that is the orthocomplementation and intersection are continuous operations.

2. The automorphisms of $\mathcal{P}(\mathcal{H})$ are continuous.

3. The continuous projective representations can be given by continuous ray representations and continuous ray representations induce continuous projective representations.

Proposition 3. There is no Hausdorff topology on the projection lattice $\mathcal{P}(\mathcal{H})$ satisfying conditions 1 and 3 .

Proof. Let e_n $(n=1,2,\ldots,\dim\mathcal{H})$ be an orthonormal basis of $\mathcal{H}$. The formula

$$U_t e_n = \begin{cases} \exp(it)\ e_n & \text{if } n = 1 \\ e_n & \text{if } n \neq 1 \end{cases} \qquad (t \in \mathbb{R})$$

determines a ray representation of the additive group of real numbers. The ray representation is continuous in Bargmann's sense. Let $\mathcal{M}$ be the subspace generated by the two vectors $e_1 + e_2$ and e_3, and let $\mathcal{N}$ be the subspace generated by e_3. If there is a topology T with the properties listed in conditions 1 and 3, then

$$\mathcal{N} = T - \lim_{\substack{t \to 0 \\ t \neq 0}} \left(U_t(\mathcal{M}) \cap \mathcal{M}\right) = \left(T - \lim_{\substack{t \to 0 \\ t \neq 0}} U_t(\mathcal{M})\right) \cap \mathcal{M} = \mathcal{M},$$

which is a contradiction.

Therefore we are forced to change condition 1 into the following weaker one:

1'. The orthocomplementation and the intersection are continuous operations in every distributive sublattice of $\mathcal{P}(\mathcal{H})$.

We say in this case that the topology is <u>partially compatible</u> with the lattice structure.

<u>Proposition 4. The topology on $\mathcal{P}(\mathcal{H})$, induced by the strong operator topology, satisfies conditions</u> 1', 2 <u>and</u> 3.

<u>Proof.</u> Condition 2 is trivially satisfied. It is also simple to see that condition 1' is fulfilled. The orthocomplementation $E \mapsto I - E \quad (E \in \mathcal{P}(\mathcal{H}))$ is evidently continuous. A sublattice of $\mathcal{P}(\mathcal{H})$ is distributive if and only if it is commutative. Furthermore, if E and F are commuting projections then their intersection $E \wedge F$ is their product EF. The multiplication in a set of ope-

rators bounded in norm is jointly continuous in the two variables with respect to the strong topology, hence the intersection is continuous operation in a distributive sublattice.

Regarding condition 3, standard arguments show that the norm topology, the strong topology and the topology of the distance d on the set of one dimensional projections are equivalent. As a consequence, the continuity of a projective representation implies the continuity of ray representations.

Conversely, let the ray representation (U,τ) be continuous. Then we get the continuity of the projective representation A induced by (U,τ) from the equality

$$A_g(F) - E = U_g(F - E)U_g^{-1} + U_g E U_g^{-1} - E$$

and from the fact that the multiplication of operators in a set bounded in norm is continuous in the two variables.

Remark 1. One can see by a simple example that the topology on $\mathcal{P}(\mathcal{H})$, given by the distance

$$d(E,F) = \\ = \inf\left\{\|x - y\|,\ x \in (E - E\wedge F)\mathcal{H},\ y \in (F - E\wedge F)\mathcal{H}, \|x\| = \|y\| = 1\right\}$$

satisfies conditions 1' and 2 but it does not satisfy condition 3 .

From now on we equip the projection lattices with the strong operator topology.

Let us now consider continuous projective representations of connected Lie groups. To every unitary cocycle continuous in a neighbourhood of the identity of the Lie group, one can assigne a closed bilinear antisymmetric real valued function on the Lie algebra of the group. Closedness means that if β is a bilinear antisymmetric function on a Lie algebra $\mathcal{G}$ then $\beta([a,b],c) + \beta([b,c],a) + \beta([c,a],b) = 0$ for all $a,b,c \in \mathcal{G}$. A closed bilinear antisymmetric function on a Lie algebra will be called a commutator cocycle. Two commutator cocycles β and β' are said cohomologous if there is a linear function α such that $\beta' = \beta + \alpha \circ [\,,\,]$ where $[\,,\,]$ denotes the commutator on the Lie algebra. β and β' are weakly cohomologous if either β and β' or $-\beta$ and β' are cohomologous.

Collecting the results of [11] we get

Corollary to Proposition 2. Let A and A' be continuous projective representations of the connected Lie group G. Assume (U,τ) and (U',τ') are realizations of A and A' as continuous ray representations. Let β and β' be the commutator cocycles on the Lie algebra $\mathcal{G}$ of G, corresponding to τ and to τ' respectively. Let H_a and H'_a denote the self-adjoint infinitesimal generators corresponding to $a \in \mathcal{G}$ in the ray representation (U,τ) and (U',τ') respectively. A and A' are equivalent if and only if there is a unitary or antiunitary map V and a linear function $\alpha : \mathcal{G} \to \mathbb{R}$, $a \mapsto \alpha_a$ such that if V is *unitary then* β *and* β' *are cohomologous by* α *and*

$$H'_a = VH_aV^{-1} + \alpha_a \qquad \text{for all} \quad a \in \mathcal{G}.$$

If V *is antiunitary then* $-\beta$ *and* β' are *cohomologous by* α *and*

$$-H'_a = VH_aV^{-1} + \alpha_a \qquad \text{for all} \quad a \in \mathcal{G}.$$

IV. Representations in classical mechanics

Let us turn to classical mechanics. The question is what objects are logics of physical systems in this theory. One knows a partial answer: the logic of a physical system is the σ-algebra $\mathcal{B}(M)$ of Borel subsets of a differentiable manifold M called the phase space (of the physical system) ([7]). Morphisms between such objects are defined as differentiable Boolean σ-homomorphisms. A Boolean σ-homomorphism $u : \mathcal{B}(M) \longrightarrow \mathcal{B}(M')$ is called differentiable if it can be given by a (necesrarily uniquely determined) differentiable map $f_u : M' \longrightarrow M$ such that $u(E) = f_u^{-1}(E)$ for all $E \in \mathcal{B}(M)$. This choice seems, however, too large, because of the [redacted] mentioned fact that all good representations on such objects are equivalent to a few ones.

We assume that on the phase spaces there is given some structure inducing a Borel measure in a canonical way. Then we can require that morphisms, besides being differentiable Boolean σ-homomorphisms, be related to this structure or only to the measure induced, but in any way in such a manner that isomorphisms, which are given by diffeomorphisms of phase spaces, preserve the measure. The most simple examples: the phase spaces can be Riemannian manifolds, contact manifolds or symplectic manifolds.

Here only one possibility will be investigated.

A closed two-form with constant rank on a differentiabl manifold will be called a <u>presymplectic</u> form. If the ra of a presymplectic form is maximal then the form is cal <u>symplectic</u>. A <u>symplectic manifold</u> is a pair (M,ω) wh M is a differentiable manifold and ω is a symplecti form on M.

<u>Definition 2.</u> Let ω and ω' be two presymplect forms with rank r and r' on the same manifold. We w $\omega < \omega'$ if $r < r'$ and if there is a presymplecti form ω_0 with rank $r_0 = r' - r$ such that $\omega + \omega_0$ We write $|\omega| < |\omega'|$ if either $\omega < \omega'$ or $-\omega$

Now we introduce a category whose objects are σ-a ras of Borel subsets in symplectic manifolds and morphi are given as follows. Let (M,ω) and (M',ω') be t symplectic manifolds. $\mathcal{B}(M,\omega)$ and $\mathcal{B}(M',\omega')$ will d the corresponding Borel σ-algebras. A morphism $\mathcal{B}(M,\omega) \rightarrow \mathcal{B}(M',\omega')$ is a differentiable map $f : M' -$ such that $|f^*\omega| \leq |\omega'|$. The morphisms will be called <u>-symplectic</u> maps. Note that if $f : M' \rightarrow M$ represents isomorphism then f is a diffeomorphism and $|f^*\omega| = |$ that is either $f^*\omega = \omega'$ (f is <u>symplectic</u>) or $f^*\omega$ $= -\omega'$ (f is <u>antisymplectic</u>).

We define that logics of physical systems in class mechanics are objects of this category.

Let us consider group representations on the objects of the above category. Let (M,ω) be a symplectic manifold; a representation A of a group G on $\mathcal{B}(M,\omega)$ is a group homomorphism $g \mapsto A_g$ from G into the group of semi-symplectic diffeomorphisms of M . That is why we shall call such representations semi-symplectic.

In other words, a semi-symplectic representation is nothing else than a special "action" of G on M , an "action" which preserves in some sense the additional structure of M . The "action" will be identified with the representation and will be denoted by the same symbol A . For topological resp. Lie groups we consider exclusively continuous resp. differentiable semi-symplectic representations, which means that the corresponding "action" is a topological resp. Lie transformation group.

We shall say for simplicity that a semi-symplectic representation is given on a symplectic manifold. One must not forget, however, about the Borel structure because the meaning of irreducibility is related to it.

If the representation A on $\mathcal{B}(M,\omega)$ is not weakly irreducible then there exists a non-trivial Borel subset invariant for all A_g . The converse is true only if there isno non-trivial Borel subset invariant for all automorphisms of $\mathcal{B}(M,\omega)$. It is the case if $\mathcal{B}(M,\omega)$ is homogeneous in the sense that $\mathrm{Aut}(\mathcal{B}(M,\omega))$ constitutes a transitive transformation group of M . Every orbit of a locally compact second countable topological transformation group is a Borel set, hence we have

<u>Proposition 5. If the "action" on M corresponding to a representation on $\mathfrak{B}(M,\omega)$ is transitive then the representation is weakly irreducible. If a continuous representation of a locally compact second countable group on a homogeneous $\mathfrak{B}(M,\omega)$ is weakly irreducible then the corresponding "action" is transitive.</u>

We shall call a semi-symplectic representation <u>transitive</u> if the corresponding "action" is transitive. We used the term action only for pointing out how we arrived at this familiar notion; quotation marks were applied to underline that, as it was mentioned earlier, an action is a special sort of representations. Later we retain the term action only for standard actions (left translation) on coset spaces.

We shall now restrict our investigations to differentiable semi-symplectic representations of Lie groups. If G is connected we have the following simple but important facts.
First: A_g is symplectic for all $g \in G$.
Secondly: if the representation is transitive then also M is connected.

Let us denote the Lie algebra of G by $\mathcal{G}$ and that of the infinitesimal generators of the representation by $\mathcal{L}_A$. Recall that the infinitesimal generator of a one-parameter group $g(\cdot)$ is the vector field defined by $x \longmapsto (d/dt)_{t=0} A_{g(t)}(x) \quad (x \in M)$. There is a surjective algebra anti-

Let $\mathcal{D}(M)$ denote the algebra of differentiable functions on M, and assume the usual notation i_X for the interior product of differential forms by a vector field X.

Proposition 6. _Consider a differentiable semi-symplectic representation of a Lie group on a simply connected symplectic manifold_ (M,ω). _Then there exists a generating function_ $H_a \in \mathcal{D}(M)$ _for all_ Z_a _such that_ $i_{Z_a}\omega = dH_a$. _Evidently,_ H_a _is determined only up to an additive constan_

Proof. Z_a is the infinitesimal generator of a one--parameter group of symplectic diffeomorphisms, hence $L_{Z_a}\omega = 0$ ([12], Chap. V.2). Now, $L_{Z_a} = i_{Z_a} \circ d + d \circ i_{Z_a}$ and thus $i_{Z_a}\omega$ is closed. We conclude from the simply connectedness of M and from Poincaré's lemma ([12], Chap.IV that $i_{Z_a}\omega$ is exact.

Remark 2. The simply connectedness of M is not necessary at all for the existence of generating functions. We suppose in the sequel that M is connected and there is a generating function for every infinitesimal generator.

We start our further investigations recalling that the symplectic form ω induces an isomorphism j_ω from the module of vector fields onto the module of one-forms on M,

$X \mapsto j_\omega(X) = i_X\omega$ ([12], Chap. VII.1).

The Lie structure is transferred by this isomorphism from the vector fields to the one-forms:

$$\left[j_\omega(X), j_\omega(Y)\right]_1 := j_\omega[X,Y]. \tag{7}$$

Furthermore one defines a Lie structure on $\mathfrak{F}(M)$ by the Poisson bracket:

$$\{F,H\} = -\omega\left(j_\omega^{-1}(dF),\ j_\omega^{-1}(dH)\right) \qquad \left(F,\ H \in \mathfrak{F}(M)\right). \tag{8}$$

One has the identity

$$d\{F,H\} = [dF,dH]_1 . \tag{9}$$

<u>Proposition 7.</u> <u>The map</u> $H : \mathcal{G} \to \mathfrak{F}(M)$, $a \mapsto H_a$ (cf. Proposition 6 and Remark 2) <u>can be chosen to be linear. Furthermore, by such a choice, there is a commutator cocycle</u> β <u>on</u> $\mathcal{G}$ <u>so that</u>

$$\{H_a, H_b\} = -H_{[a,b]} + \beta_{a,b} \quad \text{for all } a,b \in \mathcal{G} . \tag{10}$$

(We find convenient to write $\beta_{a,b}$ instead of $\beta(a,b)$).

<u>Proof.</u> The kernel in $\mathfrak{F}(M)$ of the $\mathbb{R}$-linear map d is $\mathbb{R}$, viewed as the constant functions on M. Let $\phi : \mathfrak{F}(M) \to \mathfrak{F}(M)/\mathbb{R}$ be the natural surjection. $\mathbb{R}$ is in the center of $\mathfrak{F}(M)$ with respect to the Poisson bracket, thus one can define a Lie structure on $\mathfrak{F}(M)/\mathbb{R}$ by

$$\{\phi(F), \phi(H)\}_/ := \phi\{F,H\} . \tag{11}$$

Now, d can be factored, $d = D \circ \phi$. The image of d is the submodule consisting of exact one-forms. Then D^{-1} is an $\mathbb{R}$-linear isomorphism from the module of exact one-forms onto $\mathcal{F}(M)/\mathbb{R}$. Let us transfer the Lie structure defined by (7) from the one-forms to $\mathcal{F}(M)/\mathbb{R}$.
Then we obtain by (8) and (11) :

$$\left[D^{-1}(dF), D^{-1}(dH)\right]_{/} := D^{-1}\left[dF, dH\right]_{1} = \\ = D^{-1} d\left\{F, H\right\} = \left\{\phi(F), \phi(H)\right\}_{/} \quad . \qquad (12)$$

That is, the Lie structures on $\mathcal{F}(M)/\mathbb{R}$, induced by the Poisson bracket and by the Lie structure of the vector fields, coincide.

$(\mathcal{F}(M)/\mathbb{R}) \times \mathbb{R}$ can be linearly imbedded in - and onto - $\mathcal{F}(M)$. It can be done in such a manner that the imbedding followed by the natural surjection ϕ gives the projection from $(\mathcal{F}(M)/\mathbb{R}) \times \mathbb{R}$ onto $\mathcal{F}(M)/\mathbb{R}$.

Now, by Proposition 6 , the elements of $j_{\omega}(\mathcal{L}_A)$ are exact one-forms, that is we can consider $\mathcal{F}_A := D^{-1} j_{\omega}(\mathcal{L}_A)$. It is a subalgebra of $\mathcal{F}(M)/\mathbb{R}$. Thus there is a linear imbedding $i : \mathcal{F}_A \times \mathbb{R} \to \mathcal{F}(M)$ so that $\phi(i(\bar{X}, \lambda)) = \bar{X}$ for all $\bar{X} \in \mathcal{F}_A$ and for all $\lambda \in \mathbb{R}$. Furthermore, $\phi(F) \in \mathcal{F}_A$ if and only if $F \in i(\mathcal{F}_A \times \mathbb{R})$. Thus we conclude from (11) that the Poisson bracket of two elements of $i(\mathcal{F}_A \times \mathbb{R})$ is again in $i(\mathcal{F}_A \times \mathbb{R})$. In other words, $i(\mathcal{F}_A \times \mathbb{R})$ with the Poisson bracket is a central extension of $\mathfrak{D}_A$ by $\mathbb{R}$.

Equivalence classes of such extensions are in one-to-one correspondence with cohomology classes of commutator cocycles of $\mathfrak{D}_A$ ([14], Chap.3.8). Let $\bar{\beta}$ be a commutator cocycle of $\mathfrak{D}_A$; the extension defined by $\bar{\beta}$ is given by the commutator

$$\left\{i(\bar{X},\lambda)\ ,\ i(\bar{Y},\mu)\right\} = i\left(\left[\bar{X},\bar{Y}\right]_{\nearrow},\bar{\beta}(\bar{X},\bar{Y})\right) \quad \left(\bar{X},\bar{Y}\in\mathfrak{D}_A;\lambda,\mu\in\mathbb{R}\right). \quad (13)$$

Put now $\bar{Z}_a := D^{-1}\left(j_\omega(Z_a)\right)$ and define the linear map $H : \mathcal{G} \longrightarrow \mathfrak{D}(M)$ by $H_a = i\left(\bar{Z}_a,0\right)$. Then $\phi(H_a) = \bar{Z}_a$ and by $d = D\circ\phi$ we get $dH_a = j_\omega(Z_a)$, hence H_a is a generating function for Z_a. Furthermore, it is obvious from (12) that the mapping $(a,b)\mapsto\beta_{a,b} := \bar{\beta}(\bar{Z}_a,\bar{Z}_b)$ defines a commutator cocycle β on $\mathcal{G}$. Then equality (10) follows from (13) and from the fact that Z is a Lie algebra antihomomorphism.

Later we need the following facts. Let X be a vector field on a differentiable manifold M . Consider a manifold M' and a diffeomorphism $f : M \longrightarrow M'$. Then $f_*(X) := df\circ X\circ f^{-1}$ is a vector field on M' . If φ' is a p-form on M' then

$$\left(f^*\varphi'\right)\left(X_1,X_2,\ldots,X_p\right) = \varphi'\left(f_*(X_1),f_*(X_2),\ldots,f_*(X_p)\right)\circ f \qquad (14)$$

for all vector fields $X_1,X_2,\ldots,X_p$ on M ([10], p.24). Making use of this equality one easily proves the following

Lemma. Let (M,ω) *and* (M',ω') *be symplectic manifolds,* $f : M \to M'$ *a symplectic (resp. antisymplectic) diffeomorphism. Suppose* X *and* X' *are vector fields on* M *and on* M' *respectively and put* $\alpha := j_\omega(X)$, $\alpha' := j_{\omega'}(X')$. *Then* $X' = f_*(X)$ *if and only if* $\alpha = f^*\alpha'$ (*resp.* $-\alpha = f^*\alpha'$).

We can now state

Proposition 8. Let A *and* A' *be differentiable semi-symplectic representations of the connected Lie group* G *on the connected symplectic manifolds* (M,ω) *and* (M',ω') *respectively. Let* $\mathcal{G}$ *denote the Lie algebra of* G. *Assume that* H, β *and* H', β' *corresponding to* A *and* A' *respectively, are the same as in Proposition 7.* A *and* A' *are equivalent if and only if there is a symplectic or antisymplectic diffeomorphism* $f : M \to M'$ *and a linear function* $\alpha : \mathcal{G} \to \mathbb{R}$, $a \mapsto \alpha_a$, *such that if* f *is symplectic then* β *and* β' *are cohomologous by* α *and*

$$H'_a = H_a \circ f^{-1} + \alpha_a \qquad \text{for all} \quad a \in \mathcal{G}. \tag{15}$$

If f *is antisymplectic then* $-\beta$ *and* β' *are cohomologous by* α *and*

$$-H_a = H_a \circ f^{-1} + \alpha_a \qquad \text{for all} \quad a \in \mathcal{G}.$$

Proof. Suppose first that A and A' are equivalent. Then there exists a symplectic or antisymplectic diffeomorphism $f : M \to M'$ such that $f \circ A_g = A'_g \circ f$ for all $g \in G$. Let f be symplectic; the other case can be treated similarly. The equivalence of the representations implies that the corresponding infinitesimal generators satisfy the equality $df \circ Z_a = Z'_a \circ f$, that is,

$$Z'_a = f_*(Z_a) \qquad \text{for all} \quad a \in \mathcal{G}. \tag{16}$$

Consequently, $dH_a = f^*(dH'_a)$. Now, f^* and d commute ([12], Chap.IV.2) thus there exist real numbers α_a such that $H_a = f^*(H'_a) - \alpha_a$ for all a . It can be easily seen from the linearity of H and H' and from the identity $f^*(H'_a) = H'_a \circ f$, the correspondence $a \mapsto \alpha_a$ defines a linear map on $\mathcal{G}$, whence we obtain equality (15). Furthermore, because of the identity $\{H_a, H_b\} = -\omega(Z_a, Z_b)$, we get from (10), (15), (16) that

$$\beta_{a,b} = -\omega(Z_a, Z_b) + H_{[a,b]}$$

$$\beta'_{a,b} = -\omega'(f_*(Z_a), f_*(Z_b)) + H_{[a,b]} \circ f^{-1} + \alpha_{[a,b]}$$

for all $a,b \in \mathcal{G}$. Using (14) and the fact that $\beta_{a,b} \circ f^{-1} = \beta_{a,b}$, we get the desired result concerning the cohomology of β and β' .

Suppose now that we have an f and an α with the properties listed in the proposition. Then $dH_a = f^*(dH'_a)$ or $-dH_a = f^*(dH'_a)$ which implies (16).

Let s_a be the one-parameter subgroup of G generated by $a \in \mathcal{G}$. Then we have $f \circ A_{s_a} = A'_{s_a} \circ f$. Since every element of G can be written as a product of elements taken from one-parameter subgroups, we obtain the equality $f \circ A_g = A'_g \circ f$ for all $g \in G$ which proves our assertion.

V. Discussion

There is a well known analogon of Proposition 7 for ray representations which, together with Propositon 8 and Corollary of Proposition 2, shows how a complete parallelism can be established between group representations in quantum and in classical mechanics. Unitary resp. antiunitary maps in quantum mechanics correspond to symplectic resp. antisymplectic diffeomorphisms in classical mechanics. As an essential consequence of our definition of representations and their equivalence, weak cohomology classes of commutator cocycles appear in labelling equivalence classes of representations of connected Lie groups, both in quantum and in classical mechanics.

Souriau's treatment of dynamic systems, translated to our language, is similar to our treatment of representations in classical mechanics, but he considers only symplectic representations and follows a somewhat different way. Our line was worked out to make the parallelism between quantum and classical mechanics apparent.

On the other hand, one does not see the connection between Arens' method and ours at first sight. Arens considers canonical left actions of groups on their coset spaces modulo closed subgroups and he looks for symplectic structures which make the actions symplectic. The following method would seem corresponding in quantum mechanics: one constructs some canonical linear representations of groups and then one looks for inner products which make the representations unitary. However, Arens' method can be arrived at from ours by the reasoning below.

Suppose we have to find transitive differentiable semi-symplectic representations of a connected Lie group G . Let A be such a representation of G on the symplectic manifold (M,ω). Then, as we know, there is a closed subgroup H of G and a diffeomorphism $f : G/H \to M$ such that $f \circ L_g = A_g \circ f$ for all $g \in G$, where L is the canonical action of G on G/H . Then $(G/H, f^*\omega)$ is a symplectic manifold and $L =: (L, f^*\omega)$ will be a semi-symplectic representation of G . One easily proves the following.

Proposition 6. Let A *and* A' *be two transitive differentiable semi-symplectic representations of the connected Lie group* G *on* (M,ω) *and on* (M', ω') *respectively, and let* f *and* f' *be the diffeomorphisms sending* A *and* A' *to canonical actions on the coset spaces* G/H *and* G/H' *respectively. Then* $(L, f^*\omega)$ *and* $(L', f'^*\omega')$ *are equivalent if and only if* A *and* A' *are equivalent.*

In this way we obtained, for representations in classical mechanics, a method of "symplectification" of canonical actions on coset spaces, and thus there is given directly a correspondence desired in [4] .

VI. Elementary particles

We define elementary particles both in quantum and in classical mechanics as follows. The Galilean group in non--relativistic theories and the Poincaré group in relativistic theories will be called the space-time symmetry group.

Definition 3. An elementary particle is a physical system on whose logic there is given a weakly irreducible faithful representation of the identity component of the space-time symmetry group. The properties of an elementary particle are characterized by the corresponding representation so that equivalent representations, and only those, describe the same particle.

Souriau ([1]) stated a general method of finding the representations we need for elementary particles in classical mechanics and he gave explicitly the representations both of the Galilean and of the Poincaré group. Arens ([3], [4]) obtained a similar result for the Poincaré group. There are two aspects in which the given form of representations is not sufficient. First: the equivalence classes are not the same as ours because equivalence made by an antisymplectic map is not considered. Secondly: we would like to get a member of an equivalence class of representations which reflects the dynamics of the particle in a custumary way. This means the following.

States of a physical system are defined to be probability measures on the logic of the system. Let ϱ be a probability measure on the logic $\mathcal{L}$ of a classical or a quantal system and let $t \mapsto A_t$ be a "convenient" representation on $\mathcal{L}$ of the time translation group. Then the map

$$t \mapsto \varrho_t := \varrho \circ A_t^{-1}$$

describes the dynamics of the system. We mean by "convenient" that some condition must be fulfilled; for instance, for an elementary particle, the representation of the time translation group is obtained by restricting the representation of the space-time symmetry group.

Here we mention that pure states can be identified, both in classical and in quantum mechanics, with the atoms of the logics ([7] vol.I. p.116, p.160). Consequently, the points of the phase space in classical mechanics and the vector rays of the structure space in quantum mechanics can be considered the pure states of a physical system and thus the representations on the logics can be transferred directly to the pure states.

Now I give a form of transitive semi-symplectic representations of the Galilean group and of the Poincaré group which is in perfect accordance with the results in quantum mechanics and with the canonical formulation of classical mechanics.

Let S_σ^2 denote the two dimensional sphere in $\mathbb{R}^3$ with radius $\sigma \geqslant 0$. Let $x = (q,p,s)$ be an element of $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma$ with an obvious meaning in the case $\sigma = 0$. A vector field V on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma$ is given by three components (X,Y,Z) such that $X(q,p,s)$, $Y(q,p,s)$ and $Z(q,p,s)$ are in the tangent space of the points q, p and s respectively. Let us define a symplectic form ω_σ on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma$ by

$$\omega_\sigma(V,V')(x) := $$
$$= \langle X(x), Y'(x)\rangle - \langle Y(x), X'(x)\rangle + \langle s, Z(x) \times Z'(x)\rangle$$

where $\langle , \rangle$ resp. $\times$ denotes the inner resp. exterior product in $\mathbb{R}^3$. We shall write p^2 for $\langle p,p\rangle$, and q_i, p_i, s_i $(i=1,2,3)$ for the Cartesian coordinates in $\mathbb{R}^3$.

We state our theorems based on the results in 1.

Theorem 1. There is a family of inequivalent differentiable transitive semi-symplectic representations of the connected Galilean group labelled by two real numbers $\mu > 0$ and $\sigma \geqslant 0$. The representation $A^{(\mu,\sigma)}$ can be given, up to equivalence, on $(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma, \omega_\sigma)$ by the following generating functions for the canonical base of the Lie algebra of the Galilean group:

$K_i(q,p,s) = \mu q_i$ (pure Galilean subgroup),
$P_i(q,p,s) = p_i$ (space translation subgroup),
$J_i(q,p,s) = \epsilon_{ijk} q_j p_k + s_i$ (space rotation subgroup),

$$H(q,p,s) = \frac{p^2}{2\mu} \qquad \text{(time translation subgroup)};$$

$$\left(i,j,k=1,2,3;\ (q,p,s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma\right).$$

μ enumerates the weak cohomology classes of commutator cocycles and appears in the commutator (Poisson bracket) $\{K_i, P_j\} = \mu \delta_{ij}$.

One can write down the representations explicitly:

$$A^{(\mu,\sigma)}_{(t,a,v,R)}(q,p,s) = \left(Rq + a + \frac{Rp - \mu v}{\mu} t\ ,\ \ Rp - \mu v,\ Rs\right).$$

t is a time translation, a is a space translation, v is a pure Galilean transformation and R is a rotation in space.

The representation given in Theorem 1 can be extended, in a unique way, to semi-symplectic representations of the Galilean group with space and time inversion. It is an easy task to check that the representation I_S of the space inversion S and the representation I_T of the time inversion T are obtained as follows:

$$I_S(q,p,s) = (-q,-p,s)\ , \quad I_T(q,p,s) = (q,-p,-s)\ .$$

Note that the time inversion is represented by an anti-symplectic diffeomorphism; in quantum mechanics, as it is known, the time inversion is represented by an antiunitary operator.

Theorem 2. There is a family of inequivalent differentiable transitive semi-symplectic representations of the connected Poincaré group labelled by two real numbers

$\mu > 0$, $\sigma \geq 0$ [redacted] . <u>The representations can be given, up to equivalence, on</u> $(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma , \omega_\sigma)$ <u>by the following generating functions for the canonical base of the Lie algebra of the Poincaré group:</u>

$P_i(q,p,s) = p_i$ (space translation subgroup)

$H(q,p,s) =: p_0 = \sqrt{p^2 + \mu^2}$ (time translation subgroup)

$J_i(q,p,s) = \epsilon_{ijk} q_j p_k + s_i$ (space rotation subgroup)

$K_i(q,p,s) = p_0 q_i + \epsilon_{ijk} p_j s_k / (p_0 + \mu)$ (pure Lorentz one-parameter subgroups)

$\left(i,j,k = 1,2,3 \ , \ (q,p,s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2_\sigma\right)$.

The interpretation of Theorem 1 and Theorem 2 is straightforward: the representations describe free particles with mass μ and spin σ . [redacted].

There is another family of representations which correspond to particles with zero mass. These representations, however, lie outside of the framework of mechanics, and do not describe particles in mechanical sense ([1] p. 220, [7] vol.II p. 217).

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