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Spacetime Without Reference Frames: An Application to the Kinetic Theory

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Spacetime structures defined in the language of manifolds admit an absolute formulation, i.e., a formulation which does not refer to observers (reference frames). We consider an affine structure for Galilean spacetime. As an application the Chapman–Enskog iteration for the solution of the Boltzmann equation is given in an absolute form. As a consequence, the second approximations of the stress tensor and the heat flux are obtained in a form independent of observers, which throws new light on material frame indifference.

1. INTRODUCTION

General relativity is a mathematically well developed physical theory. In the last decade several attempts have appeared to formalize nonrelativistic (Galilean, Euclidean) spacetime in a similar mathematical way (Appleby and Kadianakis, 1983, 1986; Kadianakis, 1983, 1985, 1991; Rodrigues *et al.*, 1995): spacetime is considered to be a four-dimensional manifold endowed with further structures involving absolute time and Euclidean inner product in some way. Absolute simultaneity divides spacetime into a continuous succession of instantaneous three-dimensional spaces; a working theory requires that these spaces be related to each other. The relation among the instantaneous spaces can be assured by a connection. In some approaches this connection forms part of the mathematical structure without a physical interpretation and in others it represents a gravitational field. Of course, we would like to have a theory which works without gravitation, too.

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In usual treatments of special relativity (which corresponds to general relativity without gravitation) spacetime is endowed—tacitly or explicitly—with a structure stronger than the manifold structure: the spacetime translations are meaningful and play a fundamental role as symmetries and lead to the conservation of four-momentum by the Noether theorem. This means that special relativistic spacetime is an affine space and nonrelativistic spacetime, too, is to be considered an affine space, as was stated already over 70 years ago (Weyl, 1922). The affine structure establishes a relation among the instantaneous spaces, and thus it rules out connections from the structure. Moreover, the affine structure is much simpler than the manifold structure and admits nicer applications because its notions are closer to the ones in usual treatments; of course, since an affine space is a special manifold, all the results obtained in the manifold approach hold true in this case, too.

The nonrelativistic spacetime model based on an affine structure is treated thoroughly by Matolcsi (1993). One of the greatest advantages of such a spacetime model is the fact that the intuitive notion of observer (reference frame) is ruled out; more precisely, (i) *the structure of spacetime does not involve observers*, and (ii) *the notion of observer is defined with the aid of the given structure in a mathematically exact way.*

This allows us to use only *absolute* objects in formulating physical theories, i.e., objects that are not related to observers. Of course (since a theory is justified by observations), the spacetime model must supply a rule for how observers deduce their relative objects from the absolute ones.

An important example is the *absolute velocity*, which is the analogue of the well known four-velocity in relativistic physics. Physically: a material object has some intrinsic property which is observed by different observers as different relative velocities; this intrinsic property is the absolute velocity.

We can formulate Newtonian equation, Maxwell equations, balance equations and constitutive relations of continuum physics, Boltzmann equation, etc., in absolute terms; these formulations throw new light on some usual notions involving observers. It is well known that the approximation procedures for solving the Boltzmann equation yield constitutive relations in continuum physics which cause trouble in connection with the principle of material frame indifference (Müller, 1972, 1976, 1985; Wang, 1975; Speziale, 1981; Hoover *et al.*, 1981). Examining these approximation procedures, we easily find that they are strongly related to observers.

First we will recapitulate the notions of the spacetime model based on an affine structure and then as an application we present an *absolute version* of the Chapman–Enskog iterative procedure for the Boltzmann equation resulting in an *absolute form* of the second approximation of the stress and the heat flux, which has an interesting consequence regarding the principle of material frame indifference.

2. GENERAL ASPECTS OF THE SPACETIME MODEL

2.1. Since we want to have a rigorous mathematical setting, we must take into account the physical dimensions (units of measurement) of quantities: contrary to the usual treatments, distances, time periods, and masses are not considered to be real numbers (3 m and 3 km are distances, but 3 is not). An appropriate *measure line* (an oriented one-dimensional real vector space) is to be assigned to each quantity for measuring its magnitude. The product and quotient of units—such as m/sec^2—are given by a tensor product and a tensor quotient, which obey the usual rules of multiplication and division (Matolcsi, 1993). The nth tensor power of a measure line will be denoted by an exponent (n); e.g., $\mathbf{D}^{(2)} := \mathbf{D} \otimes \mathbf{D}$. Thus if $\mathbf{D}$ and $\mathbf{I}$ are the measure lines of distances and time periods, respectively, then $\mathbf{D}/\mathbf{I}$ is the measure line of velocities, $\mathbf{D}/\mathbf{I}^{(2)}$ is the measure line of accelerations, $\mathbf{D}^{(3)}$ is the measure line of volumes, etc.; if m $\in \mathbf{D}$ and sec $\in \mathbf{I}$, then m/sec^2 $\in \mathbf{D}/\mathbf{I}^{(2)}$.

The application of measure lines makes apparent the physical dimensions, which rules out "dimension analysis."

The measure line $\mathbf{D}$ of distances and the measure line of time periods $\mathbf{I}$ are sufficient to establish a nonrelativistic theory: if the Planck constant is taken to be the real number 1, then the measure line of masses is $\mathbf{I}/\mathbf{D}^{(2)}$. This choice seems very useful in theoretical considerations. Nevertheless, we can choose the other possibility corresponding to the SI system of units: then we introduce an independent measure line of masses. In this paper we avoid the explicit use of the measure line of masses, so readers can chose it at their will.

2.2. Spacetime is considered to be a four-dimensional affine space M, which means that there is a four-dimensional vector space $\mathbf{M}$ and a map, called subtraction,

$$M \times M \to \mathbf{M}, \qquad (y, x) \mapsto y - x$$

having the following properties:

(i) $(y - x) + (x - z) + (z - y) = 0$ for all $x, y, z \in M$.

(ii) $M \to \mathbf{M}$, $x \mapsto x - y$ is a bijection for all $y \in M$.

2.3. The basic property of the nonrelativistic spacetime is that to every spacetime point x an absolute instant (time point) $\tau(x)$ can be assigned. In mathematical form: there exist an absolute time I (a one-dimensional affine space over the vector space $\mathbf{I}$) and a map τ: $M \to I$ which is supposed to be a nontrivial affine map, i.e., there is a nonzero linear map $\boldsymbol{\tau}$: $\mathbf{M} \to \mathbf{I}$ such that $\tau(x) - \tau(y) = \boldsymbol{\tau} \cdot (x - y)$ for all spacetime points x, y.

2.4. Every instant t (element of I) defines the corresponding hyperplane of simultaneous spacetime points

$$E_t := \{x \in M \mid \tau(x) = t\}$$

The differences of elements of E_t are in the three-dimensional vector space

$$\mathbf{E} := \{\boldsymbol{x} \in \mathbf{M} \mid \boldsymbol{\tau} \cdot \boldsymbol{x} = 0\}$$

which means that E_t is an affine space over $\mathbf{E}$. Then we accept that the simultaneous spacetime points are endowed with a Euclidean structure, i.e., there is given the measure line $\mathbf{D}$ of distances and a positive-definite symmetric bilinear map

$$\mathbf{b}: \mathbf{E} \times \mathbf{E} \to \mathbf{D}^{(2)}$$

2.5. Summarizing the previous facts, we define the *nonrelativistic spacetime model* as a quintuplet (M, I, τ, $\mathbf{D}$, $\mathbf{b}$) where:

- M is a four-dimensional oriented affine space (*spacetime*) over the vector space $\mathbf{M}$ (*spacetime vectors*).
- I is a one-dimensional oriented affine space (*absolute time*) over the vector space $\mathbf{I}$ (*measure line of time durations*).
- τ: $M \to I$ is an affine surjection (*time evaluation*) over the linear map $\boldsymbol{\tau}$: $\mathbf{M} \to \mathbf{I}$.
- $\mathbf{D}$ is a one-dimensional oriented vector space (*measure line of distances*).
- $\mathbf{b}$: $\mathbf{E} \times \mathbf{E} \to \mathbf{D}^{(2)}$ is a positive-definite symmetric bilinear map (*Euclidean structure*), where $\mathbf{E} := \operatorname{Ker} \boldsymbol{\tau}$.

2.6. An important notion is $\mathbf{M}^*$, the dual of $\mathbf{M}$: the set of *covectors*, i.e., real-valued linear maps on $\mathbf{M}$. Here $\mathbf{M}^*$ is a four-dimensional vector space, *distinct* from $\mathbf{M}$. Tensors and antisymmetric tensors will be denoted by the usual symbols; e.g., $\mathbf{M} \otimes \mathbf{M}$ is the space of two-tensors and $\mathbf{M}^* \wedge \mathbf{M}^*$ is the space of antisymmetric two-cotensors. Contractions of cotensors (covectors) with tensors (vectors) will be denoted by a dot.

Linear maps will be regarded as tensors, according to the following rule. If $\mathbf{V}$ and $\mathbf{U}$ are vector spaces, then a linear map $\boldsymbol{L}$: $\mathbf{V} \to \mathbf{U}$ is considered to be an element of $\mathbf{U} \otimes \mathbf{V}^*$ in such a way that the contraction $\boldsymbol{L} \cdot \boldsymbol{v}$ corresponds to the value of $\boldsymbol{L}$ at the vector $\boldsymbol{v} \in \mathbf{V}$.

In this way we have $\boldsymbol{\tau} \in \mathbf{I} \otimes \mathbf{M}^*$.

2.7. $\mathbf{E}$ is a three-dimensional linear subspace of $\mathbf{M}$ whose elements are called *spacelike* vectors. According to the previous convention, the inclusion map $\mathbf{i}$: $\mathbf{E} \to \mathbf{M}$ is an element of $\mathbf{M} \otimes \mathbf{E}^*$; its transpose $\mathbf{i}^* \in \mathbf{E}^* \otimes \mathbf{M}$ is defined by $\boldsymbol{k} \cdot \mathbf{i} \cdot \boldsymbol{q} = (\mathbf{i}^* \cdot \boldsymbol{k}) \cdot \boldsymbol{q}$ for all $\boldsymbol{k} \in \mathbf{M}^*$, $\boldsymbol{q} \in \mathbf{E}$; in other words, $\mathbf{i}^* \cdot \boldsymbol{k}$ is the restriction of the linear form $\boldsymbol{k}$ onto $\mathbf{E}$.

We shall find it convenient to write a dot instead of $\mathbf{b}$, too: $\boldsymbol{q} \cdot \boldsymbol{r} := \mathbf{b}(\boldsymbol{q}, \boldsymbol{r})$. This notation is consistent with the above one introduced for contractions because $\mathbf{b}$ allows us to identify $\mathbf{E}/\mathbf{D}^{(2)}$ with $\mathbf{E}^*$ (Matolcsi, 1993, I.1.2.5).

It is important to keep in mind that the "inner product" of two spacetime vectors is not meaningful unless they are spacelike; the "inner product" is given by the Euclidean structure **b**. The dot product (contraction), in general, is defined between a covector (element of $\mathbf{M}^*$) and a vector (element of $\mathbf{M}$) and this dot, being defined between elements of different vector spaces, does not represent an inner product. The dot between two spacelike vectors makes sense and corresponds to an inner product.

As a consequence, the magnitude of a spacetime vector is not meaningful unless the vector is spacelike. The magnitude of $\boldsymbol{q} \in \mathbf{E}$ is $|\boldsymbol{q}| := \sqrt{\boldsymbol{q}\cdot\boldsymbol{q}} \in \mathbf{D}$.

It follows from the definition that the dot product between corresponding quantities with other physical dimensions is meaningful, too. For instance, if $z \in \mathbf{M} \otimes \mathbf{D}$ and $\boldsymbol{w} \in \mathbf{M}/\mathbf{I}$, then $z\cdot\boldsymbol{w} \in \mathbf{D}/\mathbf{I}$. Consequently, the magnitude of spacelike vectors with other physical dimensions can be defined; e.g., the magnitude of $\boldsymbol{v} \in \mathbf{E}/\mathbf{I}$ is $|\boldsymbol{v}| := \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} \in \mathbf{D}/\mathbf{I}$.

It will be suitable to use

$$\mathbf{N} := \frac{\mathbf{E}}{\mathbf{D}}$$

which is the set of spacelike vectors without physical dimension. The magnitude of an element of $\mathbf{N}$ is a real number.

2.8. The history of a masspoint in the spacetime model is described by a twice-differentiable *world line function* $r\colon I \to M$ such that $\tau(r(t)) = t$ holds for all $t \in \mathbf{I}$.

The absolute velocity of the masspoint is the derivative of the world line function:

$$\dot{r}(t) = \lim_{s\to t} \frac{r(s) - r(t)}{s - t} \in \frac{\mathbf{M}}{\mathbf{I}}$$

We easily find that $\boldsymbol{\tau}\cdot\dot{r}(t) = 1$; thus we introduce the set of *absolute velocities*,

$$V(1) := \left\{\boldsymbol{w} \in \frac{\mathbf{M}}{\mathbf{I}} \,|\, \boldsymbol{\tau}\cdot\boldsymbol{w} = 1\right\}$$

The differences of elements in $V(1)$ are in $\mathbf{E}/\mathbf{I}$, which means that $V(1)$ is an affine space over $\mathbf{E}/\mathbf{I}$ which turns out to be the set of *relative velocities*.

Note the important fact that an absolute velocity is not a spacelike vector, and thus it has no magnitude. On the other hand, the relative velocities are spacelike vectors; their magnitude is meaningful as an element of $\mathbf{D}/\mathbf{I}$.

2.9. The absolute acceleration of the masspoint is the second derivative of the world line function. We have $\ddot{r}(t) \in \mathbf{E}/\mathbf{I}^{(2)}$, i.e., the absolute accelerations are spacelike vectors.

Accordingly, an absolute *force field* is a function $\mathbf{f}$ defined in $M \times V(1)$ having spacelike vector values (with appropriate physical dimension) in such a way that the absolute Newtonian equation for the world line function r of a particle with mass $\mathbf{m}$ has the form

$$\ddot{r} = \mathbf{f}(r, \dot{r})$$

2.10. An observer, physically, is a collection of masspoints; mathematically it is defined as the collection of the velocity values of the histories of the masspoints i.e., as a velocity field U: $M \to V(1)$, which is supposed to be smooth. A constant velocity field is an inertial observer.

The motion of a masspoint with history (world line function) r relative to an observer U is defined (Matolcsi, 1993, I.6.), and we find that the relative velocity of the masspoint with respect to the observer is $\dot{r}(t) - U(r(t))$.

That is why if w_1 and w_2 are absolute velocities, then $w_1 - w_2$ is called the relative velocity of w_1 with respect to w_2.

2.11. If V is an affine space (over the vector space $\mathbf{V}$), then the differentiability of a function ϕ: $M \to V$ is defined by a formula well-known for functions between vector spaces; its derivative is the function $\mathrm{D}\phi$: $M \to \mathbf{V} \otimes \mathbf{M}^*$ for which

$$\lim_{y \to x} \frac{\phi(y) - \phi(x) - \mathrm{D}\phi(x) \cdot (y - x)}{|y - x|} = 0$$

holds, where $|\cdot|$ is an arbitrary norm on $\mathbf{M}$.

If $V = M$, then $\mathrm{D}\phi$ takes values in $\mathbf{M} \otimes \mathbf{M}^*$; the self-contraction (trace) defined for the elements of $\mathbf{M} \otimes \mathbf{M}^*$ yields the *divergence* $\mathrm{D} \cdot \phi$, which is a real-valued function defined on m.

It is important that the *spacelike derivative* $\nabla\phi := \mathbf{i}^* \cdot \mathrm{D}\phi$: $M \to \mathbf{V} \otimes \mathbf{E}^*$ is an absolute object (independent of observers); on the other hand, the timelike derivative cannot be defined in an absolute manner: timelike derivatives exist only with respect to observers. More precisely, if u is a velocity field (in particular, an observer), then $D_u\phi := u \cdot \mathrm{D}\phi$ is the *u-timelike derivative* (which corresponds to the usual "substantial time derivative").

If f: $M \times V(1) \to V$ is a differentiable function, then its first partial derivative $\mathrm{D}_1 f$: $M \times V(1) \to \mathbf{V} \otimes \mathbf{M}^*$ is defined as the derivative of the function $M \to V, x \mapsto f(x, w)$ for fixed $w \in V(1)$. The second partial derivative $\mathrm{D}_2 f$: $M \times V(1) \to \mathbf{V} \otimes (\mathbf{E}/\mathbf{I})^*$ is defined similarly, according to the sense.

3. ABSOLUTE FORMULATION OF THE KINETIC THEORY

We give only a short and concise survey; the absolute formulation is a quite easy transcription of the usual one.

3.1. The Boltzmann Equation

Let us consider a gas consisting of molecules with mass $\boldsymbol{m}$ under the action of a force field. The spacetime region in which the gas molecules exist is supposed to be a connected open subset G of M. We assume that the force field has the special form

$$(x, \boldsymbol{w}) \to \boldsymbol{m}\mathbf{i}^* \cdot \boldsymbol{F}(x) \cdot \boldsymbol{w}, \qquad x \in G, \quad \boldsymbol{w} \in V(1)$$

where $\boldsymbol{mF}$: $G \to \mathbf{M}^* \wedge \mathbf{M}^*$ is a smooth map. This covers the cases when the force field has a potential or does not depend on velocity (Matolcsi, 1993, I.2.4.3 and I.9.4.6.) which is usually considered in kinetic theory.

The (molecular) *distribution function* is a differentiable map f: $G \times V(1) \to \mathbf{I}^{(3)}/\mathbf{D}^{(6)}$, subject to the integrability condition

$$\int_{G \cap E_t} \int_{V(1)} f(x, \boldsymbol{w})\, dx\, d\boldsymbol{w} = 1$$

for all $t \in I$ such that $G \cap E_t$ is nonvoid; here and in the sequel dx and $d\boldsymbol{w}$ denote the integration by the $\mathbf{D}^{(3)}$- and $\mathbf{D}^{(3)}/\mathbf{I}^{(3)}$-valued canonical translation-invariant measures of E_t and $V(1)$, respectively (Matolcsi, 1986).

The absolute Maxwell–Boltzmann equation reads (Matolcsi, 1986)

$$(\mathrm{D}_1 f(x, \boldsymbol{w}) + \mathrm{D}_2 f(x, \boldsymbol{w}) \cdot F(x)) \cdot \boldsymbol{w} = (Cf)(x, \boldsymbol{w}), \qquad (x, \boldsymbol{w}) \in G \times V(1)$$

[$\mathbf{i}^*$ can be omitted from the contraction of $\mathrm{D}_2 f(x, \boldsymbol{w})$ and $\mathbf{i}^* \cdot F(x) \cdot \boldsymbol{w}$.]

The collision integral Cf involves a scattering function and a scattering cross section detailed in the next section.

3.2. Binary Collisions and Scattering

We consider the elastic collision of two molecules with equal masses. Let the absolute velocity values of the molecules before and after collision be denoted by $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_1^+$, $\boldsymbol{w}_2^+$, respectively. Of course, we exclude that the absolute velocity values of the molecules are equal (then no collision occurs); thus we shall consider the set

$$V(1, 1) := \{(\boldsymbol{w}_1, \boldsymbol{w}_2) \in V(1) \times V(1) \mid \boldsymbol{w}_1 \neq \boldsymbol{w}_2\}$$

Furthermore, we shall also use the notation

$$S(1) := \{\boldsymbol{n} \in \mathbf{N} \mid |\boldsymbol{n}| = 1\}$$

Elastic collisions are characterized by:

(i) Conservation of absolute momentum, which results in

$$\boldsymbol{w}_1 + \boldsymbol{w}_2 = \boldsymbol{w}_1^+ + \boldsymbol{w}_2^+$$

(ii) Conservation of w_0-kinetic energy, which results in

$$|w_1 - w_0|^2 + |w_2 - w_0|^2 = |w_1^+ - w_0|^2 + |w_2^+ - w_0|^2$$

where

$$w_0 := \frac{w_1 + w_2}{2} = \frac{w_1^+ + w_2^+}{2}$$

is the velocity value of the center of mass.

It is worth mentioning that these relations imply the conservation of c-kinetic energy for all $c \in V(1)$.

From the above equalities we infer that there is an element $n \in S(1)$, called the *scattering direction*, such that

$$w_1^+ = w_0 - \frac{|w_2 - w_1|}{2} n, \qquad w_2^+ = w_0 + \frac{|w_2 - w_1|}{2} n$$

We take w_1^+ and w_2^+ as functions of w_1, w_2, and n given by the previous formulas, and we consider from now on *this scattering function*

$$(w_1^+, w_2^+): \quad V(1, 1) \times S(1) \to V(1, 1)$$

Recall that the scattering cross section is a map σ that assigns to every nonzero relative velocity value v (an element of $\mathbf{E}/\mathbf{I}$) a $\mathbf{D}^{(2)}$-valued measure σ_v on the Borel subsets of $S(1)$ (Matolcsi, 1986). Now we assume that for all Borel subsets B of $S(1)$ the map

$$V(1, 1) \to \mathbf{D}^{(2)}, \qquad (w_1, w_2) \mapsto \sigma_{w_2 - w_1}(B)$$

is locally integrable by the product measure on $V(1, 1)$ and we define the $\mathbf{D}^{(8)}/\mathbf{I}^{(3)}$-valued measure Σ on the Borel subsets of $V(1, 1) \times S(1)$ by

$$\Sigma(E_1 \times E_2 \times B) := \int_{E_2} \int_{E_1} \sigma_{w_2 - w_1}(B)\, dw_1\, dw_2$$

The bijections

$$T: \quad V(1, 1) \times S(1) \to V(1, 1) \times S(1)$$

$$(w_1, w_2, n) \mapsto \left(w_1^+(w_1, w_2, n),\, w_2^+(w_1, w_2, n),\, \frac{w_2 - w_1}{|w_2 - w_1|} \right)$$

and

$$J: \quad V(1, 1) \times S(1) \to V(1, 1) \times S(1), \qquad (w_1, w_2, n) \mapsto (w_2, w_1, n)$$

have the properties

$$T^{-1} = T, \qquad J^{-1} = J$$

T is interpreted to describe inverse scattering: if the incoming velocities are w_1 and w_2, the direction of their relative velocity is $(w_2 - w_1)/|w_2 - w_1|$ and if the outcoming velocities are w_1^+ and w_2^+, then in the inverse scattering the incoming velocities are w_1^+ and w_2^+, and the outcoming velocities are w_1 and w_2 with scattering direction $(w_2 - w_1)/|w_2 - w_1|$.

J is interpreted to describe the interchanging of the two molecules in collision.

Now we require that the scattering cross section be *invariant* under T and J; more precisely, we require

$$\Sigma \circ T^{-1} = \Sigma \circ J^{-1} = \Sigma$$

or in other words, for every function β defined on $V(1, 1) \times S(1)$ and taking values in a finite-dimensional vector space $\mathbf{V}$ and being integrable with respect to Σ, we have

$$\int_{V(1,1)\times S(1)} (\beta \circ T)\, d\Sigma = \int_{V(1,1)\times S(1)} (\beta \circ J)\, d\Sigma = \int_{V(1,1)\times S(1)} \beta\, d\Sigma$$

3.3. Balance Equations

If N is the number of molecules and m is the mass of each molecule, we introduce the *mass density*

$$\rho(x) := mN \int_{V(1)} f(x, w)\, dw \qquad (x \in G)$$

and we suppose it is nowhere zero.

If $\mathbf{V}$ is a finite-dimensional vector space, then for a function $Z\colon G \times V(1) \to \mathbf{V}$ we put

$$\langle Z \rangle^{\bullet}\colon \quad G \to \mathbf{V}, \qquad x \mapsto \frac{mN}{\rho(x)} \int_{V(1)} f(x, w) Z(x, w)\, dw$$

$$C(Z)\colon \quad G \to \mathbf{V}, \qquad x \mapsto \int_{V(1)} (Cf)(x, w) Z(x, w)\, dw$$

provided the integrals exist.

If we use the symbol

$$W\colon \quad G \times V(1) \to V(1), \qquad (x, w) \mapsto w$$

and we suppose that the usual regularity conditions hold [the integrals in question exist, the order of integration and differentiation can be interchanged,

$f(x, \cdot)$ tends to zero at infinity in a sufficient order], we get the *transport equation* for a differentiable Z:

$$\mathrm{D}\cdot(\rho\langle Z \otimes \boldsymbol{W}\rangle) - \rho\langle \mathrm{D}_1 Z\cdot \boldsymbol{W}\rangle - \rho\langle \mathrm{D}_2 Z\cdot \boldsymbol{F}\cdot \boldsymbol{W}\rangle = C(Z)$$

where $\mathrm{D}\cdot$ in the first term denotes the *divergence* of the corresponding function and, of course,

$$\begin{aligned}(Z \otimes \boldsymbol{W})(x, \boldsymbol{w}) &= Z(x, \boldsymbol{w}) \otimes \boldsymbol{w}\\ (\mathrm{D}_1 Z\cdot \boldsymbol{W})(x, \boldsymbol{w}) &= \mathrm{D}_1 Z(x, \boldsymbol{w})\cdot \boldsymbol{w}\\ (\mathrm{D}_2 Z\cdot \boldsymbol{F}\cdot \boldsymbol{W})(x, \boldsymbol{w}) &= \mathrm{D}_2 Z(x, \boldsymbol{w})\cdot \boldsymbol{F}(x)\cdot \boldsymbol{w}\end{aligned}$$

A map S: $V(1) \to \mathbf{V}$ is called *summational invariant* with respect to the scattering function $(\boldsymbol{w}_1^+, \boldsymbol{w}_2^+)$ if

$$S(\boldsymbol{w}_1) + S(\boldsymbol{w}_2) = S(\boldsymbol{w}_1^+(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{n})) + S(\boldsymbol{w}_2^+(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{n}))$$

for all $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{n}) \in V(1, 1) \times S(1)$.

The Boltzmann–Gronwall theorem (Truesdell and Muncaster, 1980, VI.(ii)) in our formulation asserts that a measurable S is a summational invariant if and only if there are an affine map L: $V(1) \to \mathbf{V}$, a $\boldsymbol{c} \in V(1)$, and a $\lambda \in \mathbf{V} \otimes \mathbf{I}^{(2)}/\mathbf{D}^{(2)}$ such that $S(\boldsymbol{w}) = L(\boldsymbol{w}) + \lambda|\boldsymbol{w} - \boldsymbol{c}|^2$ for all $\boldsymbol{w} \in V(1)$.

With the aid of the invariance properties of the scattering cross section we can prove as usual that if Z: $G \times V(1) \to \mathbf{V}$ is a function such that $Z(x, \cdot)$ is summational invariant for all $x \in G$, then

$$C(Z) = 0.$$

Introducing the *absolute gross velocity field*

$$\boldsymbol{u} := \langle \boldsymbol{W}\rangle: G \to V(1)$$

and the *random velocity* $\boldsymbol{v}_r$: $G \times V(1) \to \mathbf{E}/\mathbf{I}$,

$$\boldsymbol{v}_r(x, \boldsymbol{w}) := \boldsymbol{w} - \boldsymbol{u}(x), \qquad (x, \boldsymbol{w}) \in G \times V(1)$$

defining the usual quantities

$$\boldsymbol{P} := \rho\langle \boldsymbol{v}_r \otimes \boldsymbol{v}_r\rangle, \qquad e := \tfrac{1}{2}\langle |\boldsymbol{v}_r|^2\rangle, \qquad \boldsymbol{q} := \tfrac{1}{2}\rho\langle |\boldsymbol{v}_r|^2 \boldsymbol{v}_r\rangle$$

and in the transport equation substituting for Z the quantities $\boldsymbol{m}$ [as a constant function on $G \times V(1)$], $\boldsymbol{m}\boldsymbol{W}$, and $\boldsymbol{m}|\boldsymbol{v}_r|^2/2$, we obtain the absolute balance equations

$$\mathrm{D}\cdot(\rho\boldsymbol{u}) = 0$$

$$\mathrm{D}\cdot(\rho\boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{P}) = \rho\mathrm{i}^*\cdot \boldsymbol{F}\cdot \boldsymbol{u}$$

$$D\cdot(\rho e\boldsymbol{u} + \boldsymbol{q}) = -\boldsymbol{P} : \nabla\boldsymbol{u}$$

or in other forms

$$\begin{aligned} D_{\boldsymbol{u}}\rho &= -\rho\nabla\cdot\boldsymbol{u} \\ \rho D_{\boldsymbol{u}}\boldsymbol{u} &= -\nabla\cdot\boldsymbol{P} + \rho i^{*}\cdot\boldsymbol{F}\cdot\boldsymbol{u} \\ \rho D_{\boldsymbol{u}}e &= -\nabla\cdot\boldsymbol{q} - \boldsymbol{P} : \nabla\boldsymbol{U} \end{aligned}$$

where $D_{\boldsymbol{u}}$ is the $\boldsymbol{u}$-timelike derivation and the colon denotes a double contraction.

4. THE ABSOLUTE ITERATIVE PROCEDURE

4.1. In this section N and N_0 denote the set of positive integer numbers and the set of nonnegative integers, respectively.

Now we suppose that (1) the function $R := (\rho, \boldsymbol{u}, e, \boldsymbol{F})$ defined on G is smooth, and (2) the distribution function can be given by a series $f = \Sigma_{n=0}^{\infty} f_n$ such that:

(i) $Cf_0 = 0$.

(ii) For all $n \in$ N (and $x \in G$),

$$\int_{V(1)} f_n(x, w)\, dw = 0$$

$$\int_{V(1)} f_n(x, w)w\, dw = 0$$

$$\int_{V(1)} f_n(x, w)\boldsymbol{v}_r(x, w)\, dw = 0$$

(iii) If (for $x \in G$)

$$\begin{aligned} \boldsymbol{P}_n(x) &:= mN \int_{V(1)} f_n(x, w)\boldsymbol{v}_r(x, w) \otimes \boldsymbol{v}_r(x, w)\, dw \\ \boldsymbol{q}_n(x) &:= \frac{mN}{2} \int_{V(1)} f_n(x, w)\,|\boldsymbol{v}_r(x, w)|^2\boldsymbol{v}_r(x, w)\, dw \end{aligned}$$

then

$$\boldsymbol{P} = \sum_{n=0}^{\infty} \boldsymbol{P}_n, \qquad \boldsymbol{q} = \sum_{n=0}^{\infty} \boldsymbol{q}_n$$

i.e., the order of integration and summation can be interchanged;

(iv) For all $n \in \mathrm{N}_0$, the function $(x, \boldsymbol{w}) \mapsto f_n(x, \boldsymbol{w})$ depends on x through the spacelike derivatives of R in such a way that the order of the derivatives does not exceed n: there are smooth functions φ_n such that

$$f_n(x, \boldsymbol{w}) = \varphi_n(R(x), \nabla R(x), \ldots, \nabla^n R(x), \boldsymbol{w})$$

4.2. Let us introduce the notations

$$T_0\rho := -\rho\nabla\cdot\boldsymbol{u}, \qquad T_n\rho := 0$$

$$T_0\boldsymbol{u} := -\frac{1}{\rho}\nabla\cdot\boldsymbol{P}_0 + \mathrm{i}^*\cdot\boldsymbol{F}\cdot\boldsymbol{u}, \qquad T_n\boldsymbol{u} := -\frac{1}{\rho}\nabla\cdot\boldsymbol{P}_n$$

$$T_0\boldsymbol{F} := \mathrm{D}_{\boldsymbol{u}}\boldsymbol{F}, \qquad T_n\boldsymbol{F} := 0$$

for $n \in \mathrm{N}$ and

$$T_n e := -\frac{1}{\rho}(\nabla\cdot\boldsymbol{q}_n + \boldsymbol{P}_n : \nabla\boldsymbol{u})$$

for $n \in \mathrm{N}_0$, and let us collect them for defining $T_n R$ for $n \in \mathrm{N}_0$. [Observe that T_n corresponds to the "time derivation" $\mathrm{D}_n/\mathrm{D}t$ in Chapman (1970).]

φ_n is a function of $n + 1$ variables; $\mathrm{D}_k\varphi_n$ denotes its kth partial derivative. For the sake of simplicity we set

$$R^{[n]} := (R, \nabla R, \ldots, \nabla^n R), \qquad n \in \mathrm{N}_0$$

and, supposing that the series in question converges, we define for $m, n \in \mathrm{N}_0$

$$A_m f_n(x, \boldsymbol{w}) := \sum_{k=0}^{\infty} \mathrm{D}_k\varphi_n(R^{[n]}(x), \boldsymbol{w})\cdot((\nabla^k(\boldsymbol{v}_r\cdot\nabla R + T_m R))(x))$$

$$A_F f_n(x, \boldsymbol{w}) := \mathrm{D}_{n+1}\varphi_n(R^{[n]}(x), \boldsymbol{w})\cdot\boldsymbol{F}(x)\cdot\boldsymbol{w}$$

Using the relation $\mathrm{D}_{\boldsymbol{w}}R = (\boldsymbol{w} - \boldsymbol{u})\cdot\nabla R + \mathrm{D}_{\boldsymbol{u}}R$, we easily find that

$$\mathrm{D}_1 f_n(x, \boldsymbol{w})\cdot\boldsymbol{w} = \sum_{m=0}^{\infty} A_m f_n(x, \boldsymbol{w})$$

$$\mathrm{D}_2 f_n(x, \boldsymbol{w})\cdot\boldsymbol{F}(x)\cdot\boldsymbol{w} = A_F f_n(x, \boldsymbol{w})$$

4.3. Supposing that the order of infinite summation and differentiation can be interchanged, we derive from the Maxwell–Boltzmann equation

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} A_m f_n + A_F f_n\right) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \mathscr{C}(f_n, f_m)$$

where

$$\mathscr{C}(f_n, f_m)(x, w)$$
$$:= \int_{V(1)}\int_{S(1)} |w' - w|(f_n(x, w_1^+(w, w', n))f_m(x, w_2^+(w, w', n))$$
$$- f_n(x, w)f_m(x, w'))\, d\sigma(n)_{w'-w}\, dw'$$

provided the integral exists; the summation for m on the right-hand side of the previous equality begins from 1 because $\mathscr{C}(f_0, f_0) = Cf_0 = 0$.

4.4. Supposing that the series converge absolutely, we can compute the double sums by Cauchy's method, i.e.,

$$\sum_{n=0}^{\infty}\left(\sum_{i+k=n} A_i f_k + A_F f_n\right) = \sum_{n=0}^{\infty}\sum_{i+k=n+1} \mathscr{C}(f_i, f_k)$$

Hence, if we require that the functions f_n satisfy *Enskog's iterative system*

$$\sum_{i+k=n} A_i f_k + A_F f_n = \sum_{i+k=n+1} \mathscr{C}(f_i, f_k), \qquad n \in \mathbb{N}_0$$

then (with the assumptions made above) $\Sigma_{n=0}^{\infty} f_n$ is a solution of the Boltzmann equation.

5. APPROXIMATIONS FOR THE STRESS TENSOR AND THE HEAT FLOW

To determine $\boldsymbol{P}_n$ and $\boldsymbol{q}_n$ from Enskog's iterative system, we can follow a way similar to the usual one (Chapman, 1970); hence the details are omitted. Introducing the notations

$$\Theta := \frac{2}{3} me, \qquad p := \frac{\rho}{m}\Theta$$

$$\Lambda := \frac{\nabla u + (\nabla u)^*}{2} \qquad \text{(the symmetric part of } \nabla u\text{)}$$

$$\bar{\Lambda} := \Lambda - \frac{1}{3}(\operatorname{Tr}\Lambda)\mathbf{1} \qquad \text{(the traceless part of } \Lambda\text{)}$$

we obtain

$$P_0 = p\mathbf{1}, \qquad q_0 = 0,$$
$$P_1 = -2\mu\overline{\Lambda}, \qquad q_1 = -\lambda\nabla\Theta$$

where μ and λ are scalar-valued functions defined on G and $\mathbf{1}$ is the identity tensor (id_N).

If T is an arbitrary spacelike tensor, then $\overline{T}$ will denote the traceless part of the symmetric part of T; we have

$$P_2 = \beta_1(\mathrm{Tr}\,\Lambda)\overline{\Lambda} + \beta_2\overline{\nabla(\mathbf{i}^*\cdot F\cdot u - (1/\rho)\nabla p) - \nabla u\cdot\nabla u - 2\overline{\Lambda}\cdot\nabla u} + \beta_3\overline{\nabla\nabla\Theta}$$
$$+ \beta_4\overline{\nabla p\otimes\nabla\Theta} + \beta_5\overline{\nabla\Theta\otimes\nabla\Theta} + \beta_6\overline{\overline{\Lambda}\cdot\overline{\Lambda}} + \beta_7\overline{\mathbf{i}^*\cdot(\overline{\Lambda}\cdot F)}$$
$$q_2 = \alpha_1(\mathrm{Tr}\,\Lambda)\nabla\Theta + \alpha_2(\tfrac{2}{3}\nabla(\Theta\nabla\cdot u) - 2\nabla\Theta\cdot\nabla u) + \alpha_3\nabla p\cdot\overline{\Lambda}$$
$$+ \alpha_4\nabla\cdot\overline{\Lambda} + \alpha_5\nabla\Theta\cdot\overline{\Lambda} + \alpha_6\mathbf{i}^*\cdot(\nabla\Theta\cdot F)$$

where $\beta_1, \ldots, \beta_7$ and $\alpha_1, \ldots, \alpha_6$ are scalar-valued functions defined on G.

6. DISCUSSION

6.1. We have given an absolute version of the Chapman–Enskog iterative procedure. As a result, we have obtained absolute forms of the first and second approximations for the constitutive functions. Our zeroth and first approximations are essentially the usual ones. The second approximations contain a term (the last one) missing from the usual formulas. If the body force does not depend on velocity [mF takes values in $\mathbf{M}^* \wedge (\mathbf{I}^*\cdot\boldsymbol{\tau})$], then its contractions with spacelike vectors are zero; thus in the usually considered case the last terms vanish. Apart from these last terms, the form of the second approximations is formally the usual one; however, our formulas concern absolute quantities only, including *absolute velocity.*

6.2. Let us relate our absolute quantities to an observer U in order to obtain a comparison with the usual formulas.

Now

$$v := u - U$$

is the relative gross velocity field considered usually. To get the constitutive functions relative to the observer U, we have to use the relative velocity v, i.e., to replace u with $v + U$ (and ∇u with $\nabla v + \nabla U$) in our absolute constitutive functions.

If the observer is inertial, i.e., there is a $c \in V(1)$ such that $U = c$, then $\nabla U = 0$, $\nabla u = \nabla v$, and we get at once the formulas of Chapman (1970) in which the observer does not appear explicitly.

If the observer is rigid but noninertial, then there is a nonzero map Ω: $I \to \mathbf{E} \wedge \mathbf{E}/\mathbf{I}$ such that

$$U(x + \boldsymbol{r}) - U(x) = \Omega(\tau(x))\cdot\boldsymbol{r}, \qquad x \in M, \quad \boldsymbol{r} \in \mathbf{E}$$

(Matolcsi, 1993, I.4.2.4.). In other words

$$\nabla U(x) = \Omega(\tau(x))$$

i.e., $\nabla U(x)$ is antisymmetric for all $x \in M$; this is the angular velocity of the observer, denoted usually by $\mathbf{W}$. Since ∇U is antisymmetric, the symmetric parts of $\nabla\boldsymbol{u}$ and $\nabla\boldsymbol{v}$ coincide. Consequently, in our absolute formulas the symmetric part of $\nabla\boldsymbol{u}$ can be replaced with the symmetric part of $\nabla\boldsymbol{v}$; on the other hand, $\nabla\boldsymbol{u}$ is to be replaced with $\nabla\boldsymbol{v} + \nabla U$, so together with $\nabla\boldsymbol{v}$ *the angular velocity of the observer will be present in the relative constitutive functions*; we find at once that these relative constitutive functions contain the well-known inertial terms.

6.3. The principle of material frame indifference has been debated for more than three decades. The validity of this principle was questioned by several authors (Müller, 1972, 1976, 1985; Hoover *et al.*, 1981) according to whom material frame indifference and the kinetic theory of gases are incompatible. Other authors claimed that some arguments against frame indifference were not convincing (Wang, 1975; Speziale, 1981) and introducing some supplementary terms (Murdoch, 1983) or defining a new sort of time derivative (Boukary and Lebon, 1985), they tried to demonstrate that frame indifference is not contradicted by kinetic theory. Their arguments, however, are not convincing, and it is stated in recent publications (Müller and Ruggeri, 1993; Jou *et al.*, 1993) that material frame indifference is violated by kinetic theory and holds only approximatively.

Let us recall the main concepts of material frame indifference.

The *principle* of material frame-indifference states that "stress and heat flux are related to the fields of density, velocity and temperature in a manner dependent solely on the material" (Müller, 1972), and "the properties which characterize any given material should be independent of observer" (Murdoch, 1983).

A *mathematical formulation* was given to the principle (Noll, 1973), whose clearest composition reads as follows (Müller, 1985):

> . . .the constitutive functions . . . are the same ones for observers in inertial frames and in non-inertial ones. . . The two observers see different values of the independent variables in the constitutive equations, namely
>
> $$\rho,\ v_r,\ T,\ \frac{\partial v_r}{\partial x_s},\ \frac{\partial T}{\partial x_r} \qquad \rho,\ v_r^*,\ T,\ \frac{\partial v_r^*}{\partial x_s^*},\ \frac{\partial T}{\partial x_r^*},$$
>
> respectively. Similarly they see different values of the dependent variables stress and heat flux, namely t_{ij}, q_i and t_{ij}^*, q_i^*. It would be conceivable that the two observers also see a different relation between the different independent variables, so their respective constitutive equations would have the forms

$$t_{ij} = \hat{t}_{ij}\left(\rho, v_r, T, \frac{\partial v_r}{\partial x_s}, \frac{\partial T}{\partial x_s}\right), \qquad t_{ij}^* = \hat{t}_{ij}^*\left(\rho, v_r^*, T, \frac{\partial v_r^*}{\partial x_s^*}, \frac{\partial T}{\partial x_s^*}\right)$$

...with functions $\hat{t}_{ij}$ ... and $\hat{t}_{ij}^*$ ... respectively. However, the principle of material frame-indifference postulates that the constitutive functions are the same ones for both observers. That is to say that the principle forces us to drop the stars from the constitutive functions $\hat{t}_{ij}^*$ ....

6.4. Now let us suppose that a student says: "I do not see that the principle is well expressed by this mathematical formulation; on the contrary, I think that velocity and gradients depend so strongly on observers that the constitutive functions, too, must depend on observers in such a way that the dependence in velocity and gradients and the dependence in the constitutive functions compensate each other expressing the fact that the properties of a given material are independent of observers."

Can we refute this statement? Can the student prove it? The refutation or the proof seems impossible in the usual framework. However, if we rule out observers from the fundamentals of the theory, then the mathematical formulation of the principle will be trivial:

The properties of a given material are described by absolute objects; in other words, constitutive functions are absolute.

Then having found absolute constitutive functions and using the rule for how observers introduce their own relative objects, we can check whether the student is right or not.

6.5. In our description all the constitutive parameters—density, velocity, temperature, and their gradients—as well as the constitutive functions are absolute quantities, i.e., quantities not referring to observers (the clue is the *absolute velocity*). Since observers are not present at all in our formulas, "stress and heat flux are related to the fields of density, velocity and temperature in a manner dependent solely on the material," i.e., independently of observer: *the principle of material frame indifference is satisfied automatically.*

The absolute forms can be translated into the language of observers, i.e., constitutive functions relative to observers can be derived from the absolute ones as done in Section 6.2. and exactly the well-known inertial terms appear in the constitutive functions relative to a rotating observer.

6.6. According to these results we see that:

(i) *The kinetic theory does not contradict the principle of material frame indifference.*

(ii) *The kinetic theory contradicts the usual mathematical formulation of the principle.*

(iii) *The usual mathematical formulation does not express correctly the principle of material frame indifference:* the form of the constitutive functions *relative* to observers *does depend* on observers.

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