Lagrange-formalism of point-masses

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Abstract

We prove by symmetry properties that the Lagrangian of a free point-mass is a quadratic function of the speed in the non-relativistic case, and that the action of the free point-mass between two spacetime points is the proper time passed in the relativistic case. These well known facts are proved in a mathematically rigorous way with a frame independent treatment based on spacetime models introduced by Matolcsi. The arguments show that these results are not obvious at all, some common beliefs can be refuted by explicit counterexamples. In our treatment the similarity of non-relativistic and relativistic cases is apparent.

Introduction

Spacetime symmetries play a fundamental role in physical theories. We can distinguish between two points of view regarding their application. If governing equations are known, symmetries provide conserved quantities (symmetry charges) of the theory, conditions on their conservation and integrals of the governing equations. When the dynamics is given by variational principles the symmetries are exploited by Noether theorems. If governing equations are unknown, then symmetries are the basic tools to construct a proper dynamics (based on any kind of formulation) of the theory. The starting point is always to determine the Lagrangian of free systems (point-mass or field). A free system is invariant under all symmetries of the corresponding spacetime and it is completely determined by this property.

In classical mechanics, where the governing equation – the Newton equation – is well known one can exploit symmetries by Noether theorems (see e.g. [2, 7, 1]). On the other hand, in quantum field theories symmetries are common tools to construct appropriate Lagrangians (see. e.g. [17]). A similar – exceptional – approach in classical mechanics can be found in the books of Landau and Lifsic where the method is applied for finding the Lagrangian of a point-mass in non-relativistic dynamics [9] and partially also in relativistic dynamics [8].

That derivation (and all similar subsequent derivations in field theories) are unsatisfactory from several points of view. Although the Lagrangians are coordinate free, they are observer and therefore frame dependent. Moreover,

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it is not clear whether the action, the Lagrangian, the resulted Euler-Lagrange equations or the solutions of the Euler-Lagrange equations are the transformed objects that should be invariant under the symmetries. Different papers and textbooks give different answers, moreover these cases are frequently mixed (and contradictory) in the very same treatment.

We shall see that if the Lagrangian or the action itself were required to be invariant for all spacetime symmetries (Noether or Poincaré transformations), then we should get a constant function. Thus it is essential that instead of the Lagrangians, the invariant quantities should be the equivalence classes of the Lagrangians (two Lagrangians differing in a full time derivative are considered equivalent). Somewhere in the literature this problem is obliterated by referring to the fact that two Lagrangians in the same equivalence class can be transformed into each other with a canonical transformation [17, p26]. That is true, but it has nothing to do by invariance because we cannot transform the variables anyhow when investigating invariance under variable transformations (spacetime symmetries).

The situation is even more confused in the (special) relativistic case where the observer dependence of the Lagrangian is more apparent (there is no frame independent time). There symmetry considerations are essentially ignored [8] with the argument that the only invariant scalar is proper time. If symmetry is taken into account, the invariance of the solutions of Euler-Lagrange equations is required [18] resulting in a set of basically different Lagrange functions of a single free point-mass.

In this paper only the dynamics of *free* point-masses is considered, therefore we require invariance under the full spacetime symmetry group. Our treatment can give surprising results constructing dynamics of interacting systems, when only partial spacetime symmetries are required. First of all in our frame independent spacetime models it is apparent that some of the partial spacetime symmetries are observer dependent (there is no time translation, time inversion etc. without a frame), therefore they cannot be considered as fundamental. Independently of the above mentioned property of partial symmetries, our more exact study gives some real surprises: we shall show e.g. by an explicit example that spacetime translation symmetry alone does not imply momentum conservation.

In the present paper we investigate the precise restrictions of the possible forms of the Lagrangian of a free point-mass, implied by symmetry assumptions. We consider two Lagrangians equivalent if they result in the same Euler-Lagrange equations i.e. differ in a full time derivative. We use arguments both in relativistic and non-relativistic case which are correct from a mathematical point of view as well. We show that the problem in these two cases can in fact be handled in a very similar way. We use the spacetime model constructed without observers and reference frames, described in Matolcsi [11, 12], as a general and sophisticated tool for our observations. The formalism is based on a clever exploitation of the affine structure of non-relativistic and special relativistic spacetimes giving a method that is similar to the traditional tensorial one. A differential geometric treatment like [19] would be unnecessarily technical and not very well fitted to this problem. Another advantage of our treatment is to avoid misunderstandings based on the the well known problematic formulation of general covariance in non-relativistic and special relativistic theories [6]. Spacetime symmetries are frequently explained by the equivalence of inertial reference frames; the free point-mass is said to be invariant under translations, rotations and velocity transformations *of the reference frames*. Why just the movements of the *free* point-mass would be invariant under such changes? Working without reference frames of course excludes similar problems: spacetime transformations are really acting on the spacetime position and velocity of the particle.

2. Point-mass in non-relativistic spacetime

2.1 The non-relativistic spacetime

Here we recall some basic structures of the frame independent formalism described in Matolcsi [12]. All the vector spaces in question are real. The non-relativistic spacetime is $(M, I, \tau, \boldsymbol{D}, \boldsymbol{b})$, where

- M is *spacetime*, an oriented four-dimensional affine space over the vector space $\boldsymbol{M}$, i.e. the difference of two points in M is an element of $\boldsymbol{M}$. By orientation we mean a fixed ordering of basis vectors of $\boldsymbol{M}$.

- I is *time*, an oriented one-dimensional affine space over $\boldsymbol{I}$. The latter is a vector space, *the measure line of time intervals*. By its orientation, we have positive and negative time intervals. Between any two moments of I, the time interval is an element of $\boldsymbol{I}$.

- $\tau : M \to I$ is an affine surjection, making correspondence between a point of M and its absolute time in I. $\boldsymbol{\tau} : \boldsymbol{M} \to \boldsymbol{I}$ is the linear surjection under τ, connecting each vector to its time interval. These two functions are the *time evaluation functions*.

- $\boldsymbol{D}$ is an oriented one-dimensional vector space, *the measure line of distances*.

- $\boldsymbol{b} : \boldsymbol{E} \times \boldsymbol{E} \to \boldsymbol{D} \otimes \boldsymbol{D}$ is a positive definite symmetric bilinear mapping, the *Euclidean structure*, where $\boldsymbol{E} := \ker \boldsymbol{\tau}$ is the *subspace of the spacelike vectors*.

We obtain the usual coordinate-description by vectorizing M with an origin in M and a basis in $\boldsymbol{M}$. Coordinates are denoted by roman letters, and can have values 0, 1, 2, 3. These are written in superscript, while coordinates of the dual space $\boldsymbol{M}^*$ are put in subscripts. We use Einstein-convention for summing indices in superscripts and subscripts denoted by the same letter. Vectors in $\boldsymbol{E}$ have first coordinate zero, we denote the other three indices by Greek letters.

The history of a point-mass is described by a *world line*, a connected curve with timelike tangents (tangents not in $\boldsymbol{E}$). Such a world line can be given as the range of a *world line function*, a continuously differentiable function $r : I \to M$ defined on an interval, with $\tau(r(t)) = t$ for any $t \in \operatorname{Dom} r$. Using a construction similar to the one of tensor products, one can define the four-dimensional tensor quotient space $\frac{\boldsymbol{M}}{\boldsymbol{I}}$. The derivative of a world line function is $\dot{r} : I \to \frac{\boldsymbol{M}}{\boldsymbol{I}}$ with $\boldsymbol{\tau}(\dot{r}(t)) = 1$. Hence its values are elements of

$$V(1) := \left\{ \boldsymbol{u} \in \frac{\boldsymbol{M}}{\boldsymbol{I}} : \boldsymbol{\tau}(\boldsymbol{u}) = 1 \right\},$$

the set of *absolute velocity values*. $V(1)$ is an affine space over $\frac{\boldsymbol{E}}{\boldsymbol{I}}$, and $\ddot{r}(t) \in \frac{\boldsymbol{E}}{\boldsymbol{I}\otimes\boldsymbol{I}}$ if r is twice differentiable at $t \in I$.

We need the set of mass values, which we define as follows. $\boldsymbol{I}$ measures time lengths, it contains the *second*, while $\boldsymbol{D}$ measures distances, containing *meter*. For simplicity we choose $\hbar := 1$, hence mass values (e.g. $1\,\frac{\text{second}}{\text{meter}^2}$) are in $\frac{\boldsymbol{I}}{\boldsymbol{D}\otimes\boldsymbol{D}}$, the measure line of mass values [19].

2.2 Galilean and Noether transformations

Noether transformations are the automorphisms of the spacetime. These transformations keep the structure of spacetime, and can be described as follows. The *proper Galilean group* is

$$\mathcal{G} := \Big\{ \boldsymbol{L} \in \mathrm{Lin}(\boldsymbol{M},\boldsymbol{M}) \,:\, \boldsymbol{L} \text{ preserves orientation, } \boldsymbol{\tau}\cdot\boldsymbol{L} = \boldsymbol{\tau}, \\ \boldsymbol{L}|_{\boldsymbol{E}}{}^{+} \cdot \boldsymbol{L}|_{\boldsymbol{E}} = \mathrm{id}_{\boldsymbol{E}} \Big\}$$

acting on $\boldsymbol{M}$, and rotating spacelike vectors. The *proper Noether group* (inhomogeneous Galilean group) is

$$\mathcal{N} := \big\{ L \,:\, M \to M \,:\, L \text{ is affine, and the underlying } \boldsymbol{L} \text{ is an element of } \mathcal{G} \big\}.$$

The word proper refers to the fact that time or space inversion is not contained in these groups.

Spacetime translations i.e. transformations of the form $x \mapsto x + \boldsymbol{a}$ with a given $\boldsymbol{a} \in \boldsymbol{M}$ are Noether transformations, whose underlying linear operator is the identity of $\boldsymbol{M}$. $\mathcal{G}$ and $\mathcal{N}$ are a six dimensional and a ten dimensional Lie group, respectively. They have the Lie algebras

$$\begin{aligned} \mathrm{La}(\mathcal{G}) &= \big\{ \boldsymbol{H} \in \mathrm{Lin}(\boldsymbol{M},\boldsymbol{M}) \,:\, \boldsymbol{\tau}\cdot\boldsymbol{H} = 0,\, \boldsymbol{H}|_{\boldsymbol{E}}^{+} = -\boldsymbol{H}|_{\boldsymbol{E}} \big\} \text{ and} \\ \mathrm{La}(\mathcal{N}) &= \big\{ H \,:\, M \to \boldsymbol{M} \,:\, H \text{ affine, and the underlying } \boldsymbol{H} \text{ is in } \mathrm{La}(\mathcal{G}) \big\}, \end{aligned}$$

respectively. By its definition, an $\boldsymbol{H} \in \mathrm{La}(\mathcal{G})$ is in fact an $\boldsymbol{M} \to \boldsymbol{E}$ linear map. The Lie algebra of the subgroup of spacetime translations consists of elements H for which $\boldsymbol{H} = 0$. Such an affine map is constant, i.e. there is an $\boldsymbol{h} \in \boldsymbol{M}$ such that $Hx = \boldsymbol{h}$ for all $x \in Mw$.

Every Noether transformation L in a neighborhood of the unit element id_M has the form

$$L = \mathrm{e}^{sH} := I + \sum_{n=1}^{\infty} \frac{(s\boldsymbol{H})^{n-1}\cdot sH}{n!} \tag{1}$$

for some $s \in \mathbb{R}$ and $H \in \mathrm{La}(\mathcal{N})$. The underlying Galilean transformation is

$$\boldsymbol{L} = \mathrm{e}^{s\boldsymbol{H}} := \sum_{n=0}^{\infty} \frac{(s\boldsymbol{H})^n}{n!}.$$

2.3 Variational principle for point-masses in non-relativistic spacetime

According to variational principles of mechanics, the point-mass moves along a world line from a spacetime point x_0 to another one x_1, for which the "variation" of an "action function" is zero, possibly caused by having extremum or stationary value of the action function at this world line. In case of hamiltonian variational principles the initial and final spacetime points are fixed (the duration and the initial and final space points are not varied). We formulate this as follows.

Given a Lagrangian depending continuously on spacetime points and speed values, and mapping to the one-dimensional vector space $\frac{\mathbb{R}}{\boldsymbol{I}}$:

$$\mathfrak{L} : M \times V(1) \to \frac{\mathbb{R}}{\boldsymbol{I}},$$

the action on an r world line function is

$$S(r) := \int_{t_0}^{t_1} \mathfrak{L}\big(r(t), \dot{r}(t)\big) dt$$

with $t_0 := \tau(x_0)$ and $t_1 := \tau(x_1)$.

In order to use analysis arguments, we define differentiability of the function $r \mapsto S(r)$. Taking a norm $|\ |_{\boldsymbol{M}}$ on $\boldsymbol{M}$ and a norm $|\ |_{\frac{\boldsymbol{M}}{\boldsymbol{I}}}$ on $\frac{\boldsymbol{M}}{\boldsymbol{I}}$ (any two norms on a finite dimensional vector space are equivalent). We introduce the vector space

$$\boldsymbol{V} := \{\boldsymbol{r} : [t_0, t_1] \to \boldsymbol{E} \mid \boldsymbol{r} \text{ is continuously differentiable}, \boldsymbol{r}(t_0) = \boldsymbol{r}(t_1) = 0\} \tag{2}$$

endowed with the norm

$$\|\boldsymbol{r}\| := \max_{t \in [t_0, t_1]} \left(|\boldsymbol{r}(t)|_{\boldsymbol{M}} + |\dot{\boldsymbol{r}}(t)|_{\frac{\boldsymbol{M}}{\boldsymbol{I}}} \right).$$

Then

$$V := \{r : [t_0, t_1] \to M \mid r \text{ is a world line function},\ r(t_0) = x_0,\ r(t_1) = x_1\}$$

is an affine space over $\boldsymbol{V}$. Hence differentiability of $S : V \to \mathbb{R}$ is well-defined.

If S is differentiable, the world line function realized by the point-mass is selected by

$$\mathrm{D}S(r) = 0 \in \mathrm{Lin}(\boldsymbol{V}, \mathbb{R}),$$

i.e. the derivative of S having value zero. This corresponds to the "action having variation zero". It is well known, that if the Lagrangian $\mathfrak{L}$ is twice continuously differentiable, then S is differentiable, and in this case, $\mathrm{D}S(r) = 0$ is equivalent to twice continuous differentiability of r satisfying the Euler-Lagrange equation

$$\mathrm{D}_1\mathfrak{L}(r(t), \dot{r}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{D}_2\mathfrak{L}(r(t), \dot{r}(t)) = 0.$$

Here D_1 stands for the partial derivative according to the first variable in M, and D_2 for the derivative according to the second variable in $V(1)$. From now on, we denote elements of M by x, and elements of $V(1)$ by u, therefore we write

$$\frac{\partial \mathfrak{L}(x,u)}{\partial x} \in \mathrm{Lin}(\boldsymbol{M}, \mathbb{R}/\boldsymbol{I}), \qquad \frac{\partial \mathfrak{L}(x,u)}{\partial u} \in \mathrm{Lin}(\boldsymbol{E}/\boldsymbol{I}, \mathbb{R}/\boldsymbol{I})$$

for the partial derivatives, respectively.

By the construction of this variational principle, it is clear that adding a "full time-derivative" to $\mathfrak{L}$ only means adding a constant to S, hence leaving $\mathrm{D}S$ invariant together with the world line realized. We give precise meanings of these notions.

A function $\mathfrak{f} : M \times V(1) \to \frac{\mathbb{R}}{\boldsymbol{I}}$ is called a *full time-derivative* if there exists a $\phi : M \to \mathbb{R}$ continuously differentiable function, such that $\mathfrak{f}(x, u) = \mathrm{D}\phi(x)u$ for all $x \in M$ and $u \in V(1)$. For any world line function r, $\mathfrak{f} \circ (r, \dot{r}) = (\phi \circ r)^{\cdot}$ holds in this case, hence the action corresponding to $\mathfrak{L}$ and to $\mathfrak{L} + \mathfrak{f}$ only differ by a constant. We say that $\mathfrak{L}$ and $\mathfrak{L}'$ are *equivalent*, if $\mathfrak{L}' - \mathfrak{L}$ is a full time-derivative. This relation determines equivalence classes on the set of Lagrangians.

2.4 Symmetries and the Lagrangian

A motion of a physical system happens at the same way before and after a transformation, if the solution of the Euler-Lagrange equation is not affected by the transformation. We assume that not only these solutions, but the derivative $\mathrm{D}S$ characterizes as well the physics of the system. Hence we say that a transformation is a symmetry of the system, if it leaves the derivative $\mathrm{D}S$ invariant, i.e. it only turns the Lagrangian into an equivalent one. This is a rather restrictive standpoint, there are several examples for variational principles where only the equivalence of the solutions is required e.g. in non-equilibrium thermodynamics where the governing equations cannot be derived from a variational principle, therefore the usual variational prescriptions are shaken up [20], but also in mechanics there are attempts to find physical consequences of that fact investigating the so called s-equivalent systems ([3] and the references therein). Now we exclude transformations leaving the solution invariant, but multiplying $\mathfrak{L}$ by a constant for example.

Let $F : M \to M$ be a continuously differentiable map, for which $\boldsymbol{\tau} \cdot \mathrm{D}F(x)u \neq 0$ for any $u \in V(1)$. We say that F is a *symmetry* of the Lagrangian $\mathfrak{L}$, if there exists a full time-derivative $\mathfrak{f}_F$ such that

$$\mathfrak{L}\left(Fx,\, \frac{\mathrm{D}F(x)u}{\boldsymbol{\tau} \cdot \mathrm{D}F(x)u}\right) \boldsymbol{\tau} \cdot \mathrm{D}F(x)u = \mathfrak{L}(x,\, u) + \mathfrak{f}_F(x,\, u) \tag{3}$$

for any $x \in M,\ u \in V(1)$. The definition considers that the transformation can change the (absolute) time and assures that a transformed world line function remain a world line function after the transformation (e.g. the second variable of $\mathfrak{L}$ is an element of $V(1)$ and the integration should change to leave DS invariant after a reparameterization of the time scale). The definition is more transparent in case of symmetries that do not rescale the absolute time ($\boldsymbol{\tau} \cdot \mathrm{D}F(x)u = 1$). The above definition is valid uniformly in special relativistic and non-relativistic considerations, too. Non-relativistic space-time symmetries do not rescale the (absolute) time. Especially, a proper Noether transformation is a symmetry of $\mathfrak{L}$, if and only if

$$\mathfrak{L}(Lx,\, \boldsymbol{L}u) = \mathfrak{L}(x,\, u) + \mathfrak{f}_L(x,\, u),$$

since $\mathrm{D}L(x) = \boldsymbol{L}$, and $\boldsymbol{\tau} \cdot \boldsymbol{L} = \boldsymbol{\tau},\ \boldsymbol{\tau}u = 1$ by definition of the proper Galilean group and of $V(1)$.

2.5 Lagrangian of a free point-mass

If a point-mass is free, i.e. it is not influenced by any effect, then we "feel" that the translated, rotated etc. form of its trajectory is also a possible trajectory for it. To be more precise, we could say that by applying a spacetime-automorphism on a trajectory selected by the variational principle, we obtain another trajectory satisfying that principle. It is still not a precise statement, since it is not clear how to understand a "free" point-mass, "not influenced by any effect". We reverse the situation, and accept this concept as a definition.

We define a point-mass characterized by $\mathfrak{L}$ to be a *free point-mass*, if each proper Noether transformation is a symmetry of $\mathfrak{L}$. Hence $\mathfrak{L}$ is the Lagrangian of a free point-mass if and only if for any $L \in \mathcal{N}$ there exists a $\phi_L : M \to \mathbb{R}$, for which

$$\mathfrak{L}(Lx,\, \boldsymbol{L}u) - \mathfrak{L}(x,\, u) = \mathrm{D}\phi_L(x)u.$$

We introduce the notation $\widehat{\phi}(L,\, x) := \phi_L(x)$, and we assume that $\widehat{\phi} : \mathcal{N} \times M \to \mathbb{R}$ is smooth enough. Although $\mathcal{N} \times M$ is not an affine space thus $\widehat{\phi}$ is defined on a manifold, we only consider one-parameter subgroups of $\mathcal{N}$, hence we can use the usual differentiability notions. We take the elements in the neighborhood of $I := \mathrm{id}_M$ in the form (1), and we differentiate by the parameter s. Since

$$\left.\frac{\mathrm{d}e^{sH}}{\mathrm{d}s}\right|_{s=0} = H \quad \text{and} \quad \left.\frac{\mathrm{d}e^{s\boldsymbol{H}}}{\mathrm{d}s}\right|_{s=0} = \boldsymbol{H},$$

we obtain

(4)
$$\frac{\partial \mathfrak{L}(x,\, u)}{\partial x} \cdot H(x) + \frac{\partial \mathfrak{L}(x,\, u)}{\partial u} \cdot \boldsymbol{H} \cdot u = \left.\frac{\partial^2 \widehat{\phi}(L,\, x)}{\partial L\, \partial x}\right|_{L=I} \cdot (H,\, u) =: \frac{\partial \omega(H,\, x)}{\partial x} \cdot u$$

by letting $s \to 0$.

In order to compare our frame independent formulae with those of usual treatments, we write the coordinated forms of our expressions. We coordinate $\boldsymbol{M}$ by an appropriate basis, we vectorial M by the map $x \mapsto x - o$ with a fixed point $o \in M$, and we consider the vector-coordinates of these vectors. Then $H(x) = \boldsymbol{H}(x - o) + \boldsymbol{h}$, where $\boldsymbol{h} := H(o) \in \boldsymbol{M}$. If $\boldsymbol{H}$ has coordinates $H^i{}_j$, then $H(x)$ has coordinates $H^i{}_j x^j + h^i$. Let us remark here that sometimes one think on coordinates as a convenient tool for expressing tensorial calculations without the corresponding reference frames (see the concept of "abstract indexes" of Wald [22]). However, a formulation of general covariance (observer independence) with an observer dependent notation easily can lead to misinterpretation because a frame independent equation can lead to a formula containing observer dependent quantities in a particular reference frame (especially in non-relativistic spacetime, see the debate on the covariance of the kinetic theory e.g. [14, 15, 10, 13, 16]). Therefore, although there is no convenient notation to book the different transposes of higher order tensors without indexes, it is important to formulate the results of the calculations in our frame independent notation, too.

$V(1)$ is an affine subspace in $\frac{\boldsymbol{M}}{\boldsymbol{I}}$, the four coordinates of its elements are not independent, i.e. the zeroth coordinates are 1 in this space. Hence the derivatives by elements of $V(1)$ only contain indices 1, 2, 3. We will distinguish these possibilities in the notation. Greek letters are in $\{1,2,3\}$ and Latin indexes in $\{0,1,2,3\}$. Double indexes denote summation. Therefore, the coordinated form of (4) is

$$\frac{\partial \mathfrak{L}}{\partial x^i}(H^i{}_j x^j + h^i) + \frac{\partial \mathfrak{L}}{\partial u^\alpha} H^\alpha{}_j u^j = \frac{\partial \omega}{\partial x^i} u^i,$$

without writing the arguments of our functions.

First we consider the special case of spacetime-translations. Then $\boldsymbol{H} = 0$, hence $H(x) = \boldsymbol{h} \in \boldsymbol{M}$ is the same constant for each $x \in M$. Therefore, we can write (4) in the form

$$\frac{\partial \mathfrak{L}(x,\, u)}{\partial x} \cdot \boldsymbol{h} = \frac{\partial \omega(\boldsymbol{h},\, x)}{\partial x} \cdot u \quad , \qquad \frac{\partial \mathfrak{L}}{\partial x^i} h^i = \frac{\partial \omega}{\partial x^i} u^i.$$

The left hand-side is linear in $\boldsymbol{h}$, the right hand-side is linear in u. Hence the other sides also have these properties. Thus it follows that there are functions $l, f : M \to \boldsymbol{M}^*$ for which

$$\frac{\partial \omega(\boldsymbol{h},\, x)}{\partial x} = \boldsymbol{h} \cdot Df(x) \quad , \qquad \frac{\partial \omega}{\partial x^i} = \frac{\partial f_j}{\partial x^i} h^j,$$

and

$$\frac{\partial \mathfrak{L}(x,\, u)}{\partial x} = u \cdot Dl(x) \quad , \qquad \frac{\partial \mathfrak{L}}{\partial x^i} = \frac{\partial l_j}{\partial x^i} u^j.$$

Therefore,

$$Dl(x) = (Df)^*(x) \quad , \qquad \frac{\partial l_i}{\partial x^j} = \frac{\partial f_j}{\partial x^i}. \tag{5}$$

Assuming twice differentiability of $\mathfrak{L}$, we differentiate (5) by x^k. Changing the order of differentiation and applying (5) again we obtain by Young's theorem

$$\frac{\partial^2 f_j}{\partial x^k \partial x^i} = \frac{\partial^2 l_i}{\partial x^k \partial x^j} = \frac{\partial^2 l_i}{\partial x^j \partial x^k} = \frac{\partial^2 f_k}{\partial x^j \partial x^i}.$$

As a result, we can get that

$$D(Df - (Df)^*)(x) = 0 \;, \qquad \frac{\partial}{\partial x^i}\left(\frac{\partial f_j}{\partial x^k} - \frac{\partial f_k}{\partial x^j}\right) = 0.$$

Hence introducing the antisymmetric linear map $\boldsymbol{C} : \boldsymbol{M} \to \boldsymbol{M}^*$ with components C_{jk}, we obtain

$$\boldsymbol{C} := -\mathrm{D} \wedge f := \mathrm{D}f - (\mathrm{D}f)^* = \text{const.} \;, \qquad C_{jk} := \frac{\partial f_j}{\partial x^k} - \frac{\partial f_k}{\partial x^j} = \text{const.} \tag{6}$$

Therefore

$$Dl(x) = Df(x) + \boldsymbol{C}^* \;, \qquad \frac{\partial l_j}{\partial x_k} = \frac{\partial f_j}{\partial x_k} + C_{kj}. \tag{7}$$

We conclude

$$\mathfrak{L}(x,\, u) = f(x) \cdot u + (x - o) \cdot \boldsymbol{C} \cdot u + \varphi_f(u) \;, \quad \mathfrak{L} = f(x)_k u^k + x^k C_{kj} u^j + \varphi_f(u),$$

with an arbitrary function $\varphi_f : V(1) \to \mathbb{R}/\boldsymbol{I}$. Exploiting (7) one may write also

$$\mathfrak{L}(x,\, u) = l(x)\cdot u + \varphi_l(u) \quad , \quad \mathfrak{L} = l(x)_k u^k + \varphi_l(u),$$

with an arbitrary function $\varphi_l : V(1) \to \mathbb{R}/\boldsymbol{I}$. Hence we obtained from the spacetime translation symmetry that

$$\mathfrak{L}(x,\, u) = l(x)\cdot u + \varphi(u) \quad , \qquad \mathfrak{L} = l(x)_k u^k + \varphi(u), \tag{8}$$

where l is a function satisfying

$$D \wedge l = \mathbf{C} = \text{const.}\ , \qquad \frac{\partial^2 l_j}{\partial x^k \partial x^i} = \frac{\partial^2 l_i}{\partial x^k \partial x^j}. \tag{9}$$

This is all we can say; we could not prove the statement of [9, page 13]; from spacetime translation symmetry does not follow the independence of the Lagrangian of spacetime variables.

Now we consider general Noether transformations, and substitute the form (8) of the Lagrangian into (4) (recall that $\boldsymbol{H}$ is a map $\boldsymbol{M} \to \boldsymbol{E}$):

$$\frac{\partial l(x)\cdot u}{\partial x}\cdot H(x) + l(x)\cdot \boldsymbol{H}\cdot u + \frac{\mathrm{d}\varphi(u)}{\mathrm{d}u}\cdot \boldsymbol{H}\cdot u = \frac{\partial \omega(H,x)}{\partial x}\cdot u,$$

i.e.

$$\frac{\partial l_k}{\partial x^i} u^k (H^i{}_j x^j + h^i) + l_\alpha H^\alpha{}_k u^k + \frac{\partial \varphi}{\partial u^\alpha} H^\alpha{}_k u^k = \frac{\partial \omega}{\partial x^k} u^k. \tag{10}$$

Differentiating this by x^m leads to

$$\frac{\partial^2 l_k}{\partial x^m \partial x^i} u^k (H^i{}_j x^j + h^i) + \frac{\partial l_k}{\partial x^i} u^k H^i{}_m + \frac{\partial l_\alpha}{\mathrm{d} x^m} H^\alpha{}_k u^k = \frac{\partial^2 \omega}{\partial x^m \partial x^k} u^k.$$

The variable u is only present linearly in this equation, hence we can omit it:

$$\frac{\partial^2 l_k}{\partial x^m \partial x^i} (H^i{}_j x^j + h^i) + \frac{\partial l_k}{\partial x^i} H^i{}_m + \frac{\partial l_\alpha}{\mathrm{d} x^m} H^\alpha{}_k = \frac{\partial^2 \omega}{\partial x^m \partial x^k}. \tag{11}$$

We transpose the equation (consider it with the indices k and m interchanged), and subtract it from the original form (11). Then the right hand-side is zero by Young's theorem, and the first term on the left hand-side disappears due to (9):

$$\frac{\partial l_k}{\partial x^i} H^i{}_m - \frac{\partial l_m}{\partial x^i} H^i{}_k + \frac{\partial l_\alpha}{\partial x^m} H^\alpha{}_k - \frac{\partial l_\alpha}{\partial x^k} H^\alpha{}_m = 0.$$

As we know, $\boldsymbol{H}$ is a linear map $\boldsymbol{M} \to \boldsymbol{E}$, hence $H^0{}_k = 0$. Therefore, we can sum over indices $\alpha = 1,2,3$ instead of $i = 0,\ldots,3$ contained in the expressions $H^i{}_k$:

$$\frac{\partial l_k}{\partial x^\alpha} H^\alpha{}_m - \frac{\partial l_m}{\partial x^\alpha} H^\alpha{}_k + \frac{\partial l_\alpha}{\partial x^m} H^\alpha{}_k - \frac{\partial l_\alpha}{\partial x^k} H^\alpha{}_m = 0,$$

i.e.

$$C_{k\alpha} H^\alpha{}_m - C_{\alpha m} H^\alpha{}_k = 0. \tag{12}$$

We know that restricting $\boldsymbol{H}$ to $\boldsymbol{E}$ results in an antisymmetric map, and by the identification $\boldsymbol{E}^* \equiv \frac{\boldsymbol{E}}{\boldsymbol{D}\otimes\boldsymbol{D}}$, we can interchange subscripts and superscripts of spacelike vectors: $H^\alpha{}_\beta = -H_\beta{}^\alpha$. Thus separating the cases $k = \beta = 1, 2, 3$ and $k = 0$, we obtain for $m = \omega = 1, 2, 3$,

$$C_{\beta\alpha} H^\alpha{}_\omega - C_{\omega\alpha} H^\alpha{}_\beta = 0, \tag{13}$$

$$C_{0\alpha} H^\alpha{}_\omega + C_{\omega\alpha} H^\alpha{}_0 = 0. \tag{14}$$

Let $\boldsymbol{i} : \boldsymbol{E} \to \boldsymbol{M}$ be the embedding map; its transpose $\boldsymbol{i}^* : \boldsymbol{M}^* \to \boldsymbol{E}^*$ is a linear surjection. Then $\boldsymbol{C_E} := \boldsymbol{i}^* \boldsymbol{C} \boldsymbol{i} : \boldsymbol{E} \to \boldsymbol{E}^* \equiv \dfrac{\boldsymbol{E}}{\boldsymbol{D}\otimes\boldsymbol{D}}$ is an antisymmetric linear map. Equation (13) tells us that this commutes with any antisymmetric linear map $\boldsymbol{H}|_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E}$:

$$[\boldsymbol{C_E},\, \boldsymbol{H}|_{\boldsymbol{E}}] = 0.$$

As a consequence, $\boldsymbol{C_E}$ commutes with all the elements of the rotation group of $\boldsymbol{E}$, hence by Schur's lemma, it is a multiple of $\mathrm{id}_{\boldsymbol{E}}$, meanwhile it is antisymmetric. This is only possible if

$$\boldsymbol{C_E} = 0 \quad , \qquad C_{\alpha\beta} = 0.$$

Then, by (14), $C_{0\alpha} H^\alpha{}_\omega = 0$ for each $\boldsymbol{H}$, which implies

$$C_{0\alpha} = 0,$$

finally

$$\boldsymbol{C} = 0.$$

Then (6) implies that the derivative of l is symmetric (i.e. the antisymmetric derivative is zero), hence l is a derivative of some function $\phi : M \to \mathbb{R}$:

$$l = \mathrm{D}\phi \quad , \qquad l_k = \frac{\partial \phi}{\partial x^k}.$$

Thus the first term in expression (8) of the Lagrangian is a full time-derivative, which can be omitted. We conclude

$$\mathfrak{L}(x,\, u) = \varphi(u). \tag{15}$$

Now it is clear that we needed not only translation invariance, but rotation invariance as well in order to exclude spacetime dependence of the Lagrangian. Writing (4) again, now we get

$$\mathrm{D}\varphi(u) \cdot \boldsymbol{H} \cdot u = \frac{\partial \omega(H,\, x)}{\partial x} \cdot u \quad , \qquad \frac{\partial \varphi}{\partial u^\alpha} H^\alpha{}_j u^j = \frac{\partial \omega}{\partial x^i} u^i. \tag{16}$$

First we consider maps $\boldsymbol{H}$, for which $\boldsymbol{H}|_{\boldsymbol{E}} = 0$. For each of these maps, there exists a $\boldsymbol{v} \in \frac{\boldsymbol{E}}{\boldsymbol{I}}$, such that $\boldsymbol{H} \cdot u = \boldsymbol{v}$ for all $u \in V(1)$. Then

$$\mathrm{D}\varphi(u) \cdot \boldsymbol{v} = \frac{\partial \omega(H,\, x)}{\partial x} \cdot u \quad , \qquad \frac{\partial \varphi}{\partial u^\alpha} v^\alpha = \frac{\partial \omega}{\partial x^i} u^i.$$

The left hand-side does not depend on x, and contains $\boldsymbol{v}$ linearly. Hence the same holds for the right hand-side as well. Therefore there is a linear map $\boldsymbol{A} : \frac{\boldsymbol{M}}{\boldsymbol{I}} \to \boldsymbol{E}^* \equiv \frac{\boldsymbol{E}}{\boldsymbol{D}\otimes\boldsymbol{D}}$ such that

$$\mathrm{D}\varphi(u) = \boldsymbol{A} \cdot u \quad , \qquad \frac{\partial\varphi}{\partial u^\alpha} = A_{\alpha i} u^i \ .$$

Let $\boldsymbol{A}_{\boldsymbol{E}}$ denote the restriction of $\boldsymbol{A}$ onto $\frac{\boldsymbol{E}}{\boldsymbol{I}}$; then fixing an element c of $V(1)$ and putting $a := \boldsymbol{A}c$, we have

$$\mathrm{D}\varphi(u) = \boldsymbol{A}_{\boldsymbol{E}} \cdot (u - c) + a \ , \qquad \frac{\partial\varphi}{\partial u^\alpha} = A_{\alpha\beta}(u^\beta - c^\beta) + a_\alpha \ .$$

Differentiating by u^β, we infer from Young's theorem that $\boldsymbol{A}_{\boldsymbol{E}}$ is symmetric i.e. $A_{\alpha\beta} = A_{\beta\alpha}$ and a simple calculation results in

$$\varphi(u) = \frac{1}{2}\big((\boldsymbol{A}_{\boldsymbol{E}} \cdot (u - c)\big) \cdot (u - c) + a \cdot (u - c) \ + \mathrm{const.}.$$

The last terms here are full time-derivatives, which can be omitted. Using this form let us return to formula (4):

$$(\boldsymbol{A}_{\boldsymbol{E}}(u - c)) \cdot \boldsymbol{H} \cdot u = \frac{\partial\omega(H, x)}{\partial x} \cdot u \quad .$$

The left hand side is zero at $u = c$, thus the right hand side, too, hence the previous formula can be written as

$$(\boldsymbol{A}_{\boldsymbol{E}} \cdot (u - c)) \cdot \boldsymbol{H} \cdot (u - c) = g(H, x) \cdot (u - c) \ .$$

The left hand side is bilinear in $u - c$, the right hand side is linear; this is possible only if the right hand side is zero. Note that here in fact the restriction of $\boldsymbol{H}$ onto $\boldsymbol{E}$ appears; the well-known properties of the antisymmetric maps $\boldsymbol{H}|_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E}$ imply that $(\boldsymbol{A}_{\boldsymbol{E}} \cdot (u - c)) \cdot \boldsymbol{H} \cdot (u - c) = 0$ can hold for all $\boldsymbol{H}$ only if $\boldsymbol{A}_{\boldsymbol{E}}(u - c)$ is parallel to $(u - c)$ i.e. there is an $m \in \dfrac{\boldsymbol{I}}{\boldsymbol{D}\otimes\boldsymbol{D}}$ (mass value) such that

$$\varphi(u) = \frac{1}{2} m |u - c|^2 \quad .$$

2.6 Reformulating the variational principle

We show a method which we could have used but which would not have had any advantage. We only introduce this method here because we need to apply it later in the relativistic case, and we can show similarities to that case.

A world line of a point-mass has been given as the range of a world line function so far. This is in fact not necessary, since we can parameterize such a curve in many different ways. Let $\mathbf{1}$ be any fixed positive element of $\boldsymbol{I}$, and for a world line function r, let $p : [\mathbf{0}, \mathbf{1}] \to \mathrm{Ran}\, r$ be a parameterization for the corresponding world line. By the inverse function theorem, $\sigma := r^{-1} \circ p : [\mathbf{0}, \mathbf{1}] \to [t_0, t_1]$ is a continuously differentiable bijection, and

$$\dot{p} = (\dot{r} \circ \sigma)\dot{\sigma},$$

for which applying $\boldsymbol{\tau}$ we obtain

$$0 < \boldsymbol{\tau} \circ \dot{p} = \dot{\sigma}.$$

The function σ is given by p (as a primitive function of $\boldsymbol{\tau} \circ \dot{p}$), hence r is also determined by p. We have

$$r \circ \sigma = p \quad , \quad \dot{r} \circ \sigma = \frac{\dot{p}}{\boldsymbol{\tau} \circ \dot{p}} \quad ,$$

and

$$\ddot{r} \circ \sigma = \frac{\ddot{p}}{(\boldsymbol{\tau} \circ \dot{p})^2} - \frac{\dot{p}\, \boldsymbol{\tau} \circ \ddot{p}}{(\boldsymbol{\tau} \circ \dot{p})^3}$$

holds as well in case of twice differentiability. We see that r satisfies a second-order differential equation if and only if p does so.

The action according to this new parameterization can be computed using integral transformation:

$$\text{(17)} \qquad \int_{t_0}^{t_1} \mathfrak{L}\big(r(t),\, \dot{r}(t)\big)\, \mathrm{d}t = \int_{\mathbf{0}}^{\mathbf{1}} \mathfrak{L}\big(r(\sigma(a)),\, \dot{r}(\sigma(a))\big)\dot{\sigma}(a)\, \mathrm{d}a =$$

$$= \int_{\mathbf{0}}^{\mathbf{1}} \mathfrak{L}\left(p(a),\, \frac{\dot{p}(a)}{\boldsymbol{\tau} \cdot \dot{p}(a)}\right) \boldsymbol{\tau} \cdot \dot{p}(a)\, \mathrm{d}a.$$

According to the new parameterization, $\dot{p}$ is allowed to be an arbitrary future-like vector, not only an element of $V(1)$. Let $N^{\rightarrow} \subset \frac{\boldsymbol{M}}{\boldsymbol{I}}$ denote the set of future-like vectors. This is an open set containing $V(1)$. By the integral transformation above, the Lagrangian can be considered as a function

$$\mathfrak{L} \,:\, M \times N^{\rightarrow} \to \frac{\mathbb{R}}{\boldsymbol{I}} \quad , \quad (x,\, w) \mapsto \mathfrak{L}(x,\, w)$$

having the property

$$\mathfrak{L}(x,\, w) = \mathfrak{L}\left(x,\, \frac{w}{\boldsymbol{\tau} \cdot w}\right) \boldsymbol{\tau} \cdot w.$$

Hence the Lagrangian $\mathfrak{L} : M \times V(1) \to \frac{\mathbb{R}}{\boldsymbol{I}}$ used so far is now extended to a function $M \times N^{\rightarrow} \to \frac{\mathbb{R}}{\boldsymbol{I}}$ according to (17); the action itself only depends on the world lines, not on their parameterization.

Now we consider the vector space

$$\boldsymbol{V} := \{\boldsymbol{p} : [\mathbf{0},\, \mathbf{1}] \to \boldsymbol{M} \mid \boldsymbol{p} \text{ is continuously differentiable, } \boldsymbol{p}(\mathbf{0}) = \boldsymbol{p}(\mathbf{1}) = 0\},$$

endowed with the norm

$$||\boldsymbol{p}|| := \max_{s \in [\mathbf{0},\, \mathbf{1}]} (|\boldsymbol{p}(s)|_{\boldsymbol{M}} + |\dot{\boldsymbol{p}}(s)|_{\frac{\boldsymbol{M}}{\boldsymbol{I}}})$$

and the affine space

$$V := \{p : [\mathbf{0}, \mathbf{1}] \to M \mid p \text{ is continuously differentiable, } p(\mathbf{0}) = x_0,\, p(\mathbf{1}) = x_1\}$$

over $\boldsymbol{V}$. For the action

$$S : V \to \mathbb{R} \quad , \quad p \mapsto \int_{\mathbf{0}}^{\mathbf{1}} \mathfrak{L}(p(s),\, \dot{p}(s))\, \mathrm{d}s,$$

we can repeat everything we said before, and its extremal points give the extremal points of the original action function.

The original Lagrangian is a restriction of this new one to the set $M \times V(1)$. Hence it is clear that the extended Lagrangians $\mathfrak{L}$ and $\mathfrak{L}'$ differ by a full time-derivative if and only if their restrictions do so: $\mathfrak{L}'(x,\, u) - \mathfrak{L}(x,\, u) = \mathrm{D}\varphi(x)u$ for all $(x, u) \in M \times V(1)$ is equivalent to $\mathfrak{L}'(x,\, w) - \mathfrak{L}(x,\, w) = \mathrm{D}\varphi(x)w$ for all $(x,\, w) \in M \times N^{\rightarrow}$. By definition (3) of the symmetries, it is clear that we call F a symmetry of the extended Lagrangian, if

$$\mathfrak{L}(Fx,\, \mathrm{D}F(x)w) = \mathfrak{L}(x,\, w) + \mathfrak{f}_F(x,\, w) \qquad ((x,\, w) \in M \times N^{\rightarrow}).$$

3. Point-mass in relativistic spacetime

3.1 The relativistic spacetime

This model is also introduced and described in details in Matolcsi [12]. The relativistic spacetime is $(M,\, \boldsymbol{I},\, g)$, where

- M is *spacetime*, a four-dimensional oriented real affine space over the vector space $\boldsymbol{M}$.
- $\boldsymbol{I}$ is *the measure line of time intervals*, an oriented one-dimensional real vector space.
- g is an $\boldsymbol{M} \times \boldsymbol{M} \to \boldsymbol{I} \otimes \boldsymbol{I}$ arrow-oriented *Lorentz form*.

With the use of the speed of light, physical distances can be identified with time intervals, hence there is no need for a new measure line besides $\boldsymbol{I}$ (we use the "unit system" $\hbar = c = 1$).

The motion of a particle is described by a *world line*, a connected curve with timelike tangents. A world line is naturally given as the range of a *world line function*. The latter is a continuously differentiable function $r : \boldsymbol{I} \to M$ defined on an interval, and

$$\dot{r}(t) \in V(1) := \left\{ u \in \frac{\boldsymbol{M}}{\boldsymbol{I}} \;\middle|\; u \cdot u = -1, u \text{ is future-like} \right\}.$$

An essential difference is, compared with the non-relativistic case, that $V(1)$ is not an affine space here.

The time passed *along a world line* can be measured as follows. Let x_0 and x_1 be two points of a world line C. We consider an arbitrary parameterization p of this world line, having for simplicity the domain $[\mathbf{0},\, \mathbf{1}]$ and values $p(\mathbf{0}) = x_0,\ p(\mathbf{1}) = x_1$. The time passed between these two points on the world line C is

$$t_C(x_0,\, x_1) = \int_{\mathbf{0}}^{\mathbf{1}} \sqrt{|\dot{p}(a)\dot{p}(a)|}\, \mathrm{d}a.$$

Especially, if the parameterization is the world line function r, then $t_C(x_0,\, x_1) = r^{-1}(x_1) - r^{-1}(x_0)$.

3.2 Lorentz and Poincaré transformations

Transformations preserving the structure of the spacetime, i.e. the automorphisms of spacetime are called the *proper Poincaré transformations*. They can be described as follows.

$$\mathcal{L} := \big\{\boldsymbol{L} \in \mathrm{Lin}(\boldsymbol{M},\, \boldsymbol{M}) \,|\, \boldsymbol{L} \text{ is orientation and arrow-orientation preserving, } \boldsymbol{L}^+ \cdot \boldsymbol{L} = \mathrm{id}_{\boldsymbol{M}}\big\}$$

is *the proper Lorentz group*, and

$$\mathcal{P} := \big\{L \,:\, M \to M \,|\, L \text{ is affine, and the underlying } \boldsymbol{L} \in \mathcal{L}\big\}$$

is *the proper Poincaré group*. The word "proper" relies to the absence of time or space inversion in these groups. Spacetime translations, i.e. transformations of the form $x \mapsto x + \boldsymbol{a}$ (with $\boldsymbol{a} \in \boldsymbol{M}$) are Poincaré transformations whose underlying linear operator is the identity of $\boldsymbol{M}$.

$\mathcal{L}$ and $\mathcal{P}$ are a six dimensional and a ten dimensional Lie group. They have the Lie algebras

$$\mathrm{La}(\mathcal{L}) = \big\{\boldsymbol{H} \in \mathrm{Lin}(\boldsymbol{M},\, \boldsymbol{M}) \,|\, \boldsymbol{H}^+ = -\boldsymbol{H}\big\} \quad \text{and}$$
$$\mathrm{La}(\mathcal{P}) = \big\{H \,:\, M \to \boldsymbol{M} \,|\, H \text{ is affine, and the underlying } \boldsymbol{H} \in \mathrm{La}(\mathcal{L})\big\},$$

respectively.

The Lie algebra of the subgroup of spacetime translations consists of elements H for which $\boldsymbol{H} = 0$. Such an affine map is a constant, i.e. there is an $\boldsymbol{h} \in \boldsymbol{M}$, such that $Hx = \boldsymbol{h}$ for all $x \in M$.

Every Poincaré transformation L in the neighborhood of the unit element id_M has the form

$$L = \mathrm{e}^{sH} \;:= I + \sum_{n=1}^{\infty} \frac{(s\boldsymbol{H})^{n-1} \cdot H}{n!} \tag{18}$$

for some $s \in \mathbb{R}$ and $H \in \mathrm{La}(\mathcal{L})$. The underlying Lorentz transformation is

$$\boldsymbol{L} = \mathrm{e}^{s\boldsymbol{H}} = \sum_{n=0}^{\infty} \frac{(s\boldsymbol{H})^n}{n!}.$$

3.3 The variational principle in relativistic spacetime

In a naive way, we would say that given a continuous function $\mathfrak{L} \,:\, M \times V(1) \to \frac{\mathbb{R}}{\boldsymbol{I}}$, a point-mass propagates from a spacetime point x_0 to another one x_1 along a world line, on which the action

$$S \,:\, r \mapsto S(r) = \int_{r^{-1}(x_0)}^{r^{-1}(x_1)} \mathfrak{L}\big(r(t),\, \dot{r}(t)\big)\, \mathrm{d}t \tag{19}$$

is extremal. But this leads to a very serious problem. Along different world lines connecting x_0 and x_1, different amount of time passes, hence the domains of such world line functions are not common. The set of such world line functions can not be made an affine space directly, we cannot use the methods of differential calculus. Would we step over these problems somehow, there would be another problem with the Euler-Lagrange equation: $V(1)$ is not an affine space, hence differentiating by the speed variable would only be possible using the theory of manifolds.

We help this problem by the method shown in section 2.6. World lines will be parameterized by an arbitrary parameter having domain $[\mathbf{0}, \mathbf{1}] \subset \boldsymbol{I}$. For a world line function r, let $p : [\mathbf{0}, \mathbf{1}] \to \mathrm{Ran}r$ be a parameterization of the corresponding world line. Then $\sigma := r^{-1} \circ p : [\mathbf{0}, \mathbf{1}] \to [t_0, t_1]$ is a continuously differentiable bijection ($t_0 = r^{-1}(x_0)$ and $t_1 = r^{-1}(x_1)$), and

$$\dot{p} = (\dot{r} \circ \sigma)\dot{\sigma} \quad , \qquad 0 < |\dot{p}| := \sqrt{-\dot{p} \cdot \dot{p}} = \dot{\sigma}.$$

Therefore, σ is determined by p (as the primitive function of $|\dot{p}|$), r can also be restored from p, and

$$r \circ \sigma = p \quad , \quad \dot{r} \circ \sigma = \frac{\dot{p}}{|\dot{p}|};$$

in case of twice differentiability we have

$$\ddot{r} \circ \sigma = \frac{\ddot{p}}{|\dot{p}|^2} + \frac{\dot{p}\,(\dot{p} \cdot \ddot{p})}{|\dot{p}|^4}.$$

We see that r obeys a second-order differential equation if and only if p does so. We allow any parameterization as described above, hence its derivative can be any future-like vector, not only an element of $V(1)$. Let $N^{\rightarrow} \subset \frac{\boldsymbol{M}}{\boldsymbol{I}}$ be the set of the future-like vectors. This is an open set, containing $V(1)$.

We introduce the vector space

$$\boldsymbol{V} := \{\boldsymbol{p} : [\mathbf{0}, \mathbf{1}] \to \boldsymbol{M} \mid \boldsymbol{p} \text{ is continuously differentiable, } \boldsymbol{p}(\mathbf{0}) = \boldsymbol{p}(\mathbf{1}) = 0\}$$

endowed with the norm

$$||\boldsymbol{p}|| := \max_{a \in [\mathbf{0}, \mathbf{1}]} (|\boldsymbol{p}(a)|_{\boldsymbol{M}} + |\dot{\boldsymbol{p}}(a)|_{\frac{\boldsymbol{M}}{\boldsymbol{I}}})$$

and the affine space

$$V := \{p : [\mathbf{0}, \mathbf{1}] \to M \mid p \text{ is continuously differentiable, } p(\mathbf{0}) = x_0,\, p(\mathbf{1}) = x_1\}$$

over $\boldsymbol{V}$.

We are given a twice differentiable Lagrangian

$$\mathfrak{L} : M \times N^{\rightarrow} \to \frac{\mathbb{R}}{\boldsymbol{I}} \quad , \qquad (x, w) \mapsto \mathfrak{L}(x, w),$$

satisfying

$$\mathfrak{L}(x, w) = \mathfrak{L}\left(x, \frac{w}{|w|}\right)|w|. \tag{20}$$

We assume that the particle moves in such a way that the parameterization p of its world line between points x_0 and x_1 in spacetime makes the derivative $\mathrm{D}S(p)$ of the action

$$S : V \to \mathbb{R} \quad , \quad p \mapsto \int_{\mathbf{0}}^{\mathbf{1}} \mathfrak{L}(p(s),\, \dot{p}(s))\, \mathrm{d}s$$

zero. Equation (20) is trivial in the physically relevant cases $w \in V(1)$, since $|w| = 1$ for such speed values. Hence (20) only gives properties of the extension of the Lagrangian from $V(1)$ to $N^{\to}$, it does not mean any extra physical condition on the Lagrangian. By (20) and by replacing the integral variable, it is easy to show that the action is independent of the parameterization p of the world line.

We can repeat all our arguments we had in section 2.3; the parameterization p of the world line realized satisfies the appropriate Euler-Lagrange equation. The time scale invariance property of special relativistic point-mass dynamics is a well known problem of mechanics. The Lagrangians satisfying (20) are sometimes called "homogeneous" and this is why one cannot formulate Hamilton equations and canonical transformations in special relativistic mechanics without any further ado i. e. the formal Hamiltonian defined analogously to the non-relativistic case is identically zero (see e.g. [18]). From our treatment one can see clearly that the scale invariance property cannot be avoided, contrary to the opinion of Goldstein [7, p329]: "... it is not a sacrosanct physical law that the action integral in Hamilton's principle must have the same value wether expressed in terms of t of in terms of θ ... All that is required is that L be a world scalar that leads to the correct equations of motion." If we gave a clear meaning to the expression "leads" and without knowing it we looked for a "correct" equation of motion, then we should exclude every Lagrangian where the action would depend on the parameterization of the world lines.

The definition of symmetries (3) and the scale invariance property treated in section 2.6 was a somewhat artificial construction, because in non-relativistic spacetime the time is absolute. However, the striking similarity of the non-relativistic and relativistic concepts deserves the attention.

3.4 Symmetries and the Lagrangian

We call a *full time-derivative* a function $\mathfrak{f} : M \times N^{\to} \to \frac{\mathbb{R}}{\boldsymbol{I}}$, if there exists a function $\phi : M \to \mathbb{R}$ such that $\mathfrak{f}(x,\, w) = \mathrm{D}\phi(x)w$. Two Lagrangians determine the same variational principle (are equivalent), if and only if their difference is a full time-derivative, i.e. the derivative $\mathrm{D}S$ connected to them agree.

We say that a continuously differentiable map $F : M \to M$, for which $\mathrm{D}F(x)w \in N^{\to}$ for each $w \in N^{\to}$, is a *symmetry* of the (homogeneous) Lagrangian, if there is a full time-derivative $\mathfrak{f}_F$, such that

$$\mathfrak{L}(Fx,\, \mathrm{D}F(x)w) = \mathfrak{L}(x,\, w) + \mathfrak{f}_F(x,\, w) \qquad ((x,\, w) \in M \times N^{\to}).$$

Especially, a proper Poincaré transformation L is a symmetry of $\mathfrak{L}$, if

$$\mathfrak{L}(Lx,\, \boldsymbol{L}w) = \mathfrak{L}(x,\, w) + \mathfrak{f}_L(x,\, w).$$

3.5 Lagrangian of the free point-mass

Similarly to the non-relativistic case, we call the point-mass *free*, if each Poincaré transformation is a symmetry of the corresponding Lagrangian $\mathfrak{L}$. Hence $\mathfrak{L}$ is the Lagrangian of a free point-mass, if and only if for all $L \in \mathcal{P}$ there is a function $\phi_L : M \to \mathbb{R}$ such that

$$\mathfrak{L}(Lx,\, \boldsymbol{L}w) - \mathfrak{L}(x,\, w) = \mathrm{D}\phi_L(x)w.$$

We introduce the notation $\widehat{\phi}(L,\, x) := \phi_L(x)$, and we assume that this function $\mathcal{P} \times M \to \mathbb{R}$ is smooth enough. Although $\widehat{\phi}$ is defined on the manifold $\mathcal{P} \times M$, we only consider one-parameter subgroups of $\mathcal{P}$, hence we can use our usual differentiability notions.

We consider transformations in a neighborhood of $I := \mathrm{id}_M$ in the form (18). We obtain

(21)
$$\frac{\partial \mathfrak{L}(x,\, w)}{\partial x} \cdot H(x) + \frac{\partial \mathfrak{L}(x,\, w)}{\partial w} \cdot \boldsymbol{H} \cdot w = \left.\frac{\partial^2 \widehat{\phi}(L,\, x)}{\partial L \partial x}\right|_{L=I} \cdot (H,\, w) =: \frac{\partial \omega(H,\, x)}{\partial x} \cdot w$$

the same way as in the non-relativistic case. We write the coordinated form as well:

$$\frac{\partial \mathfrak{L}}{\partial x^i}(H^i{}_j x^j + h^i) + \frac{\partial \mathfrak{L}}{\partial w^i} H^i{}_j w^j = \frac{\partial \omega}{\partial x^i} w^i.$$

As in the non-relativistic case, we find

$$\mathfrak{L}(x,\, w) = l(x) \cdot w + \varphi(w) \quad , \qquad \mathfrak{L} = l(x)_k w^k + \varphi(w), \tag{22}$$

with an arbitrary function $\varphi : V(1) \to \mathbb{R}/\boldsymbol{I}$ and with $l : M \to \boldsymbol{M}^*$ for which

$$\boldsymbol{C} := \mathrm{D} \wedge l := (\mathrm{D}l)^* - \mathrm{D}l = \text{const.} \quad , \qquad C_{ki} := \frac{\partial l_i}{\mathrm{d}x^k} - \frac{\partial l_k}{\partial x^i} = \text{const.}$$

By considering general Poincaré transformations and substituting the form (22) to (21), we conclude

$$\frac{\partial l(x) \cdot w}{\partial x} \cdot H(x) + l(x) \cdot \boldsymbol{H} \cdot w + \frac{\mathrm{d}\varphi(w)}{\mathrm{d}w} \cdot \boldsymbol{H} \cdot w = \frac{\partial \omega(H, x)}{\partial x} \cdot w,$$

i.e.

$$\frac{\partial l_k}{\partial x^i} w^k (H^i{}_j x^j + h^i) + l_i H^i{}_k w^k + \frac{\partial \varphi}{\partial w^i} H^i{}_k w^k = \frac{\partial \omega}{\partial x^k} w^k.$$

Differentiating the latter by x^m leads to

$$\frac{\partial^2 l_k}{\partial x^m \partial x^i} w^k (H^i{}_j x^j + h^i) + \frac{\partial l_k}{\partial x^i} w^k H^i{}_m + \frac{\partial l_i}{\mathrm{d}x^m} H^i{}_k w^k = \frac{\partial^2 \omega}{\partial x^m \partial x^k} w^k.$$

By the method shown in the non-relativistic case, it follows that

$$C_{ki} H^i{}_m - C_{mi} H^i{}_k = 0.$$

$\boldsymbol{H}$ is antisymmetric and, by the identification $\boldsymbol{M}^* \equiv \frac{\boldsymbol{M}}{\boldsymbol{I}\otimes\boldsymbol{I}}$, subscripts and superscripts can be interchanged, hence $H^i{}_k = -H^k{}_i = -H_k{}^i$. Using also antisymmetry of $\boldsymbol{C}$,

$$C_{ki}H^i{}_m - H_k{}^i C_{im} = 0,$$

i.e.

$$[\boldsymbol{C},\, \boldsymbol{H}] = 0.$$

for all antisymmetric $\boldsymbol{H}$. As a consequence, $\boldsymbol{C}$ commutes with all proper Lorentz transformations, hence $\boldsymbol{C}$ is a multiple of $\mathrm{id}_{\boldsymbol{M}}$ by Schur's lemma. On the other hand, it is also antisymmetric, thus

$$\boldsymbol{C} = 0 \quad , \qquad C_{ik} = 0.$$

Based on this, as in the non-relativistic case, we conclude that the Lagrangian has the form

$$\mathfrak{L}(x,\, w) = \varphi(w)$$

by omitting a full time-derivative. According to (20) we have

$$\varphi\left(\frac{w}{|w|}\right)|w| = \varphi(w).$$

Choosing an arbitrary number $\lambda > 0$ and using the previous equality, we obtain

$$\varphi(\lambda w) = \varphi\bigg(\frac{\lambda w}{|\lambda w|}\bigg)|\lambda w| = \lambda\varphi\bigg(\frac{w}{|w|}\bigg)|w| = \lambda\varphi(w);$$

differentiating with respect to λ and substituting $\lambda = 1$, we get

$$\mathrm{D}\varphi(w)\cdot w = \varphi(w) \quad , \qquad \frac{\partial\varphi}{\partial w^k}w^k = \varphi. \tag{23}$$

Since the Lagrangian does not depend on spacetime points, considering also the equivalence of the Lagrangians by a full time derivative, its symmetry is formulated as follows: for all Lorentz transformations $\boldsymbol{L}$ there exists an $\alpha(\boldsymbol{L}) \in \boldsymbol{M}^*$ such that

$$\varphi(\boldsymbol{L}w) = \varphi(w) + \alpha(\boldsymbol{L})\cdot w + const.. \tag{24}$$

We differentiate this equation with respect to w:

$$\mathrm{D}\varphi(\boldsymbol{L}w)\boldsymbol{L} = \mathrm{D}\varphi(w) + \alpha(\boldsymbol{L}) \quad , \qquad \frac{\partial\varphi}{\partial w^i}(\boldsymbol{L}w)L^i{}_k = \frac{\partial\varphi}{\partial w^k}(w) + \alpha(\boldsymbol{L})_k.$$

Considering a Lorentz transformation for which $\boldsymbol{L}w = w$ holds, we have $\alpha(\boldsymbol{L}) = 0$ by (24), hence for all such transformations $\mathrm{D}\varphi(w)\boldsymbol{L} = \mathrm{D}\varphi(w)$, thus $\boldsymbol{L}^*\mathrm{D}\varphi(w) = \mathrm{D}\varphi(w)$. By $\boldsymbol{L}^* \equiv \boldsymbol{L}^{-1}$, this means

$$\mathrm{D}\varphi(w) = \boldsymbol{L}\mathrm{D}\varphi(w) \quad \text{if} \quad \boldsymbol{L}w = w.$$

Since $\{\boldsymbol{L} \mid \boldsymbol{L}w = w\}$ is just the rotation group of the three-dimensional subspace orthogonal to w (i.e. it is a "little group" of Wigner), the equation above can only hold in case $\mathrm{D}\varphi(w)$ is parallel to w. This means that there exists a function $\beta : \frac{\boldsymbol{M}}{\boldsymbol{I}} \to \mathbb{R}$ such that

$$\mathrm{D}\varphi(w) = \beta(w)w \quad , \qquad \frac{\partial\varphi}{\partial w^k} = \beta\, w_k,$$

hence multiplying (23) with w, we obtain

$$\varphi\, w_i = \frac{\partial\varphi}{\partial w^k} w^k\, w_i = \beta\, w_k\, w^k\, w_i = \frac{\partial\varphi}{\partial w^i} w_k\, w^k,$$

thus the differential equation

$$\mathrm{D}\varphi(w) = \frac{\varphi(w)\, w}{w \cdot w} \quad , \qquad \frac{\partial\varphi}{\partial w^i} = \frac{\varphi\, w_i}{w^k w_k}$$

holds for φ. This partial differential equation can be handled with methods of ordinary differential equations. For the zeroth variable for example, by fixing all the three other variables, with notations $x := w^0 = -w_0,\ a^2 := \sum_{i=1}^{3} w^i w_i$, we have the ordinary differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{-\psi x}{-x^2 + a^2} = \frac{\psi x}{x^2 - a^2},$$

or for the first variable, with fixed other variables and with notations $y := w^1 = w_1,\ b^2 := -(w^0 w_0 + w^2 w_2 + w^3 w_3) = -w \cdot w + w^1 w_1 > 0$,

$$\frac{\mathrm{d}\psi}{\mathrm{d}y} = \frac{\psi y}{y^2 - b^2}.$$

Then it is easy to see that the solution of (23) is

$$\varphi(w) = m\,|w|,$$

with a constant $m \in \frac{\mathbb{R}}{\boldsymbol{I}}$ (mass value).

3.6 Discussion

We obtained the Lagrangian

$$\mathfrak{L}(x,\, u) = \frac{1}{2} m |u - c|^2$$

of a non-relativistic free point-mass (with a mass value m and an absolute velocity c) by considering Noether transformations (members of the inhomogeneous Galilean group) of form e^{sH} but it is a simple fact that every Noether transformation is a symmetry of the Euler-Lagrange equations based on the above Lagrangian.

We remark that different c-s correspond to different but equivalent Lagrangians, while different m-s result in inequivalent Lagrangians. The Lagrangian given by c is the kinetic energy of the point-mass, relative to the inertial observer having absolute velocity c.

The Lagrangian does not depend on spacetime points, hence is invariant for spacetime translations. On the other hand, it is not invariant for all Noether transformations: if L is a Noether transformation whose underlying linear map is the special Galilean transformation $\boldsymbol{L}_{\boldsymbol{v}}$ with speed $\boldsymbol{v}$, then

$$\mathfrak{L}(Lx,\, \boldsymbol{L}u) = \frac{1}{2}m|(u+\boldsymbol{v})-c|^2 = \frac{1}{2}m|u-(c-\boldsymbol{v})|^2,$$

i.e. the Lagrangian turns into the (equivalent) Lagrangian given by $c-\boldsymbol{v}\in V(1)$.

As a consequence, we see that if we required that the Lagrangian itself be invariant for all Noether transformations, then we should get a constant function. This fact shows well, why our symmetry definition is preferable among all others. Evidently, we cannot require the invariance of the action, because it can be different for the very same Euler-Lagrange equations, a full time derivative gives an additional constant (in field theories it is excluded by appropriate boundary conditions). We cannot require the invariance of the solutions, because the gained freedom is too large [21]. That we really require is the invariance of the Euler-Lagrange equations that appears as an invariance of an equivalence class of the Lagrangians.

We emphasize that spacetime homogeneity (invariance for spacetime translations) alone does not imply that the Lagrangian does not contain explicit spacetime dependence, contrary to usual statements [9]: spacelike rotations, too, are necessary to deduce this result.

We give a simple counterexample. Choosing an "origin" $o\in M$ and an antisymmetric linear map $\boldsymbol{B}:\boldsymbol{M}\to\boldsymbol{M}^*$, we define the Lagrangian

$$\mathfrak{L}(x,\, u) := (x-o)\cdot\boldsymbol{B}\cdot u+\varphi(u)$$

according to (8). For $\boldsymbol{B}\neq 0$, the first term is not a full time-derivative, hence $\mathfrak{L}$ and all Lagrangians in its equivalence class depend explicitly on spacetime points. On the other hand, a spacetime translation with any $\boldsymbol{a}\in\boldsymbol{M}$ is a symmetry of this Lagrangian:

$$\mathfrak{L}(x+\boldsymbol{a},\, u)-\mathfrak{L}(x,\, u)=\boldsymbol{a}\cdot\boldsymbol{B}\cdot u$$

and the right hand-side is a full time-derivative (of the function $x\mapsto \boldsymbol{a}\cdot\boldsymbol{B}\cdot(x-o)$). The statement that spacetime homogeneity implies independence of $\mathfrak{L}$ of spacetime variables is hence not true. This Lagrangian is symmetric to spacetime translations, and contains an essential spacetime dependence. The Euler-Lagrange equation has the form

$$2\boldsymbol{B}\cdot\dot{r}-\mathrm{D}\varphi(\dot{r})\cdot\ddot{r}=0,$$

which is clearly invariant under spacetime translations but momentum is not conserved.

We obtained the Lagrangian

$$\mathfrak{L}(x,\, w) = m\,|w|.$$

of a relativistic free point-mass (with a mass value m) by considering Poincaré transformations of form e^{sH} but it is a simple fact that every Poincaré transformation is a symmetry of the above Lagrangian.

Lagrangians with different m-s are inequivalent. The Lagrangian itself and not only its equivalence class is invariant to all Poincaré transformations. There are no other possible choices with our assumptions. Goldstein [7] admits $L = mf(|w|)$ with a two times differentiable monotonous $f : \mathbb{R} \to \frac{\mathbb{R}}{\boldsymbol{I}}$ as a Lagrangian of a free point-mass. However, he requires the symmetry of the solutions of Euler-Lagrange equations in a restricted sense and admitting this possibility one can get far more general Lagrangians (see [21] on the possibilities).

As in the non-relativistic case, spacetime homogeneity (invariance for spacetime translations) alone does not imply that the Lagrangian does not contain explicit spacetime dependence. The above counterexample for spacetime dependence and non-conservation of momentum can be repeated word by word.

The relativistic action depends only on the parameterized world line, not on the parameterization itself. We conclude that the action is in fact m times the proper time passed along the world line which is usually stated as "obvious". The Lagrangian is constant on the absolute velocity values, i.e. on the set $M \times V(1)$: $\mathfrak{L}(x, u) = m$ if $u \in V(1)$. According to (19), the particle moves between two timelike-separated points by minimizing its proper time passed. This also shows problems with definition (19): a constant Lagrangian leads to a trivial Euler-Lagrange equation, which holds for any motion. We do not have hence Euler-Lagrange equation for world line functions. In [8] for example, variational principle is written for motions parameterized by the time of an observer.

For the parameterization p of a world line of a free point-mass, we have the Euler-Lagrange equation

$$\left(\frac{\dot{p}}{|\dot{p}|}\right)^{\bullet} = 0 \quad , \quad \text{i.e.} \quad \frac{\dot{p}}{|\dot{p}|} = \text{ constant},$$

hence the particle has constant velocity, the corresponding world line is a straight line.

Let us mention here, that the parameterization invariance of the relativistic Lagrangian leads to the famous difficulties of relativistic dynamics and connected to degeneracy and non-covariant property of the traditional (pseudo) energy-momentum in general relativity as well. In general relativity one can find examples to the extension of the Lagrangian (locally from $V(1)$ to $N^{\rightarrow}$ with our notation) [23], but a different possibility is to work with Dirac's formalism based on constrained variations [4, 5].

Our treatment shows clearly that variational principles and symmetries in both the non-relativistic and the relativistic case can be handled in a very similar way; most of our arguments are identical almost word by word. We have got the well-known results in a rigorous way. The spacetime models and the exact formulation of the problem helped us to see the proper reasons of these well-known results.

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