On Material Frame-Indifference

T. MATOLCSI

Communicated by W. O. WILLIAMS

1. Introduction

The principle of material frame-indifference (material objectivity) is fundamental in the theory of continuous media (thermomechanics). It states that the response of the material is independent of the observer ([1], [5], [7]). However, the usual formulation of this principle is insufficient from a mathematical point of view, because one describes mathematical formulae involving the notion of observers without defining mathematically the notion of observer. Thus the formulae have dubious mathematical base and this involves the danger of mistakes or confusion. Moreover, among the quantities in complicated calculations related to observers one often finds the objective ones with great effort ([3], [4], [8]).

The crucial point is that in the usual considerations space-time appears always in coordinates corresponding to observers. An attempt was made in [6] to get rid of this uneasiness. However, in the outlined framework of "neoclassical space-time" the observers (frames of reference) are not ruled out from the description of processes; on the contrary, the definition of processes contains an observer.

Here a new way is proposed. First a space-time model is defined mathematically (observers are only determined by the notions of the space-time model) then all physical relations—Newtonian equation, balance equations, Boltzmann equation *etc.*—must be formulated in *absolute* terms, *i.e.* only with the notions of the space-time model that are independent of observers. Moreover, in every respect, we may only use notions that are defined mathematically in the framework of the space-time model.

First we survey the nonrelativistic space-time model which is detailed in [2].

According to our conception, we have to construct a model that reflects the properties of space-time as we experience them, and that enables us to treat all physical notions connected with space-time in the structure of the model.

2. Tensor products and quotients

First of all we need an appropriate mathematical tool for physical dimensions. Although, generally, in mathematical formulae the time-lapse between two happenings is considered as a real number, we well know that this is not right. The real number 3 is not a time period, in contradistinction to 3 seconds or 3 hours. The set of time periods consists in the positive multiples of a chosen one (*e.g.* second). It is convenient to introduce the negative multiples, too, and the set of all such periods can be endowed uniquely and in a natural way with an addition and multiplication by real numbers. Thus we postulate that the set of time periods is a one-dimensional real vector space I, which is oriented, *i.e.* one of its "half lines" is chosen as consisting in the positive elements.

The same can be said about the distances between space points. The one-dimensional oriented vector space of distances is denoted by D.

The physical dimension of velocity is obtained by dividing space distances by time periods. For the physical dimension of acceleration we divide space distances by the square of time periods. Thus we need rules concerning how to multiply and divide elements of several one-dimensional vector spaces. The convenient mathematical tools are tensor products and quotients.

Avoiding the abstract definitions, here we only treat the most frequently used realizations of tensor products and quotients of finite-dimensional real vector spaces.

The dual X^* of a vector space X is the vector space of real-valued linear functionals on X. We have $(X^*)^* = X$ in the finite-dimensional case.

If V is another vector space, the tensor product $V \otimes X$ is identified with $\mathrm{Lin}\,(X^*, V)$, the vector space of linear maps from X^* into V; for $v \in V$, $x \in X$, $v \otimes x$ is the linear map that sends $p \in X^*$ into $(p \mid x)\,v$, where $(p \mid x)$ is the value of the linear functional p at x. Note that in particular, $\mathbb{R} \otimes X = \mathrm{Lin}\,(X^*, \mathbb{R}) = (X^*)^* = X$.

We often find it convenient to omit the symbol for tensor product when multiplying by elements of one-dimensional vector spaces; in other words, if A is a one-dimensional vector space, we write ax instead of $a \otimes x$ $(a \in A, x \in X)$.

If A is a one-dimensional vector space, we can form $\frac{X}{A}$, the tensor quotient of X by A, which is identified with $\mathrm{Lin}\,(A, X)$; for $x \in X$, $0 \neq a \in A$, $\frac{x}{a}$ is the linear map that sends $b \in A$ into $\frac{b}{a}x$, where $\frac{b}{a}$ is the unique real number such that $\frac{b}{a}a = b$.

There are some relations among tensor products and quotients. First of all we derive from $A = (A^*)^*$ the statement

$$\frac{X}{A} = \mathrm{Lin}\,(A, X) = X \otimes A^*,$$

which implies $\frac{\mathbb{R}}{A} = A^*$; furthermore, we have

$$A \otimes \frac{X}{A} = \frac{A \otimes X}{A} = X, \qquad a \otimes \frac{x}{a} = \frac{a \otimes x}{a} = x,$$

$$\frac{\frac{X}{A}}{B} = \frac{X}{A \otimes B}, \qquad \frac{\frac{x}{a}}{b} = \frac{x}{a \otimes b}$$

and some other natural relations.

If L is in Lin (X, Y), Z and A are vector spaces, and A is one-dimensional, we define $\mathrm{id}_Z \otimes L \in \mathrm{Lin}\,(Z \otimes X, Z \otimes Y)$ by $z \otimes x \mapsto z \otimes Lx$, and $\frac{L}{\mathrm{id}_A} \in \mathrm{Lin}\left(\frac{X}{A}, \frac{Y}{A}\right)$ by $\frac{x}{a} \mapsto \frac{Lx}{a}$.

If A and B are oriented one-dimensional vector spaces, then $A \otimes B$ and $\frac{A}{B}$ are oriented by the tensor product and the quotient of positive elements of A and B.

For natural numbers n we call

$$A^{\textcircled{n}} := \underset{(1)}{A} \otimes \underset{(2)}{A} \otimes \ldots \otimes \underset{(n)}{A}$$

the n^{th} *tensor power* of A and we write a^n for the n^{th} tensor power of $a \in A$.

If A is oriented, we can extract tensor roots: for a positive element h of $A^{\textcircled{2}}$ we can give a unique positive element $\sqrt{h}$ of A such that $(\sqrt{h})^2 = h$.

Let us return to physical dimensions. It is evident now that $\frac{D}{I}$ and $\frac{D}{I^{\textcircled{2}}}$ are the oriented vector spaces of velocity and acceleration magnitudes, respectively. If $m \in D$, $s \in I$, then $\frac{D}{I}$ and $\frac{D}{I^{\textcircled{2}}}$ consist in the multiples of $\frac{m}{s}$ and $\frac{m}{s^2}$, respectively.

It is suitable to use a system of physical dimensions such that the Planck constant is a real number (dimensionless). Then the mass values are the positive elements of $\frac{I}{D^{\textcircled{2}}}$.

3. Affine space

The simplest experience about our physical space is that two arbitrary points can be connected by an oriented straight line segment, called a "vector". Each of these vectors can be translated parallel to itself. The vectors originating at the same point form a three-dimensional vector space with natural rules of multiplication by real numbers and addition. The most suitable mathematical tool for describing such a structure is an affine space which is something like a vector space but has no distinguished origin.

An *affine space* is a triplet $(L, \boldsymbol{L}, -)$, where

(i) L is a nonvoid set,
(ii) $\boldsymbol{L}$ is a finite-dimensional real vector space,
(ii) $-$ is a map from $L \times L$ into $\boldsymbol{L}$, denoted by

$$(x, y) \mapsto x - y,$$

having the properties

a) for every $y \in L$, the map $O_y : L \to \boldsymbol{L}$, $x \mapsto x - y$ is bijective,
b) $(x - y) + (y - z) + (z - x) = \mathbf{0} \quad (x, y, z \in L)$.

O_y is called the *vectorization* of L *with origin* y. Its inverse is denoted by $\boldsymbol{L} \to L$, $\boldsymbol{x} \mapsto y + \boldsymbol{x}$. Hence, by definition,

$$y + (x - y) = x \quad (x, y \in L).$$

Furthermore we have

$$x - y = \mathbf{0} \text{ if and only if } x = y \ (x, y \in L),$$

$$(x + \boldsymbol{x}) + \boldsymbol{y} = x + (\boldsymbol{x} + \boldsymbol{y}) \quad (x \in L, \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{L}).$$

As is usual in mathematics, we shall denote an affine space by a single letter; we say that L is an affine space over the vector space $\boldsymbol{L}$, and the map $-$ is called *subtraction.*

Every vector space is an affine space over itself by the vector subtraction.

The *dimension* of an affine space is, by definition, that of the underlying vector space. The affine space is called *oriented* if the underlying vector space is oriented.

A non-void subset S of an affine space L is called an *affine subspace* if there is a linear subspace $\boldsymbol{S}$ of $\boldsymbol{L}$ such that for some (and hence for all) $y \in S$, $\{x - y \mid x \in S\} = \boldsymbol{S}$, or equivalently, $y + \boldsymbol{S} = S$.

S is said to be *directed* by $\boldsymbol{S}$ and the dimension of S is that of $\boldsymbol{S}$.

One-dimensional affine subspaces are called *straight lines.*

Points of L are zero-dimensional subspaces.

Let L and K be affine spaces over the vector spaces $\boldsymbol{L}$ and $\boldsymbol{K}$, respectively. A map $F : L \to K$ is called *affine* if there is a (necessarily unique) linear map $DF : \boldsymbol{L} \to \boldsymbol{K}$, called the *derivative* of F, such that

$$F(y) - F(x) = DF(x - y) \quad (x, y \in L)$$

or, equivalently,

$$F(x + \boldsymbol{x}) - F(x) = DF(\boldsymbol{x}) \quad (x \in L, \boldsymbol{x} \in \boldsymbol{L}).$$

A map F from a vector space $\boldsymbol{L}$ into another vector space $\boldsymbol{K}$ is affine if and only if there is a linear map $\boldsymbol{F} : \boldsymbol{L} \to \boldsymbol{K}$ and a $\boldsymbol{k} \in \boldsymbol{K}$ such that $F(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{k}$ $(\boldsymbol{x} \in \boldsymbol{L})$,; then $\boldsymbol{F} = DF$.

We can define a metric d on the affine space L with the aid of a norm $\| \ \|$ on $\boldsymbol{L}$ by $d(x, y) := \|x - y\|$ $(x, y \in L)$.

Since all norms on a finite-dimensional vector space are equivalent, we can define continuity and differentiability of maps between affine spaces in the usual way.

Let L and K be affine spaces. A map $F: L \rightarrowtail K$ (the sign $\rightarrowtail$ means that the domain of F need not be the whole of L) is called *differentiable* at an interior point x of Dom F if there are a (necessarily unique) linear map $DF[x]: \boldsymbol{L} \to \boldsymbol{K}$, called a the *derivative* of F at x, and a neighborhood $U(x) \subset$ Dom F of x such that

$$F(y) - F(x) = DF[x]\,(y - x) + \operatorname{ordo}(y - x) \qquad (y \in U(x)),$$

or, equivalently, there is a neighbourhood V of zero of $\boldsymbol{L}$ such that $x + V \subset$ Dom F and

$$F(x + \boldsymbol{x}) - F(x) = DF[x]\,\boldsymbol{x} + \operatorname{ordo}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in V),$$

where $\lim_{\boldsymbol{x} \to 0} \dfrac{\operatorname{ordo}(\boldsymbol{x})}{\|\boldsymbol{x}\|} = \boldsymbol{0}$ for some (and hence for all) norms $\|\ \|$ on $\boldsymbol{L}$.

F is *differentiable* if it is differentiable at every point of its domain. F is *twice differentiable*, if it is differentiable, and the map $L \rightarrowtail \operatorname{Lin}(\boldsymbol{L}, \boldsymbol{K})$, $x \mapsto DF[x]$ is differentiable.

Note that according to our convention $DF[x]$ is in $\boldsymbol{K} \otimes \boldsymbol{L}^*$, and the second derivative of F at x, $D^2F[x]$ is in $(\boldsymbol{K} \otimes \boldsymbol{L}^*) \otimes \boldsymbol{L}^* = \boldsymbol{K} \otimes (\boldsymbol{L}^* \otimes \boldsymbol{L}^*)$. If $\boldsymbol{L}$ is one-dimensional, by using the relation between tensor products and quotients we consider $DF[x]$ and $D^2F[x]$ as elements of $\dfrac{\boldsymbol{K}}{\boldsymbol{L}}$ and $\dfrac{\boldsymbol{K}}{\boldsymbol{L}^{②}}$, respectively.

4. The nonrelativistic space-time model

We attribute an affine structure to our physical space. Similarly, we experience an affine structure of time. Simple considerations show that absolute space does not exist, that space is related to material objects. We base our nonrelativistic space-time model on the assumption that space-time and absolute time exist and both have an affine structure. The progress of time is described in the model by an orientation. The "parity breaking" property of the interactions called "weak" indicates that the left and right orientations of our space are not equivalent; hence we postulate that space-time is oriented as well. Absolute time is related to space-time in such a way that a time point is assigned to each space-time point. We assume that this assignment is a affine map.

Thus we postulate as given the following:

M, an oriented four-dimensional affine space over the vector space $\boldsymbol{M}$; M is called *space-time* or the *world* (we avoid the usual term "event" for the elements of M, because world points are not events from a probabilistic point of view);

I, an oriented one-dimensional affine space over the vector space $\boldsymbol{I}$; I is called *time*.

$\tau: M \to I$, an affine surjection, called *time evaluation*.

If x and y are world points, then $\tau(y) - \tau(x)$, an element of $\boldsymbol{I}$, is the *time-lapse* between x and y.

$D\tau$, the derivative of τ, will be denoted by $\boldsymbol{\tau}$; it is a linear surjection from $\boldsymbol{M}$

onto I. Thus its kernel, denoted by E, is a three-dimensional linear subspace of M.

If t is a time point, the set of simultaneous world points having time value t, $E_t := \{x \in M \mid \tau(x) = t\}$ is a three-dimensional affine subspace of M, over E.

Besides the affine structure, our physical space has another property: a Euclidean structure. A vector of our space has a length, and two vectors have an angle. The lengths and angles are determined for simultaneous space points. Hence we put this structure into the space-time model by giving something like an inner product on the underlying vector space of simultaneous world points. It will not be a usual inner product because distances—or lengths of vectors—are not real numbers, as we have said.

Thus we postulate as given the following:

$\boldsymbol{D}$, an oriented one-dimensional vector space, whose elements are called *space lengths* or *distances*;

$\gamma : \boldsymbol{E} \times \boldsymbol{E} \to \boldsymbol{D}^{②}$, a positive definite symmetric bilinear map (the positive definiteness of γ means that $\gamma(q, q)$ is positive for all non-zero $q \in \boldsymbol{E}$; recall that $\boldsymbol{D}^{②}$ is oriented, and so we can speak of its positive elements).

According to the Euclidean structure γ,

$$|q| := \sqrt{\gamma(q, q)} \in \boldsymbol{D}$$

is the *length* of $q \in \boldsymbol{E}$, and the real number

$$\frac{\gamma(q_1, q_2)}{|q_1|\,|q_2|}$$

is the *cosine of the angle* between the non-zero elements q_1 and q_2 of $\boldsymbol{E}$.

Using the above notations we define $(M, I, \tau, \boldsymbol{D}, \gamma)$ to be a *nonrelativistic space-time model.* Such a nonrelativistic space-time model includes all the structures, and only those, that are tacitly applied in usual nonrelativistic physics.

Let us compare our model with that of [6]. Our world M corresponds to W there; if we identify t with the hyperplane E_t for all $t \in I$, then our time I corresponds to Γ there. From our time evaluation τ we derive the time lapse function $M \times M \to I$, $(x, y) \mapsto \tau(x) - \tau(y)$, which corresponds to the time lapse function $\hat{t}$; moreover, the subtraction $I \times I \to I$ corresponds to the time lapse function $\bar{t}$ of [6]. The fundamental difference between the two models is that our world M has an affine structure.

The simple and natural affine structure has the advantage that it allows us to differentiate both M-valued functions (world lines, see section 5), and functions defined on M (velocity fields, see section 10). As a consequence, we can rule out observers from the description of physical phenomena, and use only absolute (independent of observers) notions. For instance, the affine structure helps us to define absolute velocity which is a basic notion both in mechanics and continuum mechanics. Moreover, since observers are defined in the framework of the space-time model, we need not involve "bodies" or "material objects" from outside.

5. World lines

The motion of a masspoint usually is described by a function assigning space points to time points. Motion and space are notions related to observers (material objects). Now we wish an absolute description in our space-time model. Instead of "motion" we introduce a new term: the *existence* of a masspoint is independent of observers. The existence of a masspoint is described by what is called a *world-line function* r that assigns world points to time points. Evidently, the natural condition $\tau(r(t)) = t$ must hold for all $t \in \mathrm{Dom}\, r$. Moreover, here we require that r be twice differentiable.

The first-derivative function of r is denoted by $\dot{r}$ (instead of Dr). Recall that $\dot{r}(t)$, which we interpret as the value at t of the *absolute velocity* of the masspoint, is in $\frac{M}{I}$. We have the following important relation:

$$\frac{\tau}{\mathrm{id}_I}(\dot{r}(t)) = \frac{\tau}{\mathrm{id}_I}\left(\lim_{h\to 0}\frac{r(t+h)-r(t)}{h}\right) = \lim_{h\to 0}\frac{\tau(r(t+h))-\tau(r(t))}{h}$$

$$= \lim_{h\to 0}\frac{(t+h)-t}{h} = 1.$$

In other words, the values of absolute velocity lie in

$$V(1) := \left\{u \in \frac{M}{I} \,\middle|\, \frac{\tau}{\mathrm{id}_I}(u) = 1\right\}$$

which is a three-dimensional affine subspace of $\frac{M}{I}$, over $\frac{E}{I}$.

From now on we omit the epithet absolute, and we call the elements of $V(1)$ *velocity values*. $V(1)$ is not a vector space; there is no zero velocity value and the multiple of a velocity value and the sum of two velocity values are not velocity values. There is no Euclidean structure on $\frac{M}{I}$; thus a velocity value has no magnitude and two velocity values subtend no angle.

If u and c are in $V(1)$, then

$$v_{uc} := u - c$$

is called the *relative velocity value* of u with respect to c. v_{uc} is in $\frac{E}{I}$; that is why we call the elements of $\frac{E}{I}$ *relative velocity values*. The set of relative velocity values forms a vector space, and has a natural Euclidean structure:

$$\frac{E}{I} \times \frac{E}{I} \to \left(\frac{D}{I}\right)^{②}, \qquad (v_1, v_2) \mapsto v_1 \cdot v_2 := \frac{\gamma(v_1 t, v_2 t)}{t^2} \quad (0 \neq t \in I)$$

is a positive definite symmetric bilinear form. Thus $|v| := \sqrt{v \cdot v}$, an element of $\frac{D}{I}$, is the *magnitude* of the relative velocity value v.

The second derivative functions of r is denoted by $\ddot{r}$. $\ddot{r}(t)$, which we interpret as the value of the *absolute acceleration* of the masspoint at t, is in $\frac{E}{I^{②}}$ because it equals $\lim_{h\to 0} \frac{\dot{r}(t+h)-\dot{r}(t)}{h}$, and the differene of velocity values lies in $\frac{E}{I}$. That is why we call the elements of $\frac{E}{I^{②}}$ *acceleration values*. Acceleration values form a Euclidean vector space; their magnitudes are in $\frac{D}{I^{②}}$.

We consider mass values as elements of $\frac{I}{D^{②}}$; according to the preceding result and Newton's second law, force values are elements of $\frac{I}{D^{②}} \otimes \frac{E}{I^{②}} = \frac{E}{D^{②} \otimes I}$. In the usual formulations a force can depend on time, space and velocity; that is why we postulate that a force field is a differentiable function $F: M \times V(1) \rightarrow\!\!\!\!\rightarrow \frac{E}{D^{②} \times I}$, and that

$$m\ddot{r} = F \circ (r, \dot{r})$$

is the *absolute* Newtonian equation for the world-line function r of a material point with mass m under the action of the force field F.

Finally we mention that ranges of world-line functions are called *world lines*. World lines are curves in space-time.

6. Vectors, tensors

Let A be a vector space constructed of I, I^*, D and D^* by tensor products and quotients. Then the elements of

A	are called	*scalars* of type A,
$A \otimes M$	are called	*vectors* of type A,
$\frac{M}{A}$	are called	*vectors* of cotype A,
$A \otimes M \otimes M$	are called	*tensors* of type A.

When $A = \mathbb{R}$, *i.e.* when $A \otimes M = \frac{M}{A} = M$, we omit the terms "of type $\mathbb{R}$" and "of cotype $\mathbb{R}$".

A vector of type or cotype A is called *spacelike* if it is in $A \otimes E$ or $\frac{E}{A}$, respectively.

Let c be an element of $V(1)$. Then $c \otimes I := \{ct \mid t \in I\}$ is a one-dimensional linear subspace of M. The intersection of $c \otimes I$ and E is the zero only, thus every element x of M can be uniquely decomposed into the sum of two vectors lying in $c \otimes I$ and E, respectively: $x = c\tau(x) + (x - c\tau(x))$.

This decomposition will play a fundamental role later. $\tau(x)$ and $x\|_c := x - c\tau(x)$ are called the *timelike component* and the *c-spacelike component* of the vector x, and

$$s_c : M \to I \times E, \qquad x \mapsto (\tau(x), x\|_c)$$

is the *c-splitting* of M.

x is spacelike if and only if its timelike component is zero, or all its c-spacelike components are the same, namely x.

The so-called Galilean transformation law shows well how splittings depend on velocity values. If $c_1, c_2 \in V(1)$, then

$$(s_{c_2} \circ s_{c_1}^{-1})(t, q) = (t, -v_{c_2 c_1} t + q) \qquad ((t, q) \in I \times E).$$

This formula is well known in standard physics. We shall see in the next paragraph how it is related to observers.

Vectors of type A and vectors of cotype A are split corresponding to $c \in V(1)$ by $\mathrm{id}_A \otimes s_c$ and $\frac{s_c}{\mathrm{id}_A}$, respectively. The timelike component of a vector of type (cotype) A is in $A \otimes I \left(\frac{I}{A}\right)$, and its c-spacelike component is in $A \otimes E \left(\frac{E}{A}\right)$. In particular, $\frac{s_c}{\mathrm{id}_I}$ maps $\frac{M}{I}$ into $\mathbb{R} \otimes \frac{E}{I}$ and for all $u \in V(1)$

$$\frac{s_c}{\mathrm{id}_I}(u) = (1, v_{uc}),$$

i.e. the c-spacelike component of a velocity value u is the relative velocity value of u with respect to c.

We emphasize that there is no Euclidean structure on M; the length of a vector and the angle between two vectors make no sense unless the vectors are spacelike. A Euclidean structure can be defined on $A \otimes E$ and $\frac{E}{A}$, just as was done for relative velocity values. Moreover, we can form the *scalar product* of spacelike vectors of different types (cotype). For instance, if A and B are spaces of scalars, $h \in A \otimes E$, $g \in B \otimes E$, then

$$h \cdot g := ab\gamma\left(\frac{h}{a}, \frac{g}{b}\right) \in A \otimes B \otimes D^{②} \qquad (0 \neq a \in A, 0 \neq b \in B)$$

or, equivalently, for $aq \in A \otimes E$, $bq' \in B \otimes E$, $(aq) \cdot (bq') := ab\gamma(q, q')$.

Similarly, we can form the scalar product of spacelike tensors. If $H \in A \otimes (E \otimes E)$, $G \in B \otimes (E \otimes E)$, then $H : G$ is an element of $A \otimes B \otimes D^{④}$; if $H = a(q \otimes r)$, $G = b(q' \otimes r')$, then $H : G := ab\gamma(q, q')\gamma(r, r')$.

On the base of these formulae, the reader is asked to define the element $h \cdot G$ of $A \otimes B \otimes D^{②} \otimes E$ for $h \in A \otimes E$, $G \in B \otimes (E \otimes E)$.

7. Observers

An observer (a material object) is a continuously distributed collection of masspoints. Its existence can be given in the space-time model as a set of disjoint world lines filling an open subset of space-time. Then for all x in this subset there is a unique world line passing through x, and we can assign to x the velocity value of the corresponding world line function at $\tau(x)$. In this way we give a map $U: M \rightarrow\!\!\!\!\rightarrow V(1)$. If this map is good enough, then the world lines in question can be recovered from U as maximal integral curves of the differential equation $\dot{x} = U \circ x$.

Therefore we find it convenient to define an *observer* to be a smooth (infinitely many times differentiable) map $U: M \rightarrow\!\!\!\!\rightarrow V(1)$. The maximal integral curves of U are called *U-lines*. The U-line passing through $x \in \mathrm{Dom}\ U$ is denoted by $C_U(x)$, and E_U stands for the set of U-lines.

The material points of a physical observer constitute the space of the observer. A material point of a physical observer modelled by U is modelled by a U-line. Thus a space point of the observer is modelled by a U-line; that is why the set of U-lines, E_U is called the U-space or the *space corresponding to* U.

The observer U *splits* the corresponding space-time domain into time and space by the map

$$S_U: \mathrm{Dom}\ U \rightarrow I \times E_U, \qquad x \mapsto (\tau(x), C_U(x)).$$

An observer U which is an everywhere defined constant map is called a *global inertial observer*. Let c be the constant value of U. Then U-lines are parallel straight lines directed by $c \otimes I$. A U-line can be written in the form $x + c \otimes I$ for an arbitrary x contained in the U-line. It is not hard to see that the set of straight lines directed by $c \otimes I$, endowed with the subtraction

$$(x + c \otimes I) - (y + c \otimes I) := (x - y)\|_c \qquad (x, y \in M)$$

is an affine space over E. In other words, the space corresponding to a global inertial observer is a three-dimensional affine space. Moreover, the splitting

$$S_U: M \rightarrow I \times E_U, \qquad x \mapsto (\tau(x), x + c \otimes I)$$

is an affine map, whose derivative is s_c.

Let us emphasize that the spaces of different global inertial observers are different affine spaces over the same vector space.

If we pair the global inertial observer U with a world point o, called the *origin*, then time and U-space are vectorized with origin $\tau(o)$ and $o + c \otimes I$, respectively, and we get the splitting

$$s_{c,o}: M \rightarrow I \times E, \qquad x \mapsto (\tau(x - o), (x - o)\|_c).$$

Thus a global inertial observer with origin represents space-time points by elements of $I \times E$. This corresponds to the usual practice in physics, that time and space (corresponding to an observer) are considered vector spaces (one speaks of time zero, position vector).

Though all global inertial observers with origin use the same vector spaces I and E to represent time and space, the physical meaning of the elements of $I \times E$ depend on observers and origins. Two different global inertial observers with origin represent different world points by the same element of $I \times E$. If the observers have velocity values c_1 and c_2, respectively, and their origins coincide, then (t, q) and $(t, -v_{c_2 c_1} t + q)$ represent the same world point according to the first and second observer, respectively.

Lastly we only mention that the motion of a masspoint relative to an observer U, as a map assigning U-space points to time points, can be derived from the existence (world line) of the masspoint, and differentiation of motion yields relative velocity values, as expected (see [2], Section I.4.).

8. Differentiation

Let V be a finite-dimensional vector space and take a differentiable function F defined in space-time M and having values in V. Then $DF[x]$, the derivative of F at $x \in \operatorname{Dom} F$, is in $\operatorname{Lin}(M, V) = V \otimes M^*$.

As a consequence of the structure of our space-time model, the partial time derivative of F makes no sense. On the other hand, the space-like derivative of F is meaningful. If x is in the domain of F, then the restriction of F onto the hyperplane $E_{\tau(x)}$ of world points simultaneous with x is differentiable, and we call its derivative at x the *spacelike derivative* of F at x, denoting it by $\nabla F[x]$. In other words, $\nabla F[x]$ is the derivative of the function $E \rightarrowtail V$, $q \mapsto F(x + q)$ at zero. Evidently, $\nabla F[x]$ is in $\operatorname{Lin}(E, V) = V \otimes E^*$, and

$$\nabla F[x] = DF[x]|_E \quad (x \in \operatorname{Dom} F),$$

where the last symbol denotes the restriction of the map $DF[x]$ onto the subspace E.

Let c be an element of $V(1)$. Then for $x \in \operatorname{Dom} F$, the function $I \rightarrowtail V$, $t \mapsto F(x + ct)$ is differentiable, and we call its derivative at zero the *c-timelike derivative* of F at x, denoting it by $D_c F(x]$. Evidently, $D_c F[x]$ is in $\operatorname{Lin}(I, V) = \frac{V}{I}$, and

$$D_c F[x] = \frac{DF[x]}{\mathrm{id}_I}(c) \quad (x \in \operatorname{Dom} F).$$

To generalize this notion, let us take a *velocity field* $u : M \rightarrowtail V(1)$. Then

$$D_u F[x] := \frac{DF[x]}{\mathrm{id}_I}(u(x)) \quad (x \in \operatorname{Dom} F \cap \operatorname{Dom} u)$$

is called the *u-substantial time derivative* of F at x.

For a velocity field u and a velocity value c we have the relation

$$D_u F[x] = D_c F[x] + \frac{\nabla F[x]}{\mathrm{id}_I}(v_{u(x)c}),$$

which in the case $V = \mathbb{R}$ can be written in the familiar form

$$D_u F[x] = D_c F[x] + v_{u(x)c} \cdot \nabla F[x].$$

In the special case when there is another vector space Z such that $V = Z \otimes M$, $DF[x]$ is in $Z \otimes M \otimes M^*$, and the trace functional $\mathrm{Tr}: M \otimes M^* \to R$, $x \otimes p \mapsto (p \mid x)$ allows us to define the *divergence* of F:

$$D \cdot F[x] := (\mathrm{id}_Z \otimes \mathrm{Tr})\, DF[x] \quad (x \in \mathrm{Dom}\, F).$$

If F takes values in $Z \otimes E$, its divergence involves only spacelike derivation and equals $\nabla \cdot F$. Similar formulae can be written for $V = \frac{M}{A}$.

If $Z = \mathbb{R}$, we split the vector-valued function F into a time-like component $\tau \circ F: M \twoheadrightarrow I$ and a c-spacelike component $F\|_c: M \twoheadrightarrow E$; then for every $c \in V$(1 we have

$$D \cdot F = D_c(\tau \circ F) + \nabla \cdot (F\|_c).$$

To relate the notions introduced above to the ones usually applied, let us take a global inertial observer with velocity value c and origin o. Then $F \circ s_{c,o}^{-1}$ is a map defined in the Cartesian product of vectorized time and observer space. We can differentiate this map partially corresponding to the variables in I and E, respectively. Then the chain rule yields

$$\frac{\partial}{\partial t}(F \circ s_{c,o}^{-1})(t, q) = D_c F[s_{c,o}^{-1}(t, q)]$$

$$\frac{\partial}{\partial q}(F \circ s_{c,o}^{-1})(t, q) = \nabla F[s_{c,o}^{-1}(t, q)]$$

for all (t, q) in $\mathrm{Dom}\,(F \circ s_{c,o}^{-1}) \subset I \times E$.

That is, applying the usual imprecise notations, we can state that the global inertial observer with velocity value c and origin transforms

$$\mathrm{D}_c \quad \text{into} \quad \frac{\partial}{\partial t},$$

$$\nabla \quad \text{into} \quad \frac{\partial}{\partial q},$$

$$\mathrm{D}_u \quad \text{into} \quad \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial q}$$

where v is the relative velocity field of u with respect to c, *i.e.* $v(t, q) := u(s_{c,o}^{-1}(t, q)) - c$. Moreover, if F takes values in M, and ϕ and f denote the timelike and c-spacelike component, respectively, of F in coordinates corresponding to the observer (*i.e.* $\phi(t, q) = \tau F(s_{c,o}^{-1}(t, q))$) then $D \cdot F$ is transformed into $\frac{\partial \phi}{\partial t} + \mathrm{div}\, f$.

9. Integration

First we mention that if A is a space of scalars (a one-dimensional oriented real vector space) then we can introduce two ideal elements, the positive A-infinity ∞_A, and the negative A-infinity $-\infty_A$ in the same way as for real numbers, and we put $\overline{A} := \{-\infty_A\} \cup A \cup \{\infty_A\}$.

Two spacelike vectors e and e' are said to be orthogonal, if $\gamma(e, e') = 0$. Let e_1, e_2, e_3 be pairwise orthogonal elements of E and define $P(e_1, e_2, e_3) := \{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \mid 0 \leqq \alpha_1, \alpha_2, \alpha_3 \leqq 1\}$. For $q \in E$, $q + P(e_1, e_2, e_3)$ is called a *parallelepiped* at q. The *volume* of $q + P(e_1, e_2, e_3)$ is defined to be $|e_1|\,|e_2|\,|e_3| \in D^{③}$.

The open subsets of E are uniquely defined by an arbitrary norm on E. The σ-algebra of Borel subsets of E, $B(E)$, is the one generated by the collection of open subsets. As is well known from measure theory, there is a unique $\overline{D^{③}}$-valued positive measure μ on $B(E)$, called the *canonical measure* of E, such that the measure of an arbitrary parallelepiped equals its volume. The fundamental property of the canonical measure is translation invariance. Similarly, a $\overline{\left(\frac{D}{I}\right)^{③}}$-valued canonical measure can be defined on $B\left(\frac{E}{I}\right)$.

Take a $t \in I$ and consider E_t, which is an affine space over E. The open subsets of E_t and the Borel σ-algebra $B(E_t)$ are uniquely defined by an arbitrary norm on E. B is a Borel subset of E_t if and only if for some (and hence for all) $z \in E_t$, $B - z := \{x - z \mid x \in B\}$ is a Borel subset of E. Using the vectorization map $O_z : E_t \to E$, $x \mapsto x - z$, we can write $B - z = O_z[B]$. The translation invariance of the canonical measure of E allows us to define the *canonical measure* μ_t of E_t by

$$\mu_t(B) := \mu(B - z) = \mu(O_z[B]) \qquad (B \in B(E_t),\ z \in E_t).$$

In the same way, a $\overline{\left(\frac{D}{I}\right)^{③}}$-valued canonical measure ν can be defined on the Borel subsets of $V(1)$ with the aid of the canonical measure of $\frac{E}{I}$.

Now we wish to find the mathematical tool for the description of mass density. In standard physics, it is a real-valued function defined on time and space, and its integral over space gives the total mass. First of all, we have to pay attention to physical dimensions. Mass density cannot be a real-valued function. The volume (measure) of a bounded space region is in $D^{③}$, as it must be: if D is spanned by m, then $D^{③}$ is spanned by m^3. Furthermore, according to our convention, mass values are in $\frac{I}{D^{②}}$; as a consequence, mass density must be a function taking values in $\frac{\frac{I}{D^{②}}}{D^{③}} = \frac{I}{D^{⑤}}$. Evidently, it is defined in space-time, *i.e.* we have

$$\varrho : M \nrightarrow \frac{I}{D^{⑤}}.$$

Relative to a global inertial observer with velocity value c and origin o, $\varrho \circ s_{c,o}^{-1}$ describes mass density as a function of vectorized time and observer space. The total mass at $t \in I$ is

$$\int_E (\varrho \circ s_{c,o}^{-1})(t, q)\, d\mu(q).$$

We write $t := \tau(o) + t$ and we vectorize E_t with origin $z(c, o, t) := o + c(t - \tau(o))$, *i.e.* we give the affine bijection

$$O_{z(c,o,t)} : E_t \to E, \quad x \mapsto x - (o + c(t - \tau(o))) = (x - o)\|_c.$$

We easily find that $s_{c,o}^{-1}(t - \tau(o), q) = O_{z(c,o,t)}^{-1}(q)$. Then the well known integral transformation law makes the above integral equal

$$\int_{E_t} \varrho\, d\mu_t.$$

That is why we postulate that mass density is a measurable function $\varrho : M \rightarrowtail \frac{I}{D⑤}$ whose restriction onto E_t is integrable with respect to μ_t for all $t \in I$.

10. Balance equation

The fundamental tool for describing the existence of a continuous medium is a velocity field $u : M \rightarrowtail V(1)$ which is supposed twice differentiable. Then the points of the corresponding "body" are the maximal integral curves of u. In this repect such a body or continuous medium is nearly the same as an observer, the only difference being that observers are infinitely many times differentiable.

For a global inertial observer with velocity value c, $v_{uc} := u - c : M \rightarrowtail \frac{E}{I}$ is the c-relative velocity field. This is used in standard physics.

Since u is differentiable, its derivative at $x \in \operatorname{Dom} u$, $Du[x]$, is in $\frac{E}{I} \otimes M^* = \frac{E \otimes M^*}{I}$. Evidently, $Du = D(u - c)$ for all $c \in V(1)$. Since $u - c$ takes values in $\frac{E}{I}$, its divergence involves only space-like derivative, *i.e.* $D \cdot (u - c) = \nabla \cdot (u - c)$. Consequently, the divergence of u involves only space-like derivatives as well: $D \cdot u = \nabla \cdot u$.

To characterizatize the continuum, besides the velocity field we must give its mass density, internal energy density, *etc.*

Let A be a space of scalars. The density of an observable of type A is a measurable function $d : M \rightarrowtail \frac{A}{D③}$ whose restriction E_t is integrable with respect to μ_t for all $t \in I$.

The convective current of the quantity having the density d is $du : M \rightarrowtail \frac{A}{D③} \otimes \frac{M}{I} = \frac{A \otimes M}{D③ \otimes I}$ (note that du is not a differential but the product of the

density d and the velocity field u). Note that the timelike component of this current is the density. The conductive current of the considered quantity is a current without density; thus it must be spacelike, *i.e.* given by a map $j: M \to \frac{A \otimes E}{D^{③} \otimes I}$. The source density of the quantity is a map $q: M \times V(1) \to \frac{A}{D^{③} \otimes I}$ and the balance equation, if the functions in question are differentiable, has the form

$$D \cdot (du + j) = q \circ (\mathrm{id}_M, u).$$

Similarly, the density of a vector observable of type A is a measurable function $d: M \to \frac{A \otimes M}{D^{③}}$ whose restriction onto E_t is integrable with respect to μ_t for all $t \in I$.

The corresponding convective current is $d \otimes u: M \to \frac{A \otimes M \otimes M}{D^{③} \otimes I}$ and the conductive current is a map $J: M \to \frac{A \otimes M \otimes E}{D^{③} \otimes I}$. The source density is a map $q: M \times V(1) \to \frac{A \otimes M}{D^{③} \otimes I}$ and in the differentiable case we have the balance equation

$$D \cdot (d \otimes u + J) = q \circ (\mathrm{id}_M, u).$$

In the same way, according to the order, we can write balance equations for tensor quantities of type A.

In particular, the density of mass is a function $\varrho: M \to \frac{I}{D^{⑤}}$, its conductive current and source are zero; hence we have the continuity equation

$$D \cdot (\varrho u) = 0,$$

which can be written in the form $D_u \varrho + \nabla \cdot u = 0$, too.

Since velocity values are in $\frac{M}{I}$ and mass values are in $\frac{I}{D^{②}}$, momentum values lie in $\frac{I}{D^{②}} \otimes \frac{M}{I} = \frac{M}{D^{②}}$. We emphasize that we speak about absolute momentum. Relative momentum, the fundamental notion in usual physics, involves an observer; if $c \in V(1)$, the c-relative momentum is the product of mass and c-relative velocity, and equals the c-space-like component of absolute momentum.

The density of momentum is $\varrho u: M \to \frac{M}{D^{⑤}}$, its conductive current is $P: M \to \frac{M \otimes E}{D^{⑤} \otimes I}$ ($-P$ is the stress tensor field), its source density equals the force density $f: M \times V(1) \to \frac{E}{D^{⑤} \otimes I}$, and the balance equation reads

$$D \cdot (\varrho u \otimes u + P) = f \circ (\mathrm{id}_M, u).$$

Since P has values in $\frac{M \otimes E}{D^{⑤} \otimes I}$, we have $D \cdot P = \nabla \cdot P$, and some familiar manipulations yield $\varrho D_u u + \nabla \cdot P = f \circ (\mathrm{id}_M, u)$.

Both $D_u u$ and f have spacelike values; thus $\nabla \cdot P$ must have spacelike values as well. And if so, the timelike component of the momentum balance equation is the mass continuity equation.

The balance equation for angular momentum implies, as usual, that the stress tensor field $-P$ is symmetric, which implies that P is spacelike, *i.e.* that it takes values in $\frac{E \otimes E}{D^{⑤} \otimes I}$. Here we mention that if we consider a continuum consisting of spinning particles, P need be neither symmetric nor spacelike.

11. The energy balance equation

It is important to see that kinetic energy does not exist absolutely (independently of observers). If $c \in V(1)$, we define c-kinetic energy density (kinetic energy density relative to the inertial observer with velocity c) as $\frac{1}{2}\varrho\,|u - c|^2 : M \rightarrow\!\!\!\!\rightarrow \frac{I}{D^{⑤}} \otimes \frac{D^{②}}{I^{②}} = \frac{\mathbb{R}}{D^{③} \otimes I} = \frac{I^*}{D^{③}}$, where $|\,|$ denotes the $\frac{D}{I}$-valued Euclidean norm on $\frac{E}{I}$. The conductive current of the c-kinetic energy is $(u - c) \cdot P$.

The density of internal energy is given by the specific internal energy $e : M \rightarrow\!\!\!\!\rightarrow \frac{D^{②}}{I^{②}}$ in the form ϱe, and its conductive current is $k : M \rightarrow\!\!\!\!\rightarrow \frac{E}{D^{③} \otimes I^{②}}$.

Now we should have to take into account electromagnetic c-energy and chemical energy, but for simplicity—because my intention is only to give a general survey of an absolute formulation of thermo-mechanics—we will consider the case when electric charges and electromagnetic moments are absent, and chemical reactions do not occur. By the way, the treatment of chemical reactions in our framework is similar to the usual ones, with evident modifications. On the other hand, the treatment of electromagnetic phenomena in continuous media requires a precise formulation of electromagnetism with the notions of of our non-relativistic space-time model, which differs significantly from the usual formulations.

Now the source density of c-energy—the sum of c-kinetic energy and internal energy—is the sum of the c-energy gained from the external force, which is $(\mathrm{id}_{V(1)} - c) \cdot f$, and the internal energy gained from "heating" (absorption and emission of heat radiation) which is denoted by $s : M \times V(1) \rightarrow\!\!\!\!\rightarrow \frac{I^*}{D^{③} \otimes I}$.

Then the balance equation of c-energy and the balance equation of momentum result in the following balance equation of internal energy:

$$D \cdot (\varrho e u + k) = -P : (\nabla u) + s \circ (\mathrm{id}_M, u).$$

Note that this equation is independent of c. The left side can be transformed into the form $\varrho D_u e + \nabla \cdot k$.

12. Boundary conditions

It is necessary to associate boundary conditions to the balance equations; boundary conditions play a fundamental role both in theory and practice. Boundaries and conditions on them represent some constraints and actions of the surrounding on the continuum.

To find how to formulate boundary conditions, let us imagine a tube or a box with water in it. The tube or box confines a region of our space. We know that space exists only relative to observers. Hence boundary conditions require observers. Here we use notions that are not defined in the present paper, *e.g.* the topology of space corresponding to a non-inertial observer and the orientation of $\boldsymbol{E}$; for them we refer to [2].

Let $\boldsymbol{U}$ be an observer and G be a connected open subset of the $\boldsymbol{U}$-space E_U such that the boundary ∂G of G is an orientable two-dimensional submanifold of E_U. We assume that $\operatorname{Dom} \boldsymbol{u} = \{x \in M \mid C_U(x) \in G\}$ where $C_U(x)$ denotes the maximal integral curve of $\boldsymbol{U}$ (the $\boldsymbol{U}$-space point) containing x. Moreover, $\boldsymbol{u}$ and all other quantities describing the continuum are supposed to be continuously extended to the closure of $\operatorname{Dom} \boldsymbol{u}$.

For $t \in I$, $G_t := \{x \in E_t \mid C_U(x) \in G\} = E_t \cap \operatorname{Dom} \boldsymbol{u}$ is an open subset of the hyperplane E_t, with boundary ∂G_t an orientable two-dimensional submanifold. We put

$$B := \bigcup_{t \in I} \partial G_t .$$

Fixing an orientation of ∂G, we fix an orientation of ∂G_t for all $t \in I$, and we can give uniquely an $\boldsymbol{n}(x)$ in $\dfrac{\boldsymbol{E}}{\boldsymbol{D}}$ that has unit length, is orthogonal to $\partial G_{\tau(x)}$ and is positively oriented; $\boldsymbol{n}(x)$ is called the normal vector of B at x.

The boundary conditions are formulated with the aid of B and $\boldsymbol{n}(x)$ $(x \in B)$. It would be tedious to list all the boundary conditions. We illustrate them by the boundary condition imposed on mass if mass does not cross the boundary. In our example, the water at the wall of the tube moves parallel to the wall, *i.e.* the normal component of the relative velocity of the continuum with respect to the observer is zero at the boundary:

$$(\boldsymbol{u} - \boldsymbol{U})|_B \cdot \boldsymbol{n} = \boldsymbol{0} .$$

13. Equilibrium

Equilibrium is a basic notion in thermo-mechanics; therefore we try to formulate it with mathematical exactness in our framework.

Consider a free gas; it is never in equilibrium. A gas in a ball resting on the earth can be in equilibrium. If we rotate the ball around an axis uniformly, the gas inside can be in equilibrium with respect to the rotating ball as well, but this equilibrium differs from the previous one.

As the example shows, we must realize that equilibrium does not exist absolutely but only with respect to observers. The equilibrium depends on the action of the observer on the medium through boundary conditions.

As a consequence, there is no "trend to equilibrium"; we can speak only about a "trend to equilibrium with respect to an observer", and we can expect success in describing such trends only if we take into account the non-negligible action of the observer on the medium.

How to define equilibrium with respect to an observer $\boldsymbol{U}$? My answer is the following:

(i) the substance cannot move relative to the observer, *i.e.* the velocity field $\boldsymbol{u}$ is defined in the domain of $\boldsymbol{U}$, and $\boldsymbol{u}(x) = \boldsymbol{U}(x)$ for all $x \in \mathrm{Dom}\, \boldsymbol{u}$;

(ii) all quantities Q (mass density, internal energy density, *etc.*), except the velocity field $\boldsymbol{u}$, which is determined by the preceding item, are stationary with respect to $\boldsymbol{U}$; in other words, they do not depend on time with respect to $\boldsymbol{U}$; *i.e.* the $\boldsymbol{U}$-substantial time derivative of Q is zero, $D_{\boldsymbol{U}} Q = 0$.

Since, in equilibrium with respect to $\boldsymbol{U}$, $D_{\boldsymbol{U}} \varrho = 0$ and $\boldsymbol{u} \subset \boldsymbol{U}$, also $D_{\boldsymbol{u}} \varrho = 0$, and the continuity equation yields $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{U}|_{\mathrm{Dom}\boldsymbol{u}} = \boldsymbol{0}$. Thus if $\nabla \cdot \boldsymbol{U} \neq \boldsymbol{0}$, there cannot be equilibrium with respect to $\boldsymbol{U}$; $\nabla \cdot \boldsymbol{U} = \boldsymbol{0}$ is a necessary condition for the possibility of equilibrium with respect to $\boldsymbol{U}$. It is not hard to see that rigid observers (see [2], I.3.7) satisfy this condition.

14. The Boltzmann equation

The kinetic theory of gases is strongly related to thermo-mechanics. Some of the constitutive equations are derived with the aid of the Boltzmann equation, the usual formulation of which is based on the notion of phase space. Phase space does not exist absolutely, for it consists of position and momentum values related to observer, and the usual Boltzmann equation contains a partial time derivation which is related to observer too; that is why problems arise regarding material objectivity in connection with the Boltzmann equation ([3], [8]). To avoid such problems, we need an absolute formulation of the Boltzmann equation.

Usually the Boltzmann distribution function depends on time, space (position) and momentum values. Instead of momentum one often takes velocity. Thus we easily find that an absolute distribution function in our model is to be defined on $M \times V(1)$. Furthermore, in usual formulations the distribution function is normalized so that its integral over space and velocity equals the number 1 for all time points. Now we integrate over $E_t \times V(1)$ with respect to the tensor product of the corresponding canonical measures: $\mu_t \otimes \nu$ is the $\overline{\boldsymbol{D}^{③} \otimes \left(\frac{\boldsymbol{D}}{\boldsymbol{I}}\right)^{③}} = \overline{\frac{\boldsymbol{D}^{⑥}}{\boldsymbol{I}^{③}}}$-valued measure defined on the Borel subsets of $E_t \times V(1)$, by $(\mu_t \otimes \nu)(B \times H) := \mu_t(B)\, \nu(H)$ for all bounded Borel subsets B of E_t and H of $V(1)$.

Thus we agree that a *distribution function* is a measurable function $f: M \times V(1) \to \frac{\boldsymbol{I}^{③}}{\boldsymbol{D}^{⑥}}$ whose restriction onto $E_t \times V(1)$ is integrable with respect to $\mu_t \otimes \nu$ and

$$\int_{E_t \times V(1)} f \, d(\mu_t \otimes \nu) = 1$$

for all $t \in I$.

To set up a collision integral, we reformulate some familar notions.

Recall that we have a Euclidean structure on $\frac{E}{D}$, and the lengths of elements of $\frac{E}{D}$ are real numbers. The directions of space-like vectors can be given most conveniently by the elements of the unit sphere $S := \left\{ n \in \frac{E}{D} \,\middle|\, |n| = 1 \right\}$.

In accordance with usual considerations, a *scattering cross section* is a map that assigns to every non-zero $v \in \frac{E}{I}$ a $D^{②}$-valued non-zero positive measure σ_v on the Borel subsets of S. The meaning of the cross section is interpreted as follows: in the collision of two masspoints with relative velocity value v before collision, $\frac{\sigma_v(D)}{\sigma_v(S)}$ is the probability that the direction of the relative velocity value after collision, which is called the *scattering direction*, is in the Borel subset D of S.

A *scattering function* prescribes how the velocity values after collision of the two masspoints depend on the velocity values before collision and on the scattering direction; thus it is a function $(u_1, u_2) : V(1) \times V(1) \times S \to V(1) \times V(1)$.

The collision integral of the distribution function f corresponding to the scattering cross section σ and the scattering function (u_1, u_2) is

$$Cf(x, u) :=$$

$$\iint_{V(1)S} |u' - u| \, [f(x, u_1(u', u, \lambda)) f(x, u_2(u', u, \lambda)) - f(x, u') f(x, u)] \, d\sigma_{u'-u}(\lambda) \, dv(u')$$

$$((x, u) \in M \times V(1)).$$

provided the integral exists. Note that Cf is a function defined on $M \times V(1)$ and having values in $\frac{I^{②}}{D^{⑥}}$.

Now we suppose that f is differentiable. Then $Df[x, u]$, the derivative of f at (x, u), is in $\operatorname{Lin}\left(M \times \frac{E}{I}, \frac{I^{③}}{D^{⑥}}\right)$, and $\frac{Df[x, u]}{\mathrm{id}_I}$ is in $\operatorname{Lin}\left(\frac{M}{I} \times \frac{E}{I^{②}}, \frac{I^{②}}{D^{⑥}}\right)$. If m is the mass of the molecules of the gas to be described and the force field F acts on the molecules, then the absolute Boltzmann equation reads

$$\frac{Df[x, u]}{\mathrm{id}_I}\left(u, \frac{F(x, u)}{m}\right) = Cf(x, u) \qquad ((x, u) \in M \times V(1)).$$

It is not difficult to relate this equation to the usual one. The first partial derivative (corresponding to the "space-time variable") $D_1 f[x, u]$ of f at (x, u) is the derivative at x of the function $M \to \frac{I^{③}}{D^{⑥}}$, $y \mapsto f(y, u)$ for fixed u. We define the second partial derivative $D_2 f$ similarly. With these notations we can write the left-hand side of the Boltzmann equation in the form $\frac{D_1 f[x, u]}{\mathrm{id}_I}(u) +$

$\frac{D_2 f[x, u]}{\mathrm{id}_I}\left(\frac{F(x, u)}{m}\right)$. Then we see with the aid of the formulae in Section 8 that a global inertial observer transforms our Boltzmann equation into the customary one.

16. Constitutive relations

Up to now we have not spoken about temperature. Now we involve it as well. We take the Boltzmann constant to be the real number one, so that temperature has the same physical dimension as energy, *i.e.* temperature values are considered to be positive elements of I^*. Accordingly, the temperature field of a substance is a function $T: M \rightarrow\!\!\!\!\!\rightarrow I^*$.

To characterize the properties of the substance we establish what are called *constitutive relations* among velocity field $\boldsymbol{u}$, mass density ϱ, stress tensor field $-\boldsymbol{P}$, specific internal energy e, internal energy conductive current (heat flow) $\boldsymbol{k}$, internal energy source s, temperature field T, and the derivatives of $\boldsymbol{u}$, ϱ, T, *etc.*

Since all the quantities in question are absolute, independent of observers, if we use absolute equations for establishing constitutive relations, then the principle of material frame indifference is satisfied as a matter of course.

In particular, we can apply the Chapman-Enskog method in an appropriate absolute form to solve the Boltzmann equation in our terms; this will be done in a separate paper. Working with this method we do not get into trouble regarding material frame-indifference, *e.g.* we avoid the problem treated in [3].

References

1. GURTIN, M. E., & W. O. WILLIAMS: An axiomatic foundation for continuum thermodynamics, *Arch. Rational Mech. Anal.* **26** (1967), 83–117.
2. MATOLCSI, T.: *A Concept of Mathematical Physics, Models for space-time*, Akadémiai Kiadó, Budapest, 1984.
3. MURDOCH, A. I.: On material frame indifference, intrinsic spin, and certain constitutive relations motivated by the kinetic theory of gases, *Arch. Rational Mech. Anal.* **83** (1983), 185–194.
4. MÜLLER, I.: On the frame dependence of stress and heat flux, *Arch. Rational Mech. Anal.* **45** (1972), 241–250.
5. NOLL, W.: On the continuity of the solid and fluid states, *J. Rational Mech. Anal.* **4** (1955), 3–81.
6. NOLL, W.: Lectures on the foundations of continuum mechanics and thermodynamics, *Arch. Rational Anal. Mech.* **52** (1973), 62–92.
7. TRUESDELL, C.: *A First Course in Rational Continuum Mechanics*, Volume 1, Acad. Press. New York, San Francisco, London, 1977.
8. WANG, C. C.: On the concept of frame indifference in continuum mechanics and in the kinetic theory of gases, *Arch. Rational Mech. Anal.* **58** (1975), 381–393.

Department II. of Analysis, Eötvös Lorand University
Budapest, Hungary

(Received February 28, 1985)