A frame and gauge free formulation of quantum mechanics

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Abstract

The wave function of quantum mechanics is not a boost invariant and gauge invariant quantity. Correspondingly, reference frame dependence and gauge dependence are inherited to most of the elements of the usual formulation of quantum mechanics (including operators, states and events). If a frame dependent and gauge dependent formalism is called, in short, a relative formalism, then the aim of the paper is to establish an absolute, i.e., frame and gauge free, formalism for quantum mechanics. To fulfil this aim, we develop absolute quantities and the corresponding equations instead of the wave function and the Schrödinger equation. The absolute quantities have a more direct physical interpretation than the wave function has, and the corresponding equations express explicitly the independent physical aspects of the system which are contained in the Schrödinger equation in a mixed and more hidden form. Based on the absolute quantities and equations, events, states and physical quantities are introduced also in an absolute way. The formalism makes it possible to obtain some sharper versions of the uncertainty relation and to extend the validity of Ehrenfest's theorem. The absolute formulation allows wide extensions of quantum mechanics. To give examples, we discuss two known nonlinear extensions and, in close details, a dissipative system. An argument is provided that the absolute formalism may lead to an explanation of the Aharonov-Bohm effect purely in terms of the electromagnetic field strength tensor. At last, on special relativistic and curved spacetimes absolute quantities and equations instead of the Klein-Gordon wave function and equation are given, and their nonrelativistic limit is derived.

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I. Introduction

Quantum mechanics, being an amazingly successful, and, at the same time, a surprisingly novel theoretical framework in theoretical physics, has been investigated since its birth from innumerable aspects. As time passes, our understanding of it becomes wider and wider by the new properties explored. However, still there exist numerous questions of various kind concerning quantum mechanics that need further thorough investigation, directions in which our present understanding is not satisfactory yet. This paper is devoted to one such question, a topic that arises about the formulation and has diverse physical motivations.

The starting observation is that the usual formalism of quantum mechanics—wave functions, Hilbert space operators, Schrödinger equation, etc.—, no matter the background spacetime is nonrelativistic, special relativistic or a curved one, needs a choice of a reference frame on the spacetime in question. Based on a reference frame, spacetime is split into space and time, and the Schrödinger equation describes the time evolution of the wave function, a square integrable function of the space variable. Events, physical quantities and states (density operators) are introduced corresponding to the Hilbert space of square integrable functions. If another reference frame is chosen on the spacetime, all these quantities need to be transformed, they are not invariant under the transformation of the reference frame. In the nonrelativistic case this property is transparent, the wave function is transformed by a spacetime depending multiplying function, and the transformation of all the other quantities correspond to the transformation of the wave function. In the case of the Klein-Gordon equation this frame dependence is not so apparent since this equation has a "covariant" form; however, the way in which physical quantities can be introduced and in which the physical interpretation of the Klein-Gordon theory can be given requires a choice of a reference frame (see [1] for special relativistic spacetime and [2] for static curved spacetimes) so the full formalism proves to be frame dependent indeed.

In the case of nonrelativistic or special relativistic spacetime this feature does not have serious consequences because of the equivalence of (inertial) reference frames. However, on a curved spacetime it leads to a physically problematic situation both in quantum mechanics and in quantum field theory: On one hand, different choices of reference frames lead to different quantizations resulting physically inequivalent quantum systems, and, on the other hand, the time variable introduced this way generally cannot be regarded a physical one (the elapsed proper time between two spacelike hypersurfaces is space dependent in general). In quantum gravity, the corresponding situation is called "the problem of time" [3, 4, 5].

Another observation about the formalism of quantum mechanics is that the wave function, and, correspondingly, operators, events and states, are gauge dependent quantities. More closely, when a charged particle is considered in a given electromagnetic field, the Hamiltonian of the system is constructed from the electromagnetic four-potential. This leads to the consequence that different choices of the potential

lead to a spacetime dependent transformation of the wave function. The formalism of quantum field theory inherits this property from quantum mechanics.

In addition, the wave function possesses another, smaller but also important, arbitrariness, an ambiguity up to an arbitrary constant phase factor. It is in fact a ray in the Hilbert space, rather than an element of the Hilbert space, that bears a physical meaning.

If we call a frame dependent and/or gauge dependent formalism a relative formalism, then the aim of the approach presented in this paper can be expressed as to develop an absolute formalism for quantum mechanics. A frame free and more spacetime friendly formalism can shed new light on, and can give a better understanding of, the known troublesome properties of quantum theories on curved spacetimes. In parallel, a gauge independent formalism for quantum mechanics, and for quantum field theory, may have natural advantages, too (see, e.g., [6] and references therein). Furthermore, by their very nature, absolute quantities have a more direct physical interpretation, thus they are interesting for measurement theory as well. These are the initial motivations to seek an absolute formalism. On the other hand, it will turn out that, as we start to build up our approach—in this paper we will concentrate basically on (one-particle zero spin) nonrelativistic quantum mechanics—, new additional benefits will appear immediately.

Spacetime is a given background in quantum mechanics, and an absolute formulation of quantum mechanics needs first an absolute spacetime formalism. The first appearance of such a formalism can be found in [7]. Since then several similar treatments have been worked out, and, for example, in general relativity and manifold theory the coordinate free language is widely used. For nonrelativistic and special relativistic spacetime a profound and detailed presentation can be found in [8] (see also [9]), and in our approach we will use the treatment appearing there. The basic features of nonrelativistic spacetime are the affine structure of spacetime, an absolute time structure on it, and a Euclidean structure on the hyperplanes of simultaneous spacetime points. It is important to note that space is not absolute even nonrelativistically, different reference frames have different own spaces—for example, the formula of the special Galilean transformation rule, $t' = t$, $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t$ (with the usual notations), reflects this in such a way that under a transformation 'position is not mixed into time' (time is absolute) but 'time is mixed into position' (space is relative).

Next, an absolute formalism for classical mechanics is needed. To build up an absolute presentation of classical mechanics, world lines and the Newton equation can be formulated in an absolute way in a straightforward manner. On the other side, as space is frame dependent, and as, for conservative forces (such as the electromagnetic Lorentz force), canonical momenta are both frame and gauge dependent, phase space is not an absolute concept. As observed by Souriau [10] (see also [11]), it is the space of the solutions of the equation of motion (in short, processes) that is the corresponding absolute object. In the case of conservative forces this process space admits an absolute symplectic structure, as an absolute counterpart of the

symplectic structure of the phase space. Events, states and physical quantities (observables), whose usual introduction is based on the phase space, can be given an absolute definition, too, on the basis of the process space [11].

For quantum mechanics, a similar approach is developed in [11]. There the Schrödinger equation is understood as a partial differential equation for complex functions on spacetime, processes are the solutions of the Schrödinger equation, and the process space is found to admit a Hilbert space structure. Then, events, states and physical quantities are introduced with respect to the Hilbert space of *processes*.

Unfortunately, that formalism of quantum mechanics eliminates only a part of relativeness. The basic problem is the fact that the Schrödinger equation, even in its most absolute form, needs a choice of a gauge and a four-velocity value (the latter is mathematically equivalent to a choice of an inertial observer). Hence, to obtain an absolute approach, here our strategy is, first, to find such new quantities and equations instead of the wave function and the Schrödinger equation that do not require these choices, and second, to build from the resulting absolute process space the event space also in an absolute way, and third, to introduce states and physical quantities based on the absolute event space. Thus the whole theory is given in an absolute form.

Concerning the first step, our starting point will be to make use of the polar decomposition of the wave function. This method of introducing gauge independent (and ray ambiguity free) quantities and equations that are equivalent to the wave function and the Schrödinger equation has already been applied by several authors [12, 13, 14, 15, 16]. Since here our further requirement is to be frame free, too, we will construct such new quantities and equations that fulfil all these demands. Instead of the complex valued wave function, our absolute process function turns out to be a real cotensor field. (Actually, to offer flexibility, we will provide different equivalent sets of absolute quantities and equations to describe the absolute processes in different ways.) The obtained absolute equations exhibit the three independent physical aspects of the system. In the case of the Schrödinger equation these aspects are present in a mixed and more hidden form.

The geometry of the resulting absolute process space proves to be different from the geometry of a Hilbert space. The structure of the 'intermediate step', the space of rays of a Hilbert space—which is studied in [17]—stands closer to it. Namely, it will turn out that the absolute process space is a complex Hilbert manifold. As will be useful to observe, an equivalent description is also available: It can be considered as a real Kähler manifold, too, including a Riemannian and a symplectic structure. This latter aspect is useful, for example, for expressing the geodesic curves of the absolute process space in a transparent way.

The next step is to derive the event space from the absolute process space, based on the explored properties of the latter. The events of quantum mechanics, which in a Hilbert space formalism are linear subspaces of the Hilbert space, are formulated here as appropriate subsets of the absolute process space. The structures of the quantum mechanical event space, including the operations 'and', 'or' and 'negation',

are also given an absolute definition.

Then states and physical quantities can be defined with respect to this event space formulation. To give examples, some physical quantities—position, velocity, kinetic energy—will be discussed more closely. As a part, we show, by means of the absolute quantities, a simple derivation of the position-velocity uncertainty formula. (We prefer velocity to canonical momentum since velocity is a gauge free quantity while canonical momentum is not.) Indeed, we will obtain some sharper inequalities than the usual uncertainty formula. The Ehrenfest theorem will be presented, too, by the use of absolute quantities and equations. It will be a remarkable observation that this theorem proves to be valid not only for conservative forces (e.g., the Lorentz force) but for more general ones, too.

Actually, we will see that the whole absolute formalism accepts more general types of forces than conservative ones. One of the opening possibilities is the treatment of dissipative forces. Another opportunity is to use the absolute framework in the search of generalizations of quantum mechanics. The latter topic is also mentioned as the search for nonlinear extensions of the Schrödinger equation. A well-known such extension is the so-called nonlinear Schrödinger equation, and other ones are worked out and discussed in [18] and in [19], for example. A framework for generalizations of quantum mechanics is given by [17], in a ray ambiguity free treatment. The absolute formalism provides an approach which, in addition, is frame free and gauge free. Thus it gives a convenient and safe framework to introduce new generalizations of quantum mechanics, or to test already existing ones. We will demonstrate this by checking that both the so-called nonlinear Schrödinger equation and the equation proposed by Byalinicki-Birula and Mycielski [18] satisfy the requirement of having a ray ambiguity free, frame free, and gauge free physical content.

Concerning the treatment of dissipative forces, it is important to note that, in contrast to a Schrödinger equation with a nonhermitian Hamiltonian, the dynamics given by the absolute equations of a dissipative system is 'unitary', i.e., the total probability of finding the particle is time independent. It is the mechanical energy that decreases in time. These properties are in accord with the physical expectations.

We will investigate a dissipative example, the case of the damping force with linear velocity dependence, in detail. We will be interested in the time dependence of the expectation value and uncertainty of position and of the expectation value of kinetic energy. In particular, an interesting question is whether this damping force stops the spread of a 'wave packet' that is unavoidably present in the free particle case, whether the 'wave packet' (the process function) will tend to a stationary solution. To answer these questions (for simplicity, the calculations are done for the 1+1 dimensional system), first we determine the stationary solutions of the equations. It turns out that there is no normalizable stationary solution, hence there is no limiting stationary solution the process function could tend to. Then, for arbitrary initial conditions, we determine the exact time dependence of the expectation value of position, and the asymptotic time dependence of the uncertainty of position and of the

expectation value of kinetic energy. We find that the expectation value of position stops exponentially. Furthermore, the spread of the 'wave packet' is found to be slower than in the free particle case—the asymptotic time dependence of the uncertainty of position is proportional to $t^{1/4}$ instead of t—but, as the result shows, this spread will never stop. In parallel, the expectation value of kinetic energy dissipates completely, it tends to zero, following a $t^{-1/2}$ asymptotic time dependence.

Turning back to the case of the charged particle in an electromagnetic field, there the absolute formalism works with the electromagnetic field strength tensor instead of a corresponding four-potential. This is an interesting feature from the aspect of the Aharonov-Bohm effect (see [20] for the necessary details). In this effect the particle is excluded from a cylindrical space region, however, its motion is influenced by a magnetic field which is nonzero only within this region.The quantitative apperance of the Aharonov-Bohm effect is the integral of the vector potential along a closed curve around the cylinder which hides the magnetic field. This quantity is just the flux of the magnetic field within the cylinder, thus it can be expressed by means of the electromagnetic field strength tensor only. However, if we think in terms of the field strength, it is physically strange that the particle is excluded from the inside of the cylinder by a high potential barrier, nevertheless, it 'observes' the magnetic field which is present only inside the cylinder. It turns out that the absolute formalism seems to be promising to understand this effect only by means of the electromagnetic field strength tensor. Namely, we will show a model computation which gives the idea that while the position probability density is excluded from the inside of the cylinder, there are other absolute quantities—being also necessary for the complete description of the particle—which do not vanish within the cylinder. They seem to penetrate unavoidably into the cylinder, which suggests the picture that these quantities are those through which the particle 'feels' the inner magnetic field.

Concerning (zero spin) relativistic quantum mechanics, in this paper we take the first step: To give absolute quantities and equations equivalent to the Klein-Gordon process function and the Klein-Gordon equation. We do it both on special and general relativistic spacetime. Then in the special relativistic case we show how the nonrelativistic limit of the quantities and equations can be performed. Similarly, in the case of curved spacetime we carry out the nonrelativistic plus weak gravity limit of the quantities and equations. After finding the appropriate way to deduce the nonrelativistic quantities from the general relativistic ones, the result will be the expected one: The appearance of the nonrelativistic form of the gravitational force in addition to the effect of the electromagnetic field.

The paper is organized as follows. The elements of the absolute description of nonrelativistic spacetime necessary for our considerations are collected and summarized in Sect. II. In Sect. III. the absolute equivalents of the wave function and the Schrödinger equation are given. We explore the geometric properties of the absolute process space in Sect. IV. Events, states and physical quantities are introduced and discussed in Sect. V. and Sect. VI. Sect. VII. deals with the 'nonlinear' and

the dissipative generalizations of quantum mechanics, and Sect. VIII. presents the calculations about the dissipative example system mentioned above. The Aharonov-Bohm effect is investigated in Sect. IX. The relativistic quantities and equations, and their limiting cases are shown in Sect. X. Sect. XI. contains the Discussion and outlook.

Finally we remark that we will use a formalism which does not need choices of measurement units when dealing with physical quantities [8, 9]. For example, distance values, time period values and mass values are physically different. To reflect this, instead of using the line of real numbers, $\mathbb{R}$, for all these kinds of quantities, we will use different one dimensional vector spaces—called measure lines—for them. Products and quotients of quantities of different dimensions are established by tensorial products and quotients of one dimensional vector spaces, and a choice of a measurement unit is formulated as a choice of an element of the measure line—the choice of a basis in the one dimensional vector space, indeed. As one can check, this treatment reflects exactly the expected properties and rules concerning how to handle dimensionful quantities, for example, the ones concerning multiplication of quantities of different dimensions, or the ones about changing measurement units.

Actually, a general one dimensional vector space differs from $\mathbb{R}$ only in that in $\mathbb{R}$ there is a distinguished unit element, otherwise all the linear properties are exactly the same. Thus, while being technically easy, this use of one dimensional vector spaces ensures a correct handling ('bookkeeping') of the different dimensions and makes it possible to work with physical quantities without choosing measurement units. Actually, a choice of a measurement unit is also a relative step, not belonging to the examined physical system but to the way we are doing our observations about it. The exception is when the physical system or phenomenon itself owns some dimensionful parameters. Then these values offer distinguished identifications between different measure lines. For instance, in relativistic physics the speed of light offers an identification between space distances and time intervals. In some cases there are enough dimensionful parameters that all quantities can be made dimensionless—the dissipative system treated in detail in our Sect. VIII. will be such an example.

The measure lines of distance values, time intervals and mass values will be denoted by $\mathbb{D}$, $\mathbb{I}$, and $\mathbb{G}$, respectively. As, for instance, $\mathbb{D}^* \equiv \mathbb{R}/\mathbb{D}$, we see that inverse length values, e.g., $1cm^{-1}$, 'reside' in $\mathbb{D}^*$. As another example, a magnitude of a force is an element of $\mathbb{G} \otimes \mathbb{D}/\mathbb{I}^{(2)} = \mathbb{G} \otimes \mathbb{D} \otimes (\mathbb{I}^*)^{(2)}$ (notation: $\mathbb{I}^{(2)} = \mathbb{I} \otimes \mathbb{I}$).

II. The nonrelativistic spacetime model

A useful way to collect and present the necessary properties of nonrelativistic spacetime is to do it through a comparison with the corresponding elements of special relativistic spacetime. In this way the basic similarities and differences are well illuminated. Therefore, first we give a short overview of the special relativistic

spacetime. Both spacetimes will be treated in the absolute formalism of [8] (see also [9]).

A general relativistic, or curved, spacetime is given by M, $\mathbb{I}$, and g, where M is a four dimensional oriented manifold, $\mathbb{I}$ is the measure line of time intervals, and g is an arrow oriented Lorentz form on M. Special relativistic spacetime is the special case when M is an affine space and g is a constant tensor field on it. In this case the tangent bundle is of the form $M \times \mathbb{M}$, the tangent space at each spacetime point is $\mathbb{M}$, where $\mathbb{M}$ is the underlying vector space of M. World lines, which are the models for pointlike classical material objects, are curves on spacetime with timelike tangent vectors at each point. Between two points of a world line the proper time length (taking its value in $\mathbb{I}$) can be defined with the aid of g. Usually world lines are considered with a proper time parametrization on them. The tangent vectors of a world line obtained by derivation with respect to proper time are elements of $\mathbb{M}/\mathbb{I}$ and satisfy the following relation:

$$g(u, u) \equiv u \cdot u = 1 \tag{1}$$

[our metric convention is $(+, -, -, -)$; inner products, and the action and composition of linear maps, are denoted by a dot product]. The futurelike elements of $\mathbb{M}/\mathbb{I}$ satisfying the condition (1) are the absolute velocity (or four-velocity) values. Their set is denoted by $V(1)$. An inertial world line, i.e., a straight line, has a constant absolute velocity, a given value $a \in V(1)$. Although its elements are vectors, $V(1)$ itself is not a vector space but only a three dimensional submanifold of $\mathbb{M}/\mathbb{I}$.

An observer, usually called also an observer field or a reference frame, is in reality a collection of pointlike material objects. Any of these material points embodies a point in the observer's space. Since a material point is described by a world line, an observer is formulated mathematically as a continuous collection of world lines, a foliation of spacetime by a system of world lines. Such a system of world lines can be given conveniently by the corresponding absolute velocity field, the world lines being the integral curves of the absolute velocity field.

This definition allows very general observers, for instance, nonrigid ones (modelling nonrigid reference media, similarly as, e.g., a fluid or a dust is used as a reference medium in general relativity), however, here we will be interested only in inertial observers, for which the velocity field is a constant a. Then the world lines of the observer are parallel timelike straight lines. By means of light signals a synchronization can be carried out; the synchronous points of the different world lines prove to form parallel three dimensional spacelike hyperplanes, these hyperplanes are g-orthogonal to the world lines of the observer. The elapsed time between two such hyperplanes (actually, two such 'instants' of the inertial observer) is measured by the proper time length of the part of any of the world lines falling in between the two hyperplanes. The system of these hyperplanes gives a foliation of the spacetime. The world lines of the observer—indeed, the space points of the observer—prove to form a three dimensional affine space, E_a, over $\mathbb{E}_a$, the three dimensional subspace of $\mathbb{M}$ which is g-orthogonal to a. The restriction of g to $\mathbb{E}_a$ furnishes E_a with a

Euclidean structure. The hyperplanes (the time points of the observer) form a one dimensional affine space, I_a, over $\mathbb{I}$. The observer observes a spacetime point p as an a-time point (the hyperplane containing p) and as an a-space point (the world line of the observer that contains p).

The inertial observer a 'splits' a spacetime vector x into a time interval value and an a-space vector (an element of $\mathbb{E}_a$) by the projection-like mappings,

$$\boldsymbol{\tau}_a\, x = a \cdot x, \qquad \boldsymbol{\pi}_a x = x - (\boldsymbol{\tau}_a\, x)a,$$

respectively. After choosing a spacetime point o as a spacetime origin, spacetime points also become possible to be described by time intervals and a-space vectors: The observer can characterize a spacetime point p by $\boldsymbol{\tau}_a\,(p - o)$ and $\boldsymbol{\pi}_a(p - o)$. If, in addition, an orthonormal, positively oriented basis in $\mathbb{E}_a/\mathbb{I}$ is chosen, then a-space vectors can be described by $\mathbb{I}$ valued coordinates. Hence, an origin o and such a basis turns the observer into a (Cartesian) coordinate system.

The inertial observer a observes an arbitrary world line r as a curve $r_a : I_a \to E_a$ in its space. The connection between an absolute velocity value $u \in V(1)$ of the world line r and the corresponding relative velocity value $v \in \mathbb{E}_a/\mathbb{I}$, obtained by differentiating r_a with respect to its I_a-variable, is

$$v = \frac{u}{a \cdot u} - a.$$

The set of spacetime covectors, $\mathbb{M}^*$ (the dual of $\mathbb{M}$), can be identified with $\mathbb{M}/\mathbb{I}^{(2)}$, through the nondegenerate form g. Thus $\mathbb{M}^*$ inherits the light cone structure that $\mathbb{M}$ owns. (It is this identification through g that leads to the possibility to raise and lower the indices in formulas with indices.)

Compared to the special relativistic spacetime, nonrelativistic spacetime is also considered a (four dimensional, real, oriented) affine space M (over the linear space of spacetime vectors, $\mathbb{M}$), but in this case an absolute time / absolute simultaneity structure is assumed. (Actually, it is not an absolute *space* but an absolute *simultaneity* that is needed to formulate, e.g., gravitational force, as an action-at-a-distance.) Thus, in addition to M, a one dimensional oriented real affine space I is considered as the set of absolute instants. Its underlying vector space is $\mathbb{I}$, the set of time interval values. Absolute simultaneity is given by an affine surjection, $\tau : M \to I$, which provides a distinguished system of hyperplanes on M. One such hyperplane is formed by those spacetime points p which share the same time value $t = \tau(p)$. The linear surjection $\boldsymbol{\tau} : \mathbb{M} \to \mathbb{I}$ that underlies the affine surjection τ assigns to any spacetime vector a time interval, 'its time length'. The kernel of $\boldsymbol{\tau}$, $\mathbb{E} \subset \mathbb{M}$, is a three dimensional real oriented vector space. $\mathbb{E}$ is the set of spacelike vectors (the spacetime vectors having a zero time interval). Nonrelativistically only spacelike vectors have an inner product structure, a Euclidean one indeed: A positive definite symmetric bilinear map $b : \mathbb{E} \times \mathbb{E} \to \mathbb{D}^{(2)}$. (Here $\mathbb{D}^{(2)}$ expresses the fact that the length of a spacelike vector q, $|q| = [b(q, q)]^{1/2}$, has a dimension of length. Similarly, relativistically g maps to $\mathbb{I}^{(2)}$, instead of $\mathbb{R}$.) Spacetime vectors which are

not spacelike are called timelike, the ones that $\boldsymbol{\tau}$ assigns a positive or negative time interval value to are called future-directed or past-directed, respectively.

World lines are again curves in M having future-directed tangent vectors only. Now it is the absolute time that provides a physically distinguished, natural parametrization for the world lines. The tangent vectors obtained by differentiating with respect to this parametrization prove to be elements of the set $V(1) := \{u \in \mathbb{M}/\mathbb{I} \mid \boldsymbol{\tau}\, u = 1\}$. The elements of $V(1)$ are the absolute velocity vectors. Now $V(1)$ is a three dimensional affine subspace of $\mathbb{M}/\mathbb{I}$, over $\mathbb{E}/\mathbb{I}$. Inertial world lines are again the ones having a constant absolute velocity, $a \in V(1)$.

Observers, and, in particular, inertial ones, are defined in a way similar to that in the special relativistic case. Now synchronization is carried out not by means of light signals but with the aid of absolute time (because now it is thought that physically an absolute time is available, 'all watches go the same'). All inertial observers use τ to assign time points to the spacetime points, thus they do not have different I_as but the same I. On the other hand, different observers have different world line systems, world lines of one observer are not parallel to the world lines of the other. As a result, the spaces of the observers will be different; these E_as prove to be three dimensional Euclidean affine spaces over the set of spacelike vectors, $\mathbb{E}$.

Now an inertial observer observes a spacetime vector x as a time interval $\boldsymbol{\tau}\, x$ and as a a-space vector $\boldsymbol{\pi}_a x = x - (\boldsymbol{\tau}\, x)a$. Like in the special relativistic case, with a choice of an origin and a basis, the observer establishes a coordinate system; here the basis is chosen in $\mathbb{E}/\mathbb{D}$ and now time coordinates take their values in $\mathbb{I}$, while space coordinates take their values in $\mathbb{D}$.

The observer a observes an absolute velocity u as the relative velocity

$$v = u - a, \qquad v \in \mathbb{E}/\mathbb{I}, \qquad |v| \in \mathbb{D}/\mathbb{I}. \tag{2}$$

In a coordinate system, the time coordinate of an absolute velocity is always 1, and the time coordinate of a relative velocity is always 0.

Now the set of spacetime covectors (the dual of the linear space of spacetime vectors), $\mathbb{M}^*$, has a structure different from that of $\mathbb{M}$, because now there is no possibility for identification. In $\mathbb{M}$ a three dimensional subspace, $\mathbb{E}$, exists as a distinguished subset while in $\mathbb{M}^*$ a one dimensional subspace will be the only distinguished subset. It is formed by those covectors k for which $k \cdot q = 0$ ($\forall q \in \mathbb{E}$). These covectors are called timelike and the others are called spacelike. (Because timelike covectors are the ones that 'examine' the timelikeness of spacetime vectors, i.e., they give zero for spacelike ones and nonzero for timelike ones.)

The Euclidean form b provides here an identification not to connect $\mathbb{M}^*$ with $\mathbb{M}$ but to connect $\mathbb{E}^*$ with $\mathbb{E}$. More closely, the identification is possible between $\mathbb{E}^*$ and $\mathbb{E}/\mathbb{D}^{(2)}$. It is important to observe that $\mathbb{E}^*$ is not a subset of $\mathbb{M}^*$, but the restriction of a $k \in \mathbb{M}^*$, $k|_{\mathbb{E}}$, is an element of $\mathbb{E}^*$.

The linear map $\eta : \mathbb{M}^* \to \mathbb{E}^*$, $k \mapsto k|_{\mathbb{E}}$ plays an important role in the a-splitting of spacetime covectors: an inertial observer a splits a covector k into $k \cdot a \in \mathbb{I}^*$ and into $\eta \cdot k \in \mathbb{E}^*$. If a basis in $\mathbb{E}/\mathbb{D}$ is chosen (to describe spacelike vectors by $\mathbb{D}$

valued coordinates) then, via the dual basis, $\eta \cdot k$ will be possible to be characterized by coordinates, too (namely, by $\mathbb{D}^*$ valued ones).

How tensors and cotensors are split for an observer can also be given with the aid of $\boldsymbol{\tau}$, $\boldsymbol{\pi}_a$, and η. We will make use of the transformation rules connecting coordinates of vectors, covectors and cotensors in two different inertial coordinate systems. If the same $\mathbb{E}/\mathbb{D}$-bases are chosen for the two observers, which means physically that the coordinate axes of the two reference frames are parallel, then the transformation formulas are

$$(x^0)' = x^0, \qquad (x^\alpha)' = x^\alpha - v^\alpha x^0 \tag{3}$$

for vectors,

$$(k_0)' = k_0 + v^\alpha k_\alpha, \qquad (k_\alpha)' = k_\alpha \tag{4}$$

for covectors, and

$$(C_{00})' = C_{00} + v^\alpha(C_{0\alpha} + C_{\alpha 0}) + (1/2)v^\alpha v^\beta(C_{\alpha\beta} + C_{\beta\alpha}),$$

$$(C_{\alpha 0})' = C_{\alpha 0} + v^\beta C_{\alpha\beta}, \qquad (C_{0\beta})' = C_{0\beta} + v^\alpha C_{\alpha\beta}, \qquad (C_{\alpha\beta})' = C_{\alpha\beta} \tag{5}$$

for cotensors, where v is the relative velocity of the two observers (our notations are 0 for timelike, greek letters for spacelike and latin ones for spacetime indices). For cotensors of higher rank the transformation formula proves to be a straightforward generalization of (5).

III. Absolute quantities and equations

To find absolute quantities and equations equivalent to the wave function and the Schrödinger equation, our starting point is the Schrödinger equation given in the absolute spacetime formalism (cf. [11]). With the aid of an arbitrary four-velocity value a and a four-potential A—corresponding to the electromagnetic field strength F—, it reads

$$i\hbar(a \cdot D_A)\Psi = -\frac{\hbar^2}{2m}(\eta \cdot D_A) \cdot (\eta \cdot D_A)\Psi \tag{6}$$

with $D_A = D_M - i(e/\hbar)A$, where D_M is the derivation on the spacetime M and A denotes the operator of multiplication by A as well. The operators $(e/\hbar)A$, D_M and D_A are $\mathbb{M}^*$ valued vector operators (coordinate-freely, a V valued vector operator on a Hilbert space H is a linear map $H \to V \otimes H$, see [11]).

The solution space of (6) is not only linear but proves to be a separable Hilbert space (let us denote it by $H_{a,A}$) with respect to the scalar product

$$\langle \Psi_1, \Psi_2 \rangle = \int_{E_t} \Psi_1^* \Psi_2. \tag{7}$$

Here integration is performed on one of the spacelike hyperplanes; from (6) it follows that the definition of the scalar product does not depend on which hyperplane was chosen.

Four-velocity values and inertial observers are in a one-to-one correspondence with each other, thus a choice of a is equivalent to a choice of the corresponding inertial observer. That a choice of a a is unavoidable for the Schrödinger equation is reflected in that if another value a' is chosen, a solution of (6) does not remain invariant. Instead, it transforms as

$$\Psi_{a'}(p) = \Psi_a(p) \cdot \exp\left[i\frac{m}{\hbar}\left(\frac{v^2}{2}t - v \cdot q\right)\right] \tag{8}$$

with $v = a' - a$, and with $t = \boldsymbol{\tau}\,(p - p_0)$ as the timelike and $q = \boldsymbol{\pi}_a(p - p_0)$ as the a-spacelike component of the spacetime vector $p - p_0$, where p_0 is an arbitrary auxiliary spacetime point (different choices of it mean only constant phase factors, which is irrelevant). Similarly, a change in the choice of the four-potential, $A' = A + D_M\alpha$, involves the transformation of a solution of (6),

$$\Psi_{A'}(p) = \Psi_A(p) \cdot \exp\left[i\frac{e}{\hbar}\alpha(p)\right].$$

As a consequence, operators are also observer and gauge dependent. Different choices of an observer and a four-potential lead to the transformation $O' = TOT^{-1}$ of the operators, where T is the following multiplication operator:

$$T = \exp\left[i\frac{m}{\hbar}\left(\frac{v^2}{2}t - v \cdot q + \frac{e}{m}\alpha\right)\right].$$

After choosing a coordinate system for the inertial observer a, (6) turns into

$$i\hbar D_0\Psi = -\frac{\hbar^2}{2m}D_\alpha D_\alpha\Psi, \tag{9}$$

where $D_k = \partial_k - i(e/\hbar)A_k$. We will use the coordinate form of Eq. (8), too. To obtain this, coordinate systems are to be chosen for both the observers a and a'. If the two observers 'use' the same origin $o \in M$ and the same basis in $\mathbb{E}/\mathbb{D}$ to turn themselves into coordinate systems, one finds that the connection between $\Psi_a(t, r)$ and $\Psi_{a'}(t', r')$ is

$$\Psi_{a'}(t', r') = \Psi_a(t', r' + vt') \cdot \exp\left[i\frac{m}{\hbar}\left(-\frac{v^2}{2}t' - v \cdot r'\right)\right]. \tag{10}$$

We want to find absolute quantities instead of the relative quantity Ψ. To this end, let us consider the polar decomposition of the wave function, $\Psi = Re^{iS}$, where R and S are real functions. R is an absolute quantity as boosts, gauge transformations and the ray indefiniteness all touch only the phase of the wave function. From S we constitute

$$\varepsilon := \frac{\hbar}{m}\partial_0 S - \frac{e}{m}A_0 \tag{11}$$

and

$$u_\alpha := \frac{\hbar}{m}\partial_\alpha S - \frac{e}{m}A_\alpha. \tag{12}$$

The quantities ε and u_α are also gauge independent as well as ray ambiguity free. The definitions (11), (12) imply the following consistency conditions:

$$\partial_\alpha\varepsilon - \partial_0 u_\alpha = -\frac{e}{m}F_{\alpha 0} = -\frac{e}{m}E_\alpha, \tag{13}$$

$$\partial_\alpha u_\beta - \partial_\beta u_\alpha = -\frac{e}{m}F_{\alpha\beta} = -\frac{e}{m}\epsilon_{\alpha\beta\gamma}B_\gamma, \tag{14}$$

where the observer a splits the electromagnetic field tensor $F = dA$ into the electric field E and the magnetic field B. Conversely, the equations (13), (14) are sufficient conditions for the way in which Ψ can be reconstructed from R, ε and u_α,

$$\Psi(p) = R \cdot \exp\left[i\left\{\frac{m}{\hbar}\int_o^p\left[\left(\varepsilon + \frac{e}{m}A_0\right)dx^0 + \left(u_\alpha + \frac{e}{m}A_\alpha\right)dx^\alpha\right] + S(o)\right\}\right], \tag{15}$$

to be independent of the path of integration. The constant $S(o)$ in (15) remains undetermined, which means indefiniteness up to a constant phase factor. This is just in accord with the fact that not wave functions but rays bear a physical meaning. Thus we see that the quantities R, ε and u_α prove to be equivalent to the wave function—from the physical point of view, i.e., they are equivalent to the ray whose representative is Ψ.

The quantities ε and u_α are derived with the aid of a coordinate system. To see what coordinate and observer free quantities can be found behind them, let us compute their covariance properties under a change of the observer. A quick look at (11) and (12) might suggest that ε and u_α form a covector, but actually they do not, since S is not boost invariant [cf. (8) or (10)]. Instead, we find that they are transformed as

$$\varepsilon' = \varepsilon + v^\alpha u_\alpha - (1/2)v^2, \qquad u'_\alpha = u_\alpha - v^\alpha.$$

Hence, first, u_α can be regarded as the space components of an absolute velocity field u:

$$u^0 := 1, \qquad u^\alpha := u_\alpha$$

[cf. (3), and the remark made after (2)]. It is easy to see that the probability four-current, which, in terms of the wave function, reads

$$j^0 := \Psi^*\Psi, \qquad j^\alpha := \frac{\hbar}{2im}\left[\Psi^*\partial_\alpha\Psi - \Psi\partial_\alpha\Psi^*\right],$$

can be expressed with this u as $j = R^2 u$. Next, from ε the scalar quantity

$$s := -\varepsilon - (1/2)u_\alpha u_\alpha \tag{16}$$

can be formed. With these quantities the Schrödinger equation, imposed on the complex wave function, proves to be equivalent to the following two real equations:

$$sR + \frac{\hbar^2}{2m^2}\triangle R = 0, \tag{17}$$

$$\mathrm{Div}\, j = 0. \tag{18}$$

Concerning the first equation we remark that nonrelativistically the Laplacian, in a coordinate system, $\triangle = \partial_\alpha \partial_\alpha$, is an absolute operation. Expressing (13) and (14) with u and s, in the absolute form, yields

$$u \cdot D_M u + \eta \cdot D_M s = \frac{e}{m} \eta \cdot Fu \tag{19}$$

and

$$(\eta D_M u)^T - (\eta D_M u) = (\eta \otimes \eta) \cdot F. \tag{20}$$

In a coordinate system (19) reads

$$\partial_0 u_\alpha + u_\beta \partial_\beta u_\alpha + \partial_\alpha s = \frac{e}{m} \left(E_\alpha + \epsilon_{\alpha\beta\gamma} u_\beta B_\gamma \right). \tag{21}$$

Then an absolute process can be given by the quantities R, u, and s, satisfying the absolute equations (17), (18), (19), and (20). Naturally, wherever R will be mentioned, $\varrho = R^2$ can be taken, instead, as well. The quantity ϱ has a direct physical meaning, as expressing the position probability density.

We remark that in the hydrodynamical formalisms [12, 13, 14, 15, 16], usually the quantities R, u_α, and s, understood with the corresponding equations, are used. s is usually called the "quantum potential", because of its appearance in (21) (usually it is placed to the r.h.s. as $-\partial_\alpha s$). We do not use this misleading name because s is not an outer, given quantity but a variable, it is a part of the absolute process describing the particle.

Alternatively, ε and u_α can be viewed as components of a two-cotensor:

$$z_{00} = \varepsilon, \qquad z_{\alpha 0} = z_{0\alpha} = (1/2)u_\alpha, \qquad z_{\alpha\beta} = -(1/2)\delta_{\alpha\beta} \tag{22}$$

[cf. (5)], or components of a three-cotensor:

$$w_{000} = \varepsilon, \qquad w_{\alpha 00} = u_\alpha, \qquad w_{0\alpha 0} = w_{00\alpha} = 0,$$
$$w_{\alpha\beta 0} = w_{\alpha 0\beta} = -w_{0\alpha\beta} = -\tfrac{1}{2}\delta_{\alpha\beta}, \qquad w_{\alpha\beta\gamma} = 0. \tag{23}$$

This latter cotensor provides an elegant way to formulate the equations (13) and (14) as

$$\partial_k w_{l00} - \partial_l w_{k00} = -\frac{e}{m} F_{kl}. \tag{24}$$

The coordinate free form of this equation is

$$D_M \wedge w = -\frac{e}{m} F \otimes (\boldsymbol{\tau} \otimes \boldsymbol{\tau}), \tag{25}$$

where in $D_M \wedge w = (D_M w)^T - D_M w$ transposition occurs in the first and last variables of the four-cotensor. We remark that $(\boldsymbol{\tau} \otimes \boldsymbol{\tau})_{mn} = \delta_{m0}\delta_{n0}$ in any inertial coordinate system.

The quantity s can be considered as an invariant scalar of the cotensor w. Similarly, u is an invariant four-vector of w. Furthermore, from $W := \varrho w$ all the

quantities found above (ϱ, u, s, j, z, w, and, in a reference frame, ε) can be recovered. Therefore, in the following we will refer to an absolute process as W, a process function which is a three-cotensor field.

The complete set of absolute equations is formed by (17), (18) and (25). Eq. (25) is equivalent to the two equations (19) and (20), thus (17), (18), (19) and (20) also give a complete set of equations. The set of process functions W satisfying the absolute equations is the absolute process space (denoted by $\mathcal{S}$).

Let us investigate the physical meaning of the absolute equations. Eq. (18) is the continuity equation for the probability four-current. Eq. (25) tells how the outer field excerts its action on the particle. In Sect. X. we will see that relativistically a similar equation will hold for a covector quantity. The nonrelativistic limit of the components of that covector will lead to the components of the nonrelativistic w. At last, (17) is the nonrelativistic and quantum appearance of the mass shell condition for the particle. To see this, let us recall that in relativistic classical mechanics the connection between kinetic energy and momentum of a point particle of mass m is the mass shell condition $p^k p_k = m^2$. In Sect. X. the corresponding quantum equation will be presented. In nonrelativistic classical mechanics this mass shell condition is $2mK - p_\alpha p_\alpha = 0$ ($K = p_\alpha p_\alpha / 2m$). As we will see in Sect. VI., ε and u_α are related to the kinetic energy, respectively the velocity, physical quantities of the particle. This, together with (16) and the corresponding relativistic formula (89), makes it transparent that Eq. (17) can be interpreted as the quantum version of the nonrelativistic mass shell condition.

We remark that, in parallel to relativistic classical mechanics, where the Newton equation gives the (proper) time derivative of the kinetic energy-momentum four-vector, in nonrelativistic classical mechanics, kinetic energy and momentum form a cotensor [similarly as ε and u form w, cf. (23)], and the equation which gives how the time derivative of this cotensor is determined by the outer force is equivalent to the Newton equation.

We can observe that, instead of a single equation, the Schrödinger equation, absolutely a system of equations are present. Actually, this happens to be an advantage of the formalism. Namely, it expresses that different elements of the system exist, which are independent. The continuity equation, the way the outer field acts on the particle, and the mass shell condition are physically independent aspects of the system. As an example, in Sect. VII. we will see how the absolute formalism can be extended to be valid for not only electromagnetic (or other conservative) force fields by generalizing (25), while keeping the two other equations untouched. In contrast, in the Schrödinger equation these three elements appear in a mixed and more hidden form.

IV. The structure of the absolute process space

Let us now examine the geometric properties of the absolute process space, $\mathcal{S}$. To do this, the connection with the Hilbert space $H_{a,A}$ will be useful. However, most elements of the structure of $H_{a,A}$, including the scalar product, the distance function defined as the norm of the difference of two elements, and the topology defined by the distance function, are not absolute: They are not ray ambiguity free. To explore the absolute process space, only elements with absolute meaning can be made use of.

The most important of them is the magnitude of the scalar product. This quantity possesses direct physical meaning. We introduce the notation

$$\Gamma(W_1, W_2) := |\langle \Psi_1, \Psi_2 \rangle|,$$

where W_i is the absolute process corresponding to the normalized wave function Ψ_i. From (7) and (15), the explicit form of $\Gamma(W_1, W_2)$ in terms of absolute quantities can be obtained easily (here we will not need the concrete formula).

First, let us see how, with the aid of the function Γ, $\mathcal{S}$ becomes a topological space. For any element, W_0, of $\mathcal{S}$, and any number $\xi \in [0,1)$, let us define the neighbourhood $B_\xi(W_0)$ as formed by those processes W for which $\Gamma(W_0, W) > \xi$. Then, by taking the topology generated by all $B_\xi(W_0)$s, $\mathcal{S}$ is made a topological space.

As the next step, we show that the absolute process space is a metrizable space. To see this, let us look for all such possible distance functions on $\mathcal{S}$ that are compatible with the given topology, in other words, that define the same neighbourhoods (their open balls) as were defined before through Γ. Γ itself is not such a distance function, and ξ is not a radius-like quantity of $B_\xi(W_0)$, because $\Gamma(W, W) = 1$ instead of being zero, and 'bigger and bigger' Bs have smaller and smaller ξs (if $B_{\xi_1}(W_0) \subset B_{\xi_2}(W_0)$ then $\xi_1 > \xi_2$). Based on this observation and the expected properties of a distance function, we look for all such strictly monotonously decreasing non-negative functions f for which the distance function $l := f \circ \Gamma$ satisfies the triangle inequality.

One simple way to find all such possible fs is to work in terms of wave functions, instead of absolute processes. Then the triangle inequality reads as the condition

$$f(|\langle \Psi_1, \Psi_2 \rangle|) + f(|\langle \Psi_1, \Psi_3 \rangle|) \geq f(|\langle \Psi_2, \Psi_3 \rangle|) \tag{26}$$

for any three wave functions of unit norm. By carrying out a Schmidt-orthogonalization, the relative positions of three wave functions can be characterized by some angle parameters. By means of such parameters, the absolute values of the three scalar products in question can be expressed as

$$|\langle \Psi_1, \Psi_2 \rangle| = \cos \alpha, \qquad |\langle \Psi_1, \Psi_3 \rangle| = \cos \beta,$$

$$|\langle \Psi_2, \Psi_3 \rangle| = |\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos \gamma e^{i\delta}|,$$

where $\alpha, \beta, \gamma \in [0, \pi/2]$ and $\delta \in [0, 2\pi)$. After maximizing the r.h.s. of (26), first in δ, and then in γ, one finds that, with fixed α and β, the strongest condition (26) imposes is $f(\cos \alpha) + f(\cos \beta) \geq f[\cos(\alpha + \beta)]$. It is useful to introduce $h := f \circ \cos$, with which the condition reads

$$h(\alpha) + h(\beta) \geq h(\alpha + \beta).$$

Now let us recall that, on any metric space with a distance function l, $g \circ l$ is also a distance function and defines the same open balls, if g is such a strictly monotonously increasing function that $g(0) = 0$ and $g(a+b) \leq g(a) + g(b)$ ($\forall a, b \in \mathbb{R}_0^+$). This degree of nonuniqueness of the distance function is natural on any metrizable topological spaces.

We see that if we choose $h(\alpha) = \alpha$ (which is the most natural choice of h), i.e., we choose the distance function $l = \arccos \Gamma$, then the other allowed distance functions differ from it just to the extent of such a g arbitrariness. Therefore, to the extent any distance function can be unique on a metrizable topological space, $l = \arccos \Gamma$ is the unique distance function on $\mathcal{S}$ that satisfies our requirements. In what follows, we will consider $\mathcal{S}$ a metric space with respect to the distance function $l = \arccos \Gamma$.

Concerning the question of completeness of the metric space $\mathcal{S}$, we remark that the absolute equations, just like the Schrödinder equation, are valid only for sufficiently smooth process functions. One can prove that the absolute version of how one makes $H_{a,A}$ complete by means of Cauchy sequences of smooth wave functions is just how one makes $\mathcal{S}$ complete by means of Cauchy sequences, here with respect to the absolute distance function l.

Our next observation is that the absolute process space is a (complex) Hilbert manifold. This can be seen with the aid of the following homeomorphism between a neighbourhood $B_\xi(W_0)$ and a neighbourhood of a Hilbert space: After fixing an a and A, a ray Θ_0 of $H_{a,A}$ corresponds to W_0. Let us choose an element Ψ_0 from this ray. Similarly, let us choose a Ψ from the ray Θ corresponding to an arbitrary W belonging to the neighbourhood $B_\xi(W_0)$ of W_0. Then

$$\varphi := \frac{|\langle \Psi_0, \Psi \rangle|}{\langle \Psi_0, \Psi \rangle}[\Psi - \langle \Psi_0, \Psi \rangle \Psi_0] \tag{27}$$

is such a vector that, on the one hand, is orthogonal to Ψ_0, and, on the other hand, does not depend on how Ψ was chosen from the ray Θ. This means that, actually, φ characterizes Θ, and, correspondingly, W. The map (27) proves to provide a homeomorphism between $B_\xi(W_0)$ and $\{\varphi \in H_{a,A} \mid \varphi \perp \Psi_0, \|\varphi\| < (1 - \xi^2)^{1/2}\}$, the latter being a neighbourhood of a separable complex Hilbert space.

Based on the parametrization (27), the tangent vectors of $\mathcal{S}$ at W can be brought to a linear one-to-one correspondence with those vectors of $H_{a,A}$ which are orthogonal to Ψ_0. Moreover, if, using this correspondence, we 'transport' the scalar product of $H_{a,A}$ to the absolute tangent space, we obtain an absolute scalar product on it.

As one can check, this scalar product remains invariant under different choices of $\Psi_0 \in \Theta_0$, or different choices of a or A. This way the absolute process space proves to be a complex Hilbert manifold.

Similarly to the treatment that can be found in [17] (where the manifold of the rays of a Hilbert space is studied), this complex Hilbertian structure of $\mathcal{S}$ can be replaced by a Kähler structure. This alternative aspect is the following: Let us consider a tangent space of $\mathcal{S}$ a real vector space, instead of a complex one. The multiplication with i is replaced by the corresponding real linear operator J (having the property that $J^2 = -I$, where I is the identity operator). Then, from the complex scalar product $\langle \varphi, \psi \rangle$, we obtain a (real) Riemannian form,

$$G(\varphi, \psi) := Re\langle \varphi, \psi \rangle,$$

and a (real) symplectic form,

$$\Omega(\varphi, \psi) := -Im\langle \varphi, \psi \rangle.$$

These two forms are related to each other as

$$G(\varphi, \psi) = \Omega(\varphi, J\psi). \tag{28}$$

Hence, this way $\mathcal{S}$ can be viewed as a Kähler manifold: a real, Riemannian and symplectic manifold, having a complex structure and the property (28).

This alternative aspect can be useful, for example, if one wants to investigate the connection between the absolute quantum process space and the corresponding classical one. As we have mentioned, the classical process space is a symplectic manifold, thus the symplectic structure of the quantum process space may provide a transparent connection between the classical system and the quantum one. In terms of phase spaces such an investigation has already been carried out by [17]. Hopefully, a similar connection will turn out for the process spaces, the absolute equivalents of phase spaces, too.

Finally, we show that the distance function that can be defined by G coincides with the distance function l found above. A way to see this is to determine the G-shortest curve between two points with calculus of variation, and to compare its G-length with the l-distance of the two points. With the above notations, the G-length of a curve leading from W_0 to W, in the coordinatization (27) (obtained after a choice of an a, an A, and a Ψ_0), turns out to be

$$\int_0^1 \left\{ \|\dot{\varphi}\|^2 + \frac{1}{1 - \|\varphi\|^2} \left[Re\langle \varphi(t), \dot{\varphi}(t) \rangle\right]^2 - \left[Im\langle \varphi(t), \dot{\varphi}(t) \rangle\right]^2 \right\}^{1/2} dt \tag{29}$$

($\varphi(0) = 0$ is the coordinate of W_0, and $\varphi(1)$ is the coordinate of W). By deriving and solving the corresponding Euler-Lagrange equation, after lengthy but not particularly illuminating calculations, the geodesic between the two points can be given in the following very simple form:

$$\varphi(t) = t \cdot \varphi(1), \qquad t \in [0, 1] \tag{30}$$

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[after fixing the reparametrization ambiguity stemming from the reparametrization invariance of the integral (29)]. The G-length of this geodesic can then be calculated easily, and the result is $\arccos |\langle \Psi_0, \Psi \rangle|$, which is exactly $l(W_0, W)$.

As we see, the form of the geodesic (30) is just the one one expects on intuitive grounds. This also shows an advantage of the Kählerian aspect of the geometry of the process space.

V. Event space and states

The investigation of the mathematical description of the events of quantum mechanics and the relation of the event space to the states and physical quantities of the system was started by Birkhoff and Neumann. These results and later developments (see [21, 22, 23, 11]) led to the following picture: There is a Hilbert space H for the system in such a way that events are projections of H, states are density operators on H, and physical quantities are self-adjoint operators on H. Or, equivalently, as density operators are equivalent to probability distributions (probability laws) on the space of events, self-adjoint operators (through their spectral resolution) are equivalent to projection valued measures, and projections are in a one-to-one correspondence with the (closed) linear subspaces of H, the picture can be expressed in the following way: Events are the linear subspaces, states are probability distributions on the event space, and physical quantities are event valued measures.

This latter scheme is a close parallel of the corresponding situation in classical mechanics ([21, 11]). There events are subsets of the phase space, or, in the absolute treatment, of the process space, states are probability distributions on the event space, and physical quantities are event valued measures (as equivalents of functions defined on the phase space or the process space).

The structure of the classical mechanical event space is as follows. The operation 'and' between two events (subsets) A and B is $A \wedge B := A \cap B$, the operation 'or' is $A \vee B := A \cup B$, the 'negation' $A^\perp$ is the set theoretical complement, and the relation $A \leq B$ is the relation $A \subseteq B$. (Here we do not wish to go into mathematical details, see [11] concerning them.) In quantum mechanics, 'and' is the intersection of linear subspaces, 'or' of A and B is the linear subspace spanned by A and B, the 'negation' is the orthocomplement ($A^\perp$ consists of all the elements of H being orthogonal to A), and the relation $A \leq B$ is the relation $A \subseteq B$. (Again, see [11] for details.)

While in classical mechanics the event space derived from the process space is absolute, in quantum mechanics, even if working from $H_{a,A}$ as the Hilbert space, the resulting events are relative. In the following we establish the absolute event space (it will be denoted by $\mathcal{E}$) based on the absolute process space $\mathcal{S}$. It will be easy to check that, based on (15), both the events of $\mathcal{E}$ and the operations on $\mathcal{E}$ can be brought into one-to-one correspondence with the linear subspaces of $H_{a,A}$ and

the above mentioned operations among them, respectively. Thus the absolute event space expresses completely the same physics as the one built from $H_{a,A}$ does.

The key ingredient in constructing the absolute event space from $\mathcal{S}$ is the following relation on $\mathcal{S}$: Let us call two elements, W_1 and W_2, orthogonal to each other if $\Gamma(W_1, W_2) = 0$. In terms of the distance function l this means that those elements of $\mathcal{S}$ are orthogonal to each other which have the greatest distance 'available' on $\mathcal{S}$. Furthermore, for any subset A of $\mathcal{S}$, let $A^\perp$ denote the set of all such elements of $\mathcal{S}$ that are orthogonal to all elements of A. With the aid of this, we introduce events as such subsets of $\mathcal{S}$ that satisfy the condition $A = (A^\perp)^\perp$.

Then, the operations on $\mathcal{E}$ formed by these events are defined as follows. Let the 'and' of A and B be $A \wedge B := A \cap B$, let the 'negation' of A be $A^\perp$, let the 'or' be defined through the de Morgan formula:

$$A \vee B := (A^\perp \wedge B^\perp)^\perp,$$

and let the relation $A \leq B$ be the relation $A \subseteq B$.

As mentioned, it is easy to see that this event space with these operations is actually just the absolute counterpart of the one built from $H_{a,A}$. This, at the same time, ensures that the introduced operations satisfy the necessary properties so we do not need to check them directly.

In the absolute form of classical mechanics elementary events are those subsets of the process space which contain only one element, i.e., processes are the elementary events. In the event space of Hilbert space quantum mechanics elementary events are the one dimensional linear subspaces, or equivalently, rays, of the Hilbert space. In $\mathcal{E}$ subsets containing only one element are the elementary events. In other words, here also the processes are the elementary events.

Now, having established the absolute event space, we can introduce states as the probability distributions (probability measures) on it. As special examples, pure states are such states that take the value 1 on one elementary event. As is known from the Hilbert space formalism of quantum mechanics, a pure state is actually uniquely determined by the elementary event on which it takes the value 1. Thus a process—an element of the process space—can be regarded as an elementary event and as a pure state, too. In the pure state p_W belonging to the process W, the probability of an elementary event $A_{W'}$ naturally turns out to be $\Gamma^2(W, W')$. Mixed states can be given as convex combinations $\sum_{n \in N} \lambda_n p_n$ of pure states—this property is the absolute parallel for the corresponding property of density operators [11].

VI. Physical quantities

Next, we formulate physical quantities, also with respect to $\mathcal{E}$. We illustrate how the general 'machinery' for giving physical quantities as event valued maps (established in [21] and developed in [11]) works here through the example of the position physical quantity.

Position, which is a vector operator in the Hilbert space language, is a physical quantity that is measured at a given instant with respect to an inertial observer, supplied with a space origin. If, at a time t, the particle is found in a subset B of the space of the observer a (a detector, being at rest with respect to the observer a, observes the particle at t in the space volume B), then this observation is an event of the particle. To express the fact that these position observations are (certain) events of the considered physical system, we formulate position as a map from the subsets of the observer's space to the events of the system.

Namely, this map (let us denote it by $Q_{a,t}$) is the following: For a given instant t, let us assign to a subset B the event formed by those processes W for which

$$\varrho_W(t, \boldsymbol{r}) = 0 \qquad (\forall \boldsymbol{r} \notin B). \tag{31}$$

As in the general framework, a state, i.e., a probability distribution on the events, implies probability distributions for each physical quantity. In a state p, the probability distribution for a physical quantity F is $p \circ F$. For example, in the case of the position, $p \circ Q_{a,t}$ is a probability distribution on the observer's space. This formula is transparent: To a subset B in the observer's space, $Q_{a,t}$ assigns the event $Q_{a,t}(B)$, and $p(Q_{a,t}(B))$ tells the probability for this event, and consequently the probability belonging to B. This value is the probability for finding the particle in B at t.

In the case p is a pure state p_W, one finds that $p(Q_{a,t}(B)) = \int_B \varrho_W(t, \boldsymbol{r}) d^3 \boldsymbol{r}$. As expected, this result tells nothing else but that, in a pure state belonging to the process W, ϱ_W expresses the probability density of the position.

Other physical quantities are formulated similarly. Each physical quantity is a map from the subsets of a vector space to the event space, where the vector space in question is the one that the measured values of the physical quantity are elements of. For example, in the case of the velocity physical quantity, i.e., the relative velocity with respect to an observer a and an instant t, the possible velocity values are the elements of $\mathbb{E}/\mathbb{I}$ [cf. (2)]. Correspondingly, the velocity physical quantity (notation: $V_{a,t}$) is a map assigning events to the subsets of $\mathbb{E}/\mathbb{I}$.

There exists an alternative way to give physical quantities. This other possibility is based on the property that a physical quantity is uniquely determined by its expectation values computed in all the pure states. This fact is the absolute version of the corresponding statement in a Hilbert space formalism, where it is not hard to show that any matrix element of a self-adjoint operator can be determined from some appropriate expectation values of it.

In the case of the velocity physical quantity, which in a Hilbert space formalism leads to the (gauge invariant) velocity component operators $\frac{\hbar}{im}\partial_\alpha - \frac{e}{m}A_\alpha$, the expectation value in a pure state p_W proves to be

$$\langle V_{a,t} \rangle = \int \varrho(u - a), \tag{32}$$

or, in coordinates,

$$\langle (V_{a,t})_\alpha \rangle = \int \varrho u_\alpha = \int j_\alpha, \tag{33}$$

where ϱ, u and j are the quantities belonging to W, and the integrations are performed in the observer's space at the time point t. The formula (32) [or (33)] provides a convenient way to give the velocity physical quantity in the absolute formalism. Similarly, we give the kinetic energy physical quantity $K_{a,t}$ also by means of its expectation values in pure states. We find

$$\langle K_{a,t} \rangle = -m \int \varrho \varepsilon = m \int \varrho \left(\frac{1}{2} u_\alpha u_\alpha + s \right). \tag{34}$$

Now we can see the physical meaning of the quantities u_α and ε: They express the kinetic energy and the kinetic momentum of the particle.

If we calculate the uncertainty (standard deviation) of a velocity component, the result is

$$\left[\Delta \left(V_{a,t} \right)_\alpha \right]^2 = \int \varrho \left[u_\alpha - \langle (V_{a,t})_\alpha \rangle \right]^2 + \frac{\hbar^2}{m^2} \int (\partial_\alpha R)^2 \tag{35}$$

(here and in the following formulas of this section α is a fixed index, its double occurences do not imply summation). This formula offers to obtain some sharper inequalities than the usual position-velocity uncertainty formula. [As mentioned before, we prefer velocity to canonical momentum in uncertainty inequalities because canonical momentum is a gauge dependent quantity while velocity is not. From the aspect of the Heisenberg uncertainty inequality they behave the same way, which can be seen, for example in the Hilbert space formalism, from that—as operators—their commutator with position are the same (except for a factor m).]

To derive these sharper inequalities, by using the Cauchy-Bunyakovski-Schwartz inequality for real functions,

$$\int f^2 \cdot \int g^2 \geq \left\{ \int fg \right\}^2,$$

we find

$$\int \left\{ R \left[r_\alpha - \langle (Q_{a,t})_\alpha \rangle \right] \right\}^2 \cdot \frac{\hbar^2}{m^2} \int \left\{ \partial_\alpha R \right\}^2 \geq$$
$$\frac{\hbar^2}{m^2} \left\{ \int \left[r_\alpha - \langle (Q_{a,t})_\alpha \rangle \right] \partial_\alpha \left(\frac{1}{2} R^2 \right) \right\}^2 = \frac{\hbar^2}{4m^2}, \tag{36}$$

and

$$\int \left\{ R \left[r_\alpha - \langle (Q_{a,t})_\alpha \rangle \right] \right\}^2 \cdot \int \left\{ R \left[u_\alpha - \langle (V_{a,t})_\alpha \rangle \right] \right\}^2 \geq$$
$$\left\{ \int R^2 \left[r_\alpha - \langle (Q_{a,t})_\alpha \rangle \right] \left[u_\alpha - \langle (V_{a,t})_\alpha \rangle \right] \right\}^2 \tag{37}$$

(here and in the following surface terms are always dropped). Both the l.h.s. of (36) and the l.h.s. of (37) are less than or equal to $\left[\Delta \left(Q_{a,t} \right)_\alpha \right]^2 \left[\Delta \left(V_{a,t} \right)_\alpha \right]^2$ [cf. (35)].

Furthermore, the sum of these two l.h.s.-s is just $\left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\left[\Delta\left(V_{a,t}\right)_\alpha\right]^2$. Thus we find the following three inequalities:

$$\left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\left[\Delta\left(V_{a,t}\right)_\alpha\right]^2 \geq \frac{\hbar^2}{4m^2} + \left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\cdot\int\left\{R\left[u_\alpha - \langle(V_{a,t})_\alpha\rangle\right]\right\}^2, \quad (38)$$

$$\left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\left[\Delta\left(V_{a,t}\right)_\alpha\right]^2 \geq Y_\alpha^2 + \left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\cdot\frac{\hbar^2}{m^2}\int\left\{\partial_\alpha R\right\}^2, \quad (39)$$

$$\left[\Delta\left(Q_{a,t}\right)_\alpha\right]^2\left[\Delta\left(V_{a,t}\right)_\alpha\right]^2 \geq \frac{\hbar^2}{4m^2} + Y_\alpha^2 \quad (40)$$

with

$$Y_\alpha = \int R^2\left[r_\alpha - \langle(Q_{a,t})_\alpha\rangle\right]\left[u_\alpha - \langle(V_{a,t})_\alpha\rangle\right]. \quad (41)$$

From these inequalities (38) and (40) are stronger (or, at least, not weaker) than the usual uncertainty formula $\Delta\left(Q_{a,t}\right)_\alpha\Delta\left(V_{a,t}\right)_\alpha \geq \hbar/2m$, while (39) is an independent inequality.

Finally we prove Ehrenfest's theorem, in terms of the absolute formalism. First calculate the first time-derivative of the expectation value of the position of the particle:

$$\frac{\partial}{\partial t}\langle(Q_{a,t})_\alpha\rangle = \frac{\partial}{\partial t}\int \varrho r_\alpha = \int r_\alpha\frac{\partial\varrho}{\partial t}. \quad (42)$$

Using the continuity equation, (18), we get

$$\frac{\partial}{\partial t}\langle(Q_{a,t})_\alpha\rangle = \langle(V_{a,t})_\alpha\rangle. \quad (43)$$

The second derivative of the expectation value of the position can be written now as

$$\frac{\partial^2}{\partial t^2}\langle(Q_{a,t})_\alpha\rangle = \frac{\partial}{\partial t}\int \varrho u_\alpha = \int u_\alpha\frac{\partial\varrho}{\partial t} + \int \varrho\frac{\partial u_\alpha}{\partial t}. \quad (44)$$

The first term of the right hand side can be calculated again from the continuity equation. For the second term, we make use of (21) and (17). The result is:

$$m\frac{\partial^2}{\partial t^2}\langle(Q_{a,t})_\alpha\rangle = \int \varrho\mathcal{F}_\alpha^L, \quad (45)$$

where $\mathcal{F}^L$ has the form of the Lorentz force:

$$\mathcal{F}_\alpha^L = e(E_\alpha + \epsilon_{\alpha\beta\gamma}u_\beta B_\gamma).$$

It is important to remark that in proving the theorem no special property of F was used. Therefore, it remains valid even if F is not the electromagnetic field strength tensor but has any other origin. In that case $\mathcal{F}_\alpha = (\eta F u)_\alpha = F_{\alpha k}u^k$ appears in the r.h.s. of (45).

VII. Nonlinear and nonconservative extensions

Previously we found that each of the absolute equations (17), (18), and (25), express independent physical aspects of the quantum system. This makes it possible to extend quantum mechanics to treat more general force fields than conservative ones. The extension can be done by extending or altering Eq. (25), the equation which describes how the outer force field acts on the particle, while keeping the mass shell condition (17) and the continuity equation (18) untouched. One can check that, to prove the time independence of Γ by means of the absolute quantities and equations, only the equations (17) and (18) are needed. Hence, any alteration in the form of the outer action means no change in how the structure of the event space is established via Γ, and the 'unitarity' of the extended system is ensured. Also, the considerations about states and physical quantities, including the uncertainty inequalities and a generalized Ehrenfest's theorem, remain valid. On the other hand, in replacing (25) with a new equation, one has to check whether the resulting system of equations is self-consistent and not too restrictive, and whether the process space of the new system is also a Hilbert manifold.

One possible type of extension is when one keeps the form of (25) but replaces eF with a process dependent $\tilde{F}$. For example, one can take $\tilde{F} = \tilde{F}(W)$ in the form $\tilde{F}(W) = dK(W)$, where $K = K(W)$ is a process dependent covector function. (K has spacetime dependence through W, however, in addition, it can contain explicit spacetime dependence, too.) If $\tilde{F}$ is given in this way then, after a choice of an observer a, and a spacetime origin o, a quasi Schrödinger equation can be obtained from the absolute equations, if one introduces a 'quasi wave function' as

$$\Psi(p) = R \cdot \exp\left\{i\frac{m}{\hbar}\int_o^p \left[\left(\varepsilon + \tfrac{1}{m}K_0(W)\right)dx^0 + \left(u_\alpha + \tfrac{1}{m}K_\alpha(W)\right)dx^\alpha\right]\right\} \qquad (46)$$

[cf. (15)]. This quasi Schrödinger equation will have the same form as (6), with A replaced with $K(W)$ or, based on (46), with $K(\Psi)$. In a coordinate system, it reads

$$i\hbar\left[\partial_0 - \tfrac{i}{\hbar}K_0(\Psi)\right]\Psi = -\frac{\hbar^2}{2m}\left[\partial_\alpha - \tfrac{i}{\hbar}K_\alpha(\Psi)\right]\left[\partial_\alpha - \tfrac{i}{\hbar}K_\alpha(\Psi)\right]\Psi. \qquad (47)$$

The well-known, so-called nonlinear Schrödinger equation differs from the usual Schrödinger equation in that an additional, $k|\Psi|^2\Psi$ term is included. Another known nonlinear modification of the Schrödinger equation is given by Byalinicki-Birula and Mycielski [18], where the additional term is $k_1 \ln(k_2|\Psi|)\Psi$. (k, k_1 and k_2 are constants here.) From (47) we can see that both these modifications are special cases of a process dependent $\tilde{F} = dK$, with a ϱ dependent, timelike covector function K (recall that timelike covectors have zero space components in any inertial coordinate system, and an observer invariant time component).

The quantum version of the linearly velocity dependent damping can be formulated by a process dependent $\tilde{F}$, too. Let the inertial observer corresponding to $a \in V(1)$ describe an inertial medium with respect to which the damping will occur,

and we consider $\tilde{F} = -k(u-a) \wedge \boldsymbol{\tau}$, with a positive constant $k \in \mathbf{G}/\mathbf{I}\!\mathbf{I}$. We see that this $\tilde{F}$ is not given in the form $\tilde{F}(W) = dK(W)$.

For this system, the generalized (25) has its simplest relative form [the generalized equations (13), (14)] with respect the inertial observer a; we find

$$\partial_0 u_\alpha = -\frac{k}{m} u_\alpha + \partial_\alpha \varepsilon, \tag{48}$$

$$\partial_\alpha u_\beta - \partial_\beta u_\alpha = 0. \tag{49}$$

We remark that if a constant force is present in addition to the damping force—e.g., a homogeneous gravitational force—, then from $\tilde{F} = [mg - k(u-a)] \wedge \boldsymbol{\tau}$, with respect to the inertial observer $a' = a + (m/k)g$, one arrives at the same equations. Thus all our following considerations will be applicable to this more general situation, too.

With respect to the observer a (or a'), the formulas (43) and (45) concerning Ehrenfest's theorem give

$$\dot{Q}_\alpha = -\frac{k}{m} V_\alpha, \qquad \ddot{Q}_\alpha = -\frac{k}{m} \dot{Q}_\alpha, \tag{50}$$

(for simplicity, the position and velocity expectation values will be denoted by Q_α and V_α in the following). This shows that the classical limit of this quantum system is the case of the damping which linearly depends on velocity.

From (48) it follows that if (49) holds at a time point then it will hold later, too. This ensures the consistency of the equations of the system. Furthermore, (48) and (49) yield that there exists a real valued spacetime function S such that

$$u_\alpha = \frac{\hbar}{m} \partial_\alpha S, \qquad \varepsilon = \frac{\hbar}{m} \partial_0 S + k \frac{\hbar}{m} S. \tag{51}$$

Hence, by introducing the quasi wave function $\Psi = R \exp(iS)$, here we also find that a quasi Schrödinger equation is available for the system. Now this equation reads

$$i\hbar \partial_0 \Psi = -\frac{\hbar^2}{2m} \triangle \Psi + \frac{\hbar}{m} S\Psi. \tag{52}$$

Remarkably, here the extra, nonlinear, 'potential-like' term contains the phase of the wave function, rather than its absolute value.

In all these three example systems, the found quasi wave function offers one way to check that the process space is a Hilbert manifold.

VIII. The dissipative system

In the following we answer some interesting physical questions concerning the behavior of the above mentioned dissipative system. Namely, we are interested in how damping 'stops the particle', whether the spread of a 'wave packet' is also stopped by the damping, and how the kinetic energy of the particle dissipates. Our findings will be valid for any initial conditions (more precisely, for all those in which the

integrals giving the expectation values and standard deviations are finite and all surface terms are zero). For simplicity, we will perform the calculations for the one space dimensional version of the system. It is worth observing that the dimensionful parameters m, $\hbar$, and k, of the system define the characteristic length, time and mass scales $(\hbar/k)^{1/2} \in \mathbb{D}$, $m/k \in \mathbb{I}$, and $m \in \mathbb{G}$. These distinguished units allow us to make the quantities of the system completely dimensionless. Dot and prime will mean derivations with respect to the dimensionless time and space variables, respectively.

To obtain a convenient starting point, we reformulate the absolute equations in terms of R, u_1, and $j_1 = R^2 u_1$, and turn them into the following practical form:

$$\left(R^2\right)^{\cdot} + (j_1)' = 0, \tag{53}$$

$$(j_1)^{\cdot} = -j_1 + \frac{1}{2}\left\{RR'' - R'^2 - 2R^2 u_1^2\right\}'. \tag{54}$$

First let us see whether stationary solutions of these equations exist, since if the spread of the 'wave packet' stops then it is plausible that a process will tend to a stationary one, as time tends to infinity.

Now, for absolute quantities, stationarity means real time independence. For a stationary solution, from (53) we find $(j_1)' = 0$, which means that j_1 is both time and space independent. As the space integral of $j_1 = R^2 u_1$ gives the expectation value of the velocity, which is assumed to be a finite value, this constant value of j_1 must be zero. The second equation then gives:

$$\left\{RR'' - R'^2\right\}' = 0.$$

This equation can be solved, its general solution is:

$$R(x) = c_1 e^{c_0 x} + c_2 e^{-c_0 x},$$

where c_0, c_1 and c_2 are arbitary complex numbers such that R must be real. It is easy to see that for no values of these numbers will $\varrho = R^2$ be normalizable (i.e., having a finite space integral). So there exists no normalizable stacionary solution of equations (53) and (54).

Next, we determine the time dependence of the position and velocity expectation values of the particle. This can be done easily from (50). The result is

$$Q = Q(0) + V(0)\left(1 - e^{-t}\right), \qquad V = V(0)e^{-t}. \tag{55}$$

Then we are looking for the asymptotic behavior of the position expectation value and standard deviation and of the kinetic energy expectation value. Let X denote the squared standard deviation of the position, let Y be as in (41), and let us introduce

$$T := \int \varrho \left(u_1 - V\right)^2, \qquad P := \int R'^2.$$

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Actually, T and P are nothing else but the two terms on the r.h.s. of (35), in the one dimensional case. The expectation value of the kinetic energy can then be given as

$$K = \tfrac{1}{2}\left(V^2 + T + P\right). \tag{56}$$

From (53) and (54) we can deduce the following equations:

$$\dot{X} = 2Y, \tag{57}$$

$$\dot{Y} = -Y + T + P, \tag{58}$$

$$(T + P)\dot{} = -2T. \tag{59}$$

The inequalities (36) and (37) now read

$$PX \geq \tfrac{1}{4}, \tag{60}$$

$$TX \geq Y^2. \tag{61}$$

In our considerations we will several times make use of the facts that for any functions f and g

$$\dot{f} + \lambda f = e^{-\lambda t}[e^{\lambda t} f]\dot{}; \tag{62}$$

$$f \leq g, \quad t_1 \leq t_2 \;\Rightarrow\; \int_{t_1}^{t_2} f \leq \int_{t_1}^{t_2} g. \tag{63}$$

The key to find good estimates for X is to rewrite (60) and (61) in terms of X. In terms of the auxiliary quantity

$$Z := (X^2)\ddot{} + (X^2)\dot{} - 3\dot{X}^2, \tag{64}$$

(60) and (61) can be written as simply as

$$\dot{Z} \leq 0, \tag{65}$$

$$\dot{Z} + 2Z \geq 2. \tag{66}$$

It follows from these inequalities that $Z(t)$ is decreasing and converges to a Z^*, $Z^* \geq 1$, as t goes to infinity. Then

$$(X^2)\ddot{} + (X^2)\dot{} - 3\dot{X}^2 \geq Z^*, \tag{67}$$

or, after omitting a negative term from the left hand side,

$$(X^2)\ddot{} + (X^2)\dot{} \geq Z^*. \tag{68}$$

From this, after two integrations and using (62) and (63), we get

$$X^2 \geq Z^* t + C_1 - C_2 e^{-t}, \tag{69}$$

where C_1 and C_2 are the constants

$$C_1 = X^2(0) + (X^2)\dot{}(0) - Z^*, \qquad C_2 = (X^2)\dot{}(0) - Z^*.$$

Now we give an upper bound for X. First we prove that $\dot{X}$ is bounded. It follows from (69) that there exists a time point t_1 such that, for each $t > t_1$, $(X^2)\dot{} > 0$. Since $(X^2)\dot{} = 2X\dot{X}$ and $X \geq 0$, $\dot{X}$ will be positive for each $t > t_1$. On the other hand, $\dot{X}$ has an upper bound, which can be proven as follows. $(T + P)\dot{} = -2T \leq 0$, so $(T + P)(t) \leq (T + P)(0)$ for each $t > 0$. Then

$$\dot{Y} + Y \leq (T + P)(0), \tag{70}$$

and applying (62) and (63) to this inequality we get

$$\tfrac{1}{2}\dot{X} = Y \leq (T + P)(0) + (Y - T - P)(0)e^{-t}. \tag{71}$$

Thus $\dot{X}$ is bounded, i.e., there exists a positive number K_1 such that $\dot{X}^2 < K_1$ for each $t > t_1$. Using this result we can find an upper bound for $Z + 3\dot{X}^2$. Let it be K_2. Then we can write:

$$(X^2)\ddot{} + (X^2)\dot{} \leq K_2 \tag{72}$$

for each $t > t_1$. After applying (62) and (63) to (72) we get the following estimate for $\dot{X}$:

$$0 \leq \dot{X} \leq \frac{K_3}{t^{1/2}} \qquad (t > t_2), \tag{73}$$

where K_3 and t_2 are appropriate constants. We can see that $\dot{X}$ is not only bounded but it goes to zero when t goes to infinity. This means that, for every $\epsilon > 0$, there exists a t_3 such that

$$(X^2)\ddot{} + (X^2)\dot{} \leq Z^* + \epsilon. \tag{74}$$

Using (62) and (63) again we get:

$$X^2 \leq (Z^* + \epsilon)t + C_3 - C_4 e^{t_3 - t}, \tag{75}$$

where $C_3 = X^2(t_3) + \dot{X}^2(t_3) - (Z^* + \epsilon)(t_3 + 1)$ and $C_4 = \dot{X}^2(t_3) - (Z^* + \epsilon)$.

If we compare this to (69) then we can see how X behaves for great values of t:

$$\lim_{t \to \infty} \frac{X^2(t)}{t} = Z^*. \tag{76}$$

Hence the standard deviation of the position behaves for great t values as the function $\sqrt[4]{Z^*}\, t^{1/4}$. This result means that although the spread of the 'wave packet' is slower than in the free particle case—there $\sqrt{X} \sim t$ asymptotically—, this spread never stops.

In parallel, $(X^2)\dot{}$ tends to Z^* asymptotically. From this we find $\dot{X} \approx \sqrt{Z^*/4}\,t^{-1/2}$ ($f \approx g$ denotes $f/g \to 1$ as $t \to \infty$), and, from (64), that $X\ddot{X} \to 0$. Thus $\ddot{X}/\dot{X} \to 0$, which, together with (57) and (58), gives $T + P \approx \sqrt{Z^*/16}\,t^{-1/2}$. As a consequence,

$$K \approx (\sqrt{Z^*}/8)\,t^{-1/2}. \tag{77}$$

Therefore, we find that the kinetic energy expectation value dissipates completely, following an asymptotic behavior $t^{-1/2}$.

IX. The Aharonov-Bohm effect

In this section let us turn back to the case in which the outer force field is an electromagnetic field. As we have seen, the absolute formalism—as a consequence of being gauge ambiguity free—works with the electromagnetic field strength tensor. This property is an interesting feature from the aspect of the Aharonov-Bohm effect (see [20]). In this effect the particle is excluded from a cylindrical region with an infinite potential wall, however, its motion is influenced by a magnetic field which is nonzero only within this region. Here we provide an argument that supports that the absolute formalism may prove to be a good framework to understand the effect purely by means of the field strength tensor. Namely, although the probability density is zero within the region in question, the following computation will suggest that not all absolute quantities needed for the complete description of the particle vanish within that domain. Some seem to penetrate into the region, thus it can be expected that it is these quantities through which the particle 'feels' the action of the inner electromagnetic field.

Usually, in regions where the electromagnetic field is a finite and smooth function, R is zero only on a set of measure zero. In the Aharonov-Bohm effect, however, $R = 0$ in a whole domain. Thus it is a question whether the other absolute quantities are well defined within this domain. That's why we apply a limiting procedure. We investigate the set-up with a finite potential wall, determine the absolute quantities inside and outside the cylinder, and then we send the potential wall to infinity. We remark that in the wave function formalism of quantum mechanics a similar limiting procedure is needed to derive the boundary conditions the wave function has to satisfy at an infinite potential wall. We will find that, in the infinite potential wall limit, some of the absolute quantities have a nonzero limit inside the cylinder. This suggests that if the absolute formalism is extended to treat infinite electromagnetic fields, too, the direct treatment of the Aharonov-Bohm effect will give that the absolute quantities are well defined inside, too, and some of them are nonzero.

Let us consider the following set-up. There is an inertial observer a and in its space there is an infinitely long cylinder. Let the radius of the cylinder be b. Let our z-axis be the axis of the cylinder. If we fix a space origin on this axis and a direction, perpendicular to the axis, then every space point can be characterized by its usual

cylindrical coordinates r, θ and z. Let the magnetic field be zero outside the cylinder and a constant value, B_0, inside. Let the scalar potential be zero outside and some function $\phi(r)$ inside. For our purposes it will be enough to look for stationary, cylindrically symmetric and z independent solutions of the absolute equations.

Now, in this case it follows from the continuity equation that u may not have a radial component, otherwise the outcoming flux of $j = R^2 u$ on any cylinder concentric to the original one would not be zero. So u has only a tangential component which will be denoted by u_θ and a z-component which will be denoted by u_z. The continuity equation then always holds, and the other equations lead to

$$\frac{1}{2}m(u_\theta^2 + u_z^2) + e\phi - \frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{R'}{R} + \frac{R''}{R}\right) = E, \tag{78}$$

$$u_\theta' + \frac{1}{r}u_\theta = \frac{e}{m}B_0, \tag{79}$$

$$u_z' = 0 \tag{80}$$

inside the cylinder and

$$\frac{1}{2}m(u_\theta^2 + u_z^2) - \frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{R'}{R} + \frac{R''}{R}\right) = E, \tag{81}$$

$$u_\theta' + \frac{1}{r}u_\theta = 0, \tag{82}$$

$$u_z' = 0 \tag{83}$$

outside the cylinder, where $'$ denotes derivatives with respect to r, and the integration constant E is actually the total mechanical energy of the particle. It follows from (80) and (83) that u_z is constant.

The solution of (79) is $u_\theta(r) = \frac{e}{2m}B_0 r + C_1/r$ (the constant C_1 being related to angular momentum, actually), while, from (82), outside the cylinder $u_\theta(r) = C_2/r$. Continuity of u_θ at $r = b$ gives $C_2 = C_1 + \frac{e}{2m}B_0 b^2$.

Then, to obtain a convenient way to solve (78) now we specify the electromagnetic scalar potential as follows. Let ϕ be $\phi(r) = \phi_0 - \frac{eB_0^2}{8m}r^2$, $(0 \leq r \leq b)$, where ϕ_0 is a constant. If ϕ_0 is sent to infinity then $\phi(r)$ goes to infinity in a uniform way, thus providing the desired infinitely high potential wall. Eq. (78) then reads

$$r^2 R'' + rR' - \frac{2m}{\hbar^2}\left(e\phi_0 - E + \frac{1}{2}mu_z^2 + \frac{e}{2}B_0 C_1\right)r^2 R - \frac{m^2}{\hbar^2}C_1^2 R = 0. \tag{84}$$

With $\kappa = \frac{\sqrt{2m}}{\hbar}\left(e\phi_0 - E + \frac{1}{2}mu_z^2 + \frac{e}{2}B_0 C_1\right)^{1/2}$, this equation is the modified Bessel equation in the variable κr. Its general solution is $R(r) = C_3 I_\mu(\kappa r) + C_4 K_\mu(\kappa r)$, where I_μ and K_μ are the modified Bessel functions and $\mu = mC_1/\hbar$. Similarly, Eq. (81) can be written as

$$r^2 R'' + rR' + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}mu_z^2\right)r^2 R - \frac{m^2}{\hbar^2}C_2^2 R = 0, \tag{85}$$

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the Bessel equation in the variable λr with $\lambda = \frac{\sqrt{2m}}{\hbar}\left(E - \frac{1}{2}mu_z^2\right)^{1/2}$. Its solution is $R(r) = C_5 J_\nu(\lambda r) + C_6 Y_\nu(\lambda r)$, where $\nu = mC_2/\hbar$. Matching of R at $r = b$ and the normalization of R in the variable r determine the constants C_3–C_6.

By using the asymptotic properties of the Bessel functions, it can be investigated how R tends to zero inside the cylinder in the limit $\phi_0 \to \infty$. The important issue for our present purpose is to observe that, in the limit $\phi_0 \to \infty$, u_θ (and probably u_z) remains finite and nonzero inside. This makes it plausible that in the Aharonov-Bohm situation it is the quantity u, penetrating into the cylinder, through which the particle is affected by the inner magnetic field.

X. Relativistic considerations

Now let us turn to special relativistic spacetime, and introduce absolute quantities and equations concerning the Klein-Gordon equation. In

$$(\hbar^2 D^k D_k + m^2 c^4)\Psi = 0 \tag{86}$$

$[D_k = \partial_k - i(e/\hbar)A_k$, in this section we will work only with indexed formulas] Ψ is a Lorentz scalar, but has a gauge dependence similar to that of the nonrelativistic wave function. Using again the polar decomposition, $\Psi = Re^{iS}$, let us define

$$u_k := \frac{\hbar}{m}\partial_k S - \frac{e}{m}A_k. \tag{87}$$

Here the arising consistency relation for u reads

$$\partial_k u_l - \partial_l u_k = -\frac{e}{m}F_{kl}. \tag{88}$$

With R and u, the complex equation (86) will be equivalent to the following two real ones:

$$m^2 c^4 R - m^2 g^{kl} u_k u_l R + \hbar^2 g^{kl}\partial_k\partial_l R = 0, \tag{89}$$

$$\partial_k(R^2 u^k) = 0. \tag{90}$$

We see that $j^k := R^2 u^k$ satisfies a continuity equation. As indicated before, (89) is the quantum analogue of the classical mass shell relation $p^k p_k = m^2 c^2$.

It will be interesting to see how the nonrelativistic limit of these equations leads to the nonrelativistic absolute equations.

In general, nonrelativistic limit can be valid only at a neighbourhood of a space-time point p, and needs an inertial coordinate system with respect to $\left|1 + \frac{u_0(p)}{c^2}\right| \ll 1$. In the following we will restrict ourselves to an appropriate neighbourhood of p. To be in conform with the aimed nonrelativistic quantities, we will have to work with both $\mathbb{D}$ and $\mathbb{I}$, rather than only with $\mathbb{I}$. Then $\{g_{kl}\} = \mathrm{diag}[1, -1/c^2, -1/c^2, -1/c^2]$, and, for example, u_α is $\mathbb{D}/\mathbb{I}$ valued, and u_0 is $\mathbb{D}^{(2)}/\mathbb{I}^{(2)}$ valued.

By introducing

$$\varepsilon = u_0 + c^2,$$

we can see that ε and u_α satisfy the nonrelativistic equations (13), (14). $|\varepsilon/c^2| \ll 1$, and if all terms in (90) with higher orders of $1/c^2$ are negligible, then (90) leads to the nonrelativistic continuity equation

$$\partial_0(R^2) + \partial_\alpha(R^2 u_\alpha) = 0.$$

In (89), written as

$$\left[c^2 - c^2\left(\frac{u_0}{c^2}\right)^2 + u_\alpha u_\alpha\right]R + \frac{\hbar^2}{m^2}\left[\frac{1}{c^2}\partial_0^2 R - \partial_\alpha\partial_\alpha R\right] = 0, \tag{91}$$

$(u_0/c^2)^2 = [1-(\varepsilon/c^2)]^2 \approx 1-2\varepsilon/c^2$, and if the term proportional to $1/c^2$ is negligible again, then we arrive at

$$[2\varepsilon + u_\alpha u_\alpha]R + \frac{\hbar^2}{m^2}\partial_\alpha\partial_\alpha R = 0, \tag{92}$$

which is just the nonrelativistic mass shell condition (17) [cf. (16)].

We can see that the special relativistic $u_0 + c^2$, u_α and R became the nonrelativistic ε, u_α and R.

On a general relativistic spacetime, the straightforward generalization of the special relativistic Klein-Gordon equation is to replace partial derivatives with covariant derivatives in (86). After polar decomposition, we arrive at quantities and formulas similar to those in the special relativistic case, again with covariant derivatives instead of partial derivatives. After some known properties of the covariant derivative, the equations can be written as

$$\partial_k u_l - \partial_l u_k = -\frac{e}{m}F_{kl}, \tag{93}$$

$$m^2 c^4 R - m^2 g^{kl} u_k u_l R + \hbar^2 \frac{1}{\sqrt{-g}}\partial_k\left(\sqrt{-g}g^{kl}\partial_l R\right) = 0, \tag{94}$$

$$\frac{1}{\sqrt{-g}}\partial_k\left(\sqrt{-g}g^{kl}R^2 u_l\right) = 0. \tag{95}$$

In this case we consider the 'nonrelativistic + weak gravity' limit of these equations. In the weak gravity limit, in an appropriate coordinate system, the matrix of the metric tensor differs from the Minkowskian only slightly, in the form

$$\{g_{kl}\} = \text{diag}\left[1 + \frac{2\phi}{c^2}, -\frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right), -\frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right), -\frac{1}{c^2}\left(1 - \frac{2\phi}{c^2}\right)\right],$$

for details, see, e.g., [24]. In parallel, $\left|1 + \frac{u_0}{c^2}\right| \ll 1$ will be needed to hold, at least in a domain of spacetime, now with respect to this coordinate system.

In this case we introduce ε as

$$\varepsilon = u_0 + c^2 + \phi.$$

After such steps as were taken in the special relativistic case, the calculation gives that, if all terms proportional to higher orders of $1/c^2$ are negligible, then (95) leads to the nonrelativistic continuity equation, and (94) to the nonrelativistic mass shell condition, with u_α and R becoming the corresponding nonrelativistic quantities. Furthermore, from (93) we can see that ε and u_α satisfy the nonrelativistic (14), and

$$\partial_0 u_\alpha - \partial_\alpha \varepsilon = \frac{e}{m} F_{\alpha 0} - \partial_\alpha \phi. \tag{96}$$

Thus ϕ appears as an outer scalar potential acting on the particle, in addition to the action of the electromagnetic field. This result is just the expected one: For a classical particle, the presence of a weak gravity appears as ϕ being an outer scalar potential (the Newtonian gravitational potential) acting on the particle (cf., e.g., [24]).

XI. Discussion and outlook

As we have seen, the presented approach has the following advantages. It provides a reference frame free and gauge free formulation of quantum mechanics. The arising quantities are more spacetime friendly and have a more direct physical interpretation, compared to the wave function. The three independent physical aspects of the system—namely, the way the outer field acts on the particle, the mass shell condition, and the conservation of probability—become transparent, while in the wave function formalism they remain implicit and hidden. The absolute framework is applicable to treat nonconservative situations or 'nonlinear' generalizations of quantum mechanics.

The found results motivate further investigation in diverse directions. One of them is to apply the absolute framework for further dissipative systems. Another one is to study the conjectured connection between the symplectic structure of the quantum process space and the symplectic structure of the corresponding classical one. To find applications of the obtained uncertainty relations is a further arising possibility. Besides, by extending the absolute formalism for systems where the given electromagnetic field contains singularities, a direct way will be available to study the Aharonov-Bohm situation. This step is important to check the validity of the indirect treatment presented here.

To continue establishing the absolute formulation of quantum mechanics, the future tasks are to extend the presented formalism for multiparticle quantum mechanics and for particles with spin, in the nonrelativistic case; and to continue the relativistic cases with the study of the process space, the states, and the physical quantities, for zero and nonzero spin particles as well. After these steps the possibility will open to start building an absolute formalism for quantum field theories.

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