Can material time derivative be objective?

T. Matolcsi* and P. Ván**

*Eötvös Lornd University, Department of Applied Analysis
Budapest, Hungary
**(corresponding author) KFKI Research Institute for Particle and Nuclear Physics,
Department of Theoretical Physics
1125 Budapest 141, Pf. 49., Hungary
T: +36-12922222/3397
F: +36-13922727*

(Dated: December 19, 2010)

The concept of objectivity in classical field theories is traditionally based on time dependent Euclidean transformations. In this paper we treat objectivity in a four-dimensional setting, calculate Christoffel symbols of the spacetime transformations, and give covariant and material time derivatives. The usual objective time derivatives are investigated.

PACS numbers: 46.05.+b, 83.10.Ff

I. ABOUT OBJECTIVITY

The usual concept of objectivity in classical field theories is based on time-dependent Euclidean transformations. The importance of these transformations was recognized by Noll in 1958 [1] and later on became an important tool to restrict constitutive functions through the principle of material frame-indifference (see e.g. [2, 3, 4, 5]).

Later on the principle of material frame-indifference was criticized by several authors from different points of view [6, 7, 8]. Several authors argued that some consequences of the usual mathematical formulation of the principle contradict the experimental observations. We emphasize that one should make a clear distinction between the *principle* of material frame-indifference (which is physically well-justified) and its *mathematical formulation* (see e.g. [9, 10, 11, 12, 13]). There are several different opinions in the literature concerning the mathematical formulation and recent research papers indicate that the discussion does not seem to settle [14, 15, 16, 17].

The concept of material frame-indifference is inherently related to the notion of objectivity. In this paper, as a first step towards a possible solution of the problems of material frame-indifference mentioned above, we investigate the concept of objectivity.

Some well-known problems arise from the definition of objectivity which mainly concern quantities containing derivatives. It seems to us that the problems take their origin from the fact that objectivity is defined for three-dimensional vectors but differentiation – with respect to time and space together – results in a four-dimensional covector. Therefore, we propose to extend the definition of objectivity to a four-dimensional setting.

We start with the usual transformation rules of Noll [1] for spacetime variables:

$$\hat{t} = t, \qquad \hat{\mathbf{x}} = \mathbf{h}(t) + \mathbf{Q}(t)\mathbf{x}. \tag{1}$$

Though time is not transformed, the transformation of space variables contains time, so this is in fact a four-dimensional transformation. Let us write in the form

$$\hat{x}^0 = x^0, \quad \hat{x}^\alpha = h^\alpha + Q^\alpha{}_\beta x^\beta; \qquad \text{in short} \qquad \hat{x}^k = \hat{x}^k(x)$$

where Latin indices run trough $0, 1, 2, 3$, Greek indices run through $1, 2, 3$ and the Einstein summation rule is used.

It is well-known from differential geometry (see e.g. in [18]) that C is a four-dimensional objective vector if it transforms according to

$$\hat{C}^i = \hat{J}^i_j C^j$$

where

$$\hat{J}^i_j := \frac{\partial \hat{x}^i}{\partial x^j}$$

---

*Electronic address: vpet@eik.bme.hu

is the Jacobian matrix of the transformation.

In the present case, in a block matrix form,

$$\hat{J} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix}.$$

Accordingly, we say that a four-vector $(C^0, \mathbf{C})$ is objective if it transforms according to the rule

$$\begin{pmatrix} \hat{C}^0 \\ \hat{\mathbf{C}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} C^0 \\ \mathbf{C} \end{pmatrix},$$

i.e.

$$\hat{C}^0 = C^0, \qquad \hat{\mathbf{C}} = (\dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x})C^0 + \mathbf{Q}\mathbf{C}. \tag{2}$$

If the four-vector is in fact a three-vector, i.e. $C^0 = 0$, then we get back the usual formula.

$$\hat{\mathbf{C}} = \mathbf{Q}\mathbf{C}. \tag{3}$$

However, some important physical quantities are four-vectors. Namely, consider a motion $\mathbf{r}$ of a mass point. Then in a four-dimensional setting it is described by $(t, \mathbf{r}(t))$. Its time derivative is $(1, \mathbf{v})$ where $\mathbf{v} = \dot{\mathbf{r}}$. According to (1) we have

$$\hat{\mathbf{r}} = \mathbf{h} + \mathbf{Q}\mathbf{r},$$

thus

$$\hat{\mathbf{v}} = \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{r} + \mathbf{Q}\dot{\mathbf{r}}.$$

As a consequence, we conclude from (2) that the four-velocity

$$V := (1, \mathbf{v}) \tag{4}$$

is an objective four-vector. The three-velocity – i.e. the quantity $(0, \mathbf{v})$ – is not objective.

II. COVARIANT DERIVATIVES

Since nonrelativistic spacetime is flat (has an affine structure), it is known from differential geometry that a distinguished covariant differentiation D is assigned to it. (In differential geometry the covariant differentiation is usually denoted by ∇ but in spacetime this symbol usually refers to spacelike derivatives.) This means that if a is a scalar field in spacetime, then Da is a covector field, if C is a vector field, then DC is mixed tensor field; in coordinates

$$\mathrm{D}a \sim \mathrm{D}_i a = \partial_i a,$$

$$\mathrm{D}C \sim \mathrm{D}_i C^j = \partial_i C^j + \Gamma^i_{jk} C^k,$$

where Γ^i_{jk} are the Christoffel symbols of the coordinatization, which are all zero if and only if the coordinatization is linear (affine). It is worth emphasizing: *the coordinates of the covariant derivative of a vector field do not equal the partial derivatives of the vector field if the coordinatization is not linear.* The spacetime coordinatization is linear if and only if the underlying observer is inertial.

If $\hat{x}$-s are inertial, linear coordinates, then the Christoffel symbols with respect to the coordinates x have the form

$$\Gamma^i_{jk} := \frac{\partial^2 \hat{x}^m}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \hat{x}^m} = -\frac{\partial^2 x^i}{\partial \hat{x}^m \partial \hat{x}^l} \frac{\partial \hat{x}^m}{\partial x^j} \frac{\partial \hat{x}^l}{\partial x^k}. \tag{5}$$

(See the Appendix.)

A straightforward calculation yields for the coordinatization (1) that

$$\Gamma^0_{jk} = 0, \qquad \Gamma^\alpha_{00} = \left(\mathbf{Q}^{-1}(\ddot{\mathbf{h}} + \ddot{\mathbf{Q}}\mathbf{x})\right)^\alpha = \left(\mathbf{Q}^{-1}\ddot{\mathbf{h}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}\mathbf{\Omega})\mathbf{x})\right)^\alpha, \tag{6}$$

$$\Gamma^\alpha_{0\beta} = \Omega^\alpha{}_\beta, \qquad \Gamma^\alpha_{\gamma\beta} = 0, \tag{7}$$

where $\mathbf{\Omega} := \mathbf{Q}^{-1}\dot{\mathbf{Q}}$ is the angular velocity of the observer.

III. MATERIAL TIME DERIVATIVE

Let us describe a continuum in spacetime in such a way that to each spacetime point x we assign the absolute velocity (four-velocity) $V(x)$ of the particle at x. Then V is an objective vector field.

The flow generated by the vector field V is a well-known notion in differential geometry: $F_t(x)$ is the point at time t of the integral curve of V passing through x (Figure 1). Physically: an integral curve of V is the history of a particle of the continuum.
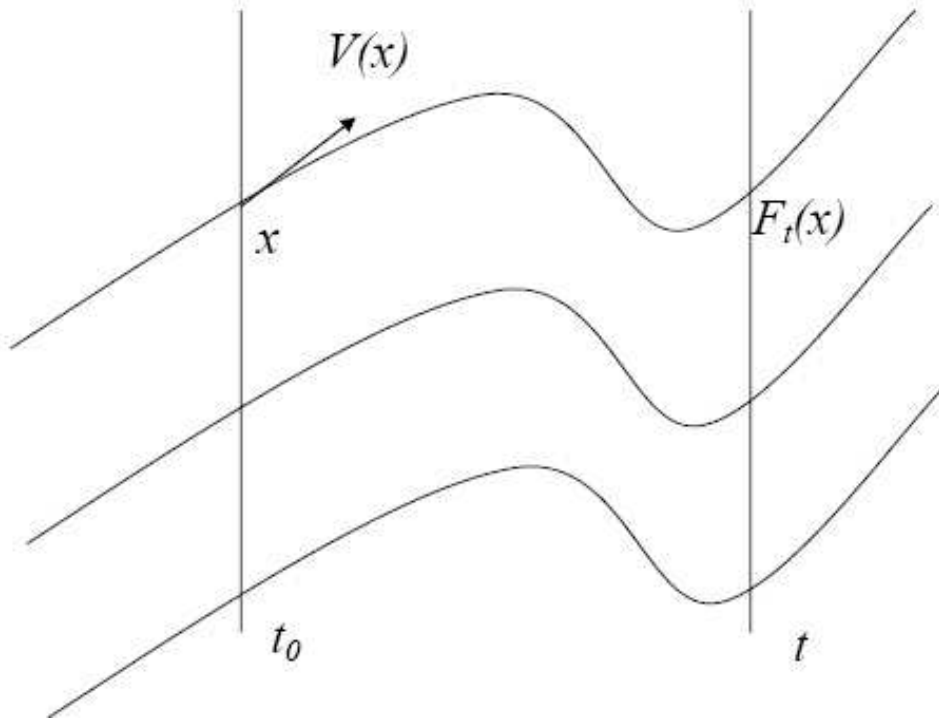

FIG. 1: Integral curves and flow of the vector field V.

Let us consider a quantity Φ of any tensorial order defined in spacetime. Then the function $t \mapsto \Phi(F_t(x))$ is the change in time of the quantity along an integral curve i.e. at a particle of the continuum. Then it is known from differential geometry that

$$\frac{d}{dt}\Phi(F_t(x)) = (\mathrm{D}_V \Phi)(F_t(x)), \tag{8}$$

where $\mathrm{D}_V \Phi$ is the covariant derivative of Φ according to V.

We call $\mathrm{D}_V \Phi$ the *material time derivative* of Φ.

Mathematically D_V is a scalar operation, i.e. the tensorial rank of $\mathrm{D}_V \Phi$ equals that of Φ. In coordinates:

$$\mathrm{D}_V a = V^j \mathrm{D}_j a = V^j \partial_j a \qquad a \text{ is scalar} \tag{9}$$

$$(\mathrm{D}_V C)^i = V^j \mathrm{D}_j C^i = V^j (\partial_j C^i + \Gamma^i_{jk} C^k) \qquad C \text{ is vector.} \tag{10}$$

In view of (6), (7) and (4), we have

$$\mathrm{D}_V a = (\partial_0 + \mathbf{v} \cdot \nabla)a \tag{11}$$

and if $C = (0, \mathbf{C})$ is a spacelike vector field, then

$$\mathrm{D}_V \mathbf{C} = (\partial_0 + \mathbf{v} \cdot \nabla + \mathbf{\Omega})\mathbf{C}. \tag{12}$$

(If $C^0 \neq 0$, the timelike component is not zero, then a further term containing C^0 enters the expression of the spacelike component.)

The material time derivative of a vector – even if it is spacelike – is not given by $\partial_0 + \mathbf{v} \cdot \nabla$.

IV. OBJECTIVE TIME DERIVATIVES

In usual literature $\partial_0 + \mathbf{v} \cdot \nabla$ is considered to be the material time derivation. This applied to scalars results in scalars but applied to vectors does not result in vectors; that is why it is always stated that this operation is not objective. Since the problem of proper objective time derivatives relates to several phenomena in physical theories (e.g. in rheology [19, 20]), 'objective time derivatives' are looked for in such a way that the above operation is supplemented by some terms which formally do not refer to the observer [21]. This means in our formalism that one looks for an objective expression, containing $\partial_0 + \mathbf{v} \cdot \nabla$, in which the Christoffel symbols do not appear. For instance, let us take the objective quantity

$$C^j \mathrm{D}_j V^i = C^j \partial_j V^i + C^j \Gamma^i_{jk} V^k \tag{13}$$

Evidently, the difference of the true material time derivative (10) and the above expression is objective as well (being the difference of two objective quantities). Since Γ is symmetric in its lower indices, we obtain that

$$V^j \mathrm{D}_j C^i - C^j \mathrm{D}_j V^i = V^j \partial_j C^i - C^j \partial_j V^i.$$

The right-hand side does not contain the Christoffel symbols, it is given by partial derivatives only, nevertheless it is objective. For a spacelike vector $(0, \mathbf{C})$ the right-hand side can be written in the form

$$(\partial_0 + \mathbf{v} \cdot \nabla)\mathbf{C} - \mathbf{L} \cdot \mathbf{C}, \tag{14}$$

where $\mathbf{L}$ is the velocity gradient. This expression is just the 'upper convected derivative' of $\mathbf{C}$. In a similar way we get the lower convected derivative and the Jaumann derivative as well.

V. CONCLUSIONS

1. We propose that objectivity be extended to a four-dimensional setting.

2. The four-dimensional covariant differentiation is a fundamental fact of nonrelativistic spacetime. *The coordinates of the covariant derivative of a vector field do not equal the partial derivatives of the vector field if the coordinatization is not linear; the Christoffel symbols enter.* The essential part of the Christoffel symbols is the angular velocity of the observer.

3. Usual treatments leave the covariant differentiation out of consideration; they involve only partial derivatives which, of course, are not objective. A number of problems arise from this fact.

4. Material time derivative of a quantity, in a physically proper sense, ought to be defined by the time derivative of the quantity along the particles of a continuum.

5. The mathematical expression of material time differentiation concerns the covariant differentiation, so its form relative to an observer is not given by partial derivatives only, the Christoffel symbols (the angular velocity of the observer) enter; for spacelike vectors the material time differentiation has the form

$$\partial_0 + \mathbf{v} \cdot \nabla + \boldsymbol{\Omega}.$$

6. In the literature Jaumann derivative, upper and lower convected derivatives are introduced because instead of the above true material time derivative authors look for objective expressions containing $\partial_0 + \mathbf{v} \cdot \nabla$ and partial derivatives only.

VI. ACKNOWLEDGEMENTS

VII.   APPENDIX

Let $\hat{x}^j$ denote inertial (i.e. affine) coordinates of spacetime and let x^i be arbitrary coordinates. Then

$$J(x)^i_j := \frac{\partial x^i}{\partial \hat{x}^j}, \qquad \hat{J}(\hat{x})^j_i := \frac{\partial \hat{x}^j}{\partial x^i} \tag{15}$$

and we know that $J(\hat{x}(x))^i_l \hat{J}(x)^l_k = \delta^i_k$ from which it follows that

$$0 = \frac{\partial^2 x^i}{\partial \hat{x}^m \partial \hat{x}^l} \frac{\partial \hat{x}^m}{\partial x^j} \frac{\partial \hat{x}^l}{\partial x^k} + \frac{\partial x^i}{\partial \hat{x}^l} \frac{\partial^2 \hat{x}^l}{\partial x^j \partial x^k} \tag{16}$$

For the coordinates of a vector field C we have

$$C^i(x) = J^i_l(\hat{x}(x))\hat{C}^l(\hat{x}(x)), \qquad \hat{C}^l(\hat{x}) = \hat{J}^l_k(x(\hat{x}))C^k(x(\hat{x})). \tag{17}$$

The covariant derivative $\mathrm{D}C$ of a vector field is a tensor field whose coordinates in affine coordinatization are just the partial derivatives of the vector field: $(\mathrm{D}\hat{C})^l_m = \mathrm{D}_m\hat{C}^l = \frac{\partial \hat{C}^l}{\partial \hat{x}^m}$. According to the transformation of tensors, the arbitrary coordinates of this tensor field are

$$(\mathrm{D}_j C^i)(x) = J^i_l(\hat{x}(x))(\mathrm{D}\hat{C})^l_m(\hat{x}(x)))\hat{J}^m_j(x) \tag{18}$$

For the partial derivatives of C^i, from (17) we infer (with a loose notation, omitting the variables)

$$\frac{\partial C^i}{\partial x^j} = \frac{\partial J^i_l}{\partial \hat{x}^m} \frac{\partial \hat{x}^m}{\partial x^j} \hat{C}^l + J^i_l \frac{\partial \hat{C}^l}{\partial \hat{x}^m} \frac{\partial \hat{x}^m}{\partial x^j}.$$

Then with the aid of (15), (18) and (17) we obtain

$$\mathrm{D}_j C^i = \frac{\partial C^i}{\partial x^j} + \Gamma^i_{jk} C^k.$$

Here we have got the first form of the Christoffel symbol given in (5); the second form comes from (16).

  2. The inverse of the transformation (1) is

$$t = \hat{t}, \qquad \mathbf{x} = \mathbf{Q}(\hat{t})^{-1}(\hat{\mathbf{x}} - \mathbf{h}(\hat{t})).$$

Then

$$J = \begin{pmatrix} 1 & 0 \\ -\mathbf{Q}^{-1}\dot{\mathbf{Q}}\mathbf{Q}^{-1}(\hat{\mathbf{x}} - \mathbf{h}) - \mathbf{Q}^{-1}\dot{\mathbf{h}} & \mathbf{Q}^{-1} \end{pmatrix}. \tag{19}$$

[1] W. Noll. A mathematical theory of the mechanical behavior of continuous media. *Archives of Rational Mechanics and Analysis*, 2:197–226, 1958.

[2] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics*. Springer Verlag, Berlin-Heidelberg-New York, 1965. Handbuch der Physik, III/3.

[3] I. Müller. *Thermodynamics*. Pitman, Toronto, 1985.

[4] M. Šilhavý. *The Mechanics and Thermodynamics of Continuous Media*. Springer Verlag, Berlin-etc., 1997.

[5] C. G. Speziale. A review of material frame-indifference in mechanics. *Applied Mechanical Reviews*, 51(8):489–504, 1998.

[6] I. Müller. On the frame dependence of stress and heat flux. *Archive of Rational Mechanics and Analysis*, 45:241–250, 1972.

[7] D. G. B. Edelen and J. A. McLennan. Material indifference: a principle or a convenience. *International Journal of engineering Science*, 11:813–817, 1973.

[8] A. I. Murdoch. On material frame-indifference, intrinsic spin and certain constitutive relations motivated by the kinetic theory of gases. *Archive of Rational Mechanics and Analysis*, 83:185–194, 1983.

[9] G. Ryskin. Misconception which led to the "material frame indifference" controversy. *Physical Review E*, 32(2):1239–1240, 1985.

[10] T. Matolcsi. On material frame-indifference. *Archive of Rational Mechanics and Analysis*, 91(2):99–118, 1986.

[11] T. Matolcsi and T. Gruber. Spacetime without reference frames: An application to the kinetic theory. *International Journal of Theoretical Physics*, 35(7):1523–1539, 1996.

[12] A. Bertram and B. Svendsen. On material objectivity and reduced constitutive equations. *Archive of Mechanics*, 53:653–675, 2001.

[13] W. Muschik. Objectivity and frame indifference, revisited. *Archive of Mechanics*, 50:541–547, 1998.

[14] A. I. Murdoch. Objectivity in classical continuum physics: a rationale for discarding the 'principle of invariance under superposed rigid body motions' in favour of purely objective considerations. *Continuum Mechanics and Thermodynamics*, 15:309–320, 2003.

[15] I-S. Liu. On Euclidean objectivity and the principle of material frame-indifference. *Continuum Mechanics and Thermodynamics*, 16:177–183, 2003.

[16] A. I. Murdoch. On criticism of the nature of objectivity in classical continuum physics. *Continuum Mechanics and Thermodynamics*, 17:135–148, 2005.

[17] I-S. Liu. Further remarks on Euclidean objectivity and the principle of material frame-indifference. *Continuum Mechanics and Thermodynamics*, 17:125–133, 2005.

[18] Y. Choquet-Bruhat, C. DeWitt-Morette, and M. Dillard-Bleick. *Analysis, Manifolds and Physics*. North-Holland Publishing Company, Amsterdam - New York - Oxford, 2nd, revised edition, 1982.

[19] J. G. Oldroyd. On the formulation of rheological equations of state. *Proceedings of Royal Society of London, A*, 200:523–541, 1949.

[20] J. Verhás. *Thermodynamics and Rheology*. Akadémiai Kiadó and Kluwer Academic Publisher, Budapest, 1997.

[21] F. Bampi and A. Morro. Objectivity and objective time derivatives in continuum physics. *Foundations of Physics*, 10(11/12):905–920, 1980.