ALLEN STAIRS

ON THE LOGIC OF PAIRS OF QUANTUM SYSTEMS

Even a nodding acquaintance with the history of the philosophical discussion of quantum mechanics makes clear that many of the theory's puzzles have to do with pairs (and larger collections) of systems. The EPR "paradox", the question of local hidden variables, Schrödinger's cat, the measurement problem, and the question of holism all have essentially to do with the way in which collections of systems are represented in quantum mechanics. The aim of this paper is to provide a mathematical framework for approaching these problems from within the quantum logical tradition. Before presenting the details, however, it will be helpful to set the problem in its proper context.

Although there have been various discussions of the logical aspects of collections of quantum systems, none of which the author is aware is as complete mathematically as one might wish and most restrict themselves entirely to the mathematics. Among the papers which treat the problem within lattice theory, A. Zecca's "On the coupling of logics" and T. Matolcsi's "Tensor product of Hilbert lattices and free orthodistributive product of orthomodular lattices" appear to be the most complete. Zecca defines a product of proposition systems (in the Jauch and Piron sense) and proves that if the factors have trivial centers (and hence are isomorphic to lattices of subspaces of vector spaces) then so is the product. However, he does not show that if the factors are lattices of subspaces of Hilbert spaces, then the product is itself a lattice of subspaces of a Hilbert space. Also, Zecca's definition of a product, although mathematically elegant, is in terms of conditions many of which seem to lack an intuitive motivation.[1] Matolcsi, on the other hand, proves very strong theorems within the category of lattices of subspaces of Hilbert spaces (henceforth *Hilbert space lattices*) but proves no results for the more general category of proposition systems (or, for that matter, orthomodular lattices).

The desirability of connecting the product with Hilbert space is obvious enough, as is the desirability of motivating the conditions embodied in a definition. However, since Matolcsi's conditions are not lacking in motivation, it may seem that the fact that he restricts himself to

Synthese **56** (1983) 47–60. 0039–7857/83/0561–0047 $01.40

Hilbert space lattices does not constitute a serious shortcoming. After all, it is quantum mechanics that gives point to quantum logic, and quantum mechanics is a Hilbert space theory. Therefore, why not simply begin by considering Hilbert space lattices? The resulting theorems will be weaker, since they make stronger assumptions, but it may seem that the difference is not philosophically significant. Though tempting, I believe that this conclusion would be a mistake. To see why, we need to consider briefly the aims of quantum logic as it is understood in this paper and to say a bit about just what it is that makes collections of quantum systems puzzling.

Quantum logic is many things to many people. To some, it is simply a means of encoding certain facts about quantum systems in a way which permits the making of certain analogies with logic, but which really has nothing to do with logic properly understood. The point of view taken here, however, is much more robust. It is held that logical structure is a real feature of systems, physical and otherwise, and that a salient feature of quantum mechanics is the novel logical structure which it attributes to events. Moreover, it is held that logic plays an *explanatory* role – that the nonstandard logical aspects of quantum systems help to account for their puzzing behavior. To develop and defend this view in detail would be inappropriate here. However, one important tenet of the so-called "realist" program just described is that quantum logic is a *generalization* of classical logic and includes classical logic as a special case. Failure to appreciate this point has, I believe, led to various important misunderstandings of the realist position in quantum logic. For this reason alone, then, it is important that my notion of a product of logical structures in quantum logic be a generalization of the classical notion.

On the matter of explanation, it is safe to say that in all of the cases noted in the first paragraph, the puzzles arise because of the existence of pure states of collections of systems which are not simply products of pure states of the subsystems. From one point of view, the existence of these so-called 'type-two' states is readily accounted for: they are a natural consequence of the use of the tensor product Hilbert space in the treatment of families of quantum systems. However, from the point of view of quantum logic, the notion of the tensor product of Hilbert spaces is clearly extrinsic. It obviously would aid the explanatory ambitions of quantum logic if it could be shown that the existence of type-two states is the inevitable consequence of generalizing the product of classical logical systems to the quantum mechanical case. We show that this desideratum is fulfilled and provide a discussion of the logical differences between classical and quantum systems on this point.

Before turning to the mathematical treatment of products, it should be noted that definitions of standard lattice-theoretic notions are presupposed. Readers unfamiliar with these notions may consult the works of Jauch, Varadarajan, and Maeda and Maeda listed among the references. It also should be noted that I rely heavily on results proved in Varadarajan and direct the reader to that source for details.

1. CLASSICAL VS. QUANTUM

We begin with a review of those features of quantum systems which distinguish them from classical systems and give rise to the difficulties indicated above. In the case of a classical system, the event space of a physical system has the structure of a Boolean algebra. In the usual case, one begins with a set of E of states or maximal events and constructs the event space by taking the Boolean algebra $B(E)$ of measurable subsets of E. Let S_1 and S_2 be two classical systems with associated event spaces or logics $B(E_1)$ and $B(E_2)$. $B(E_1)$ and $B(E_2)$ will be σ-fields of sets and the appropriate event space for the union of the two systems will be the field σ-product of $B(E_1)$ and $B(E_2)$.

The simplest way of viewing the field product is to note that it is isomorphic to $B(E_1 \times E_2)$, i.e., the σ-field of measurable subsets of $E_1 \times E_2$, the cartesian product of E_1 and E_2. More revealing for our purposes, however, is the characterization in terms of the notion of a *Boolean product*.[2] Let B_1 and B_2 be a pair of Boolean σ-algebras. The Boolean product of B_1 and B_2 is a triple (i_1, i_2, B) such that

(a) B is a Boolean σ-algebra,
(b) i_1 and i_2 are σ-isomorphisms of B_1, B_2 respectively into B,
(c) $i_1(B_1)$ and $i_2(B_2)$ are independent; i.e., if b_1, b_2 are nonzero elements respectively of B_1, B_2 then $i_1(b) \wedge i_2(b_2) \neq 0$,
(d) $i_1(B_1) \cup i_2(B_2)$ σ-generates B.

The field product is just a special case of the Boolean product. (More precisely, it is the *least* Boolean product of B_1 and B_2, where the order is given by imbeddability.) Let us briefly examine these four conditions. Condition (a) needs no comment. Condition (b) guarantees that the logic of $S_1 + S_2$ contains isomorphic copies of the logics of S_1 and S_2. (c) assures us that S_1 and S_2 are independent in the following sense. Any combination of an S_1 event with an S_2 event is a possible event for $S_1 + S_2$. Thus, no event associated with one system excludes any event associated with the other, nor does it permit any *inference* to an event of the other system. Condition (c) also entails the existence of statistical states with the property that every event of one system is probabilistically independent of every event of the other. (States which lack this feature are, of course, possible as well.) Condition (d) insures that the parts determine the whole in a certain clear sense. In the case of classical systems, this is a rather strong sense. A consequence of the above definition is that if a is an atom, or maximal event, then $a = i_1(a_1) \wedge i_2(a_2)$ where a_1 is an atom of B_1 and a_2 an atom of B_2. In the case of pairs of classical systems, then, we can always think of the subsystems as being in definite states determined by elements e_1, e_2 of E_1, E_2.

As is well known, the quantum mechanical case is more complex. The states of a quantum system are represented by unit vectors in Hilbert space. The event space is the lattice of closed linear subspaces of Hilbert space. If S_1 and S_2 are two quantum systems with event spaces $L(H_1)$ and $L(H_2)$, then the event space for $S_1 + S_2$ is $L(H_1 \otimes H_2)$, the lattice of subspaces of the tensor product of H_1 and H_2. $H_1 \otimes H_2$ is *generated* by the elements $\{\alpha_i \times \beta_j\}$, where $\alpha_i \in H_1$, $\beta_j \in H_2$ and unit vectors of the form $\alpha \otimes \beta$ are referred to as type-I states. However, a general vector in $H_1 \otimes H_2$ has the form

$$\sum_{ij} c_{ij}\alpha_i \otimes \beta_j,$$

and not all such unit vectors are expressible as type-I states. Those which are not are referred to as type-II states and it is the type-II states which give rise to many of the characteristic difficulties of interpretation which beset quantum mechanics. From a lattice-theoretic point of view, the notable feature of these states is that the corresponding propositions are not of the form $a \wedge b$, where a is a proposition concerning S_1 and b a proposition concerning S_2.

2. PROPOSITION SYSTEMS

Classical event spaces and lattices of subspaces of Hilbert space are both instances of *proposition systems*.[3] Following Jauch and Prion, a proposition system is a complete orthomodular lattice L satisfying the *atomicity axiom*:

A(i) For every element $a = \emptyset$ of L there exists an atom p such that $p \leqslant a$.

A(ii) If p is an atom then for all $a, x \in L$,

$$a \leqslant x \leqslant (a \vee p) \Rightarrow x = a \quad \text{or} \quad x = a \vee p.$$

A(ii) is usually referred to as the covering law. We recall an important fact concerning proposition systems which is a consequence of the covering law. If $k \in L$ is finite dimensional, then the elements of the lattice $[k]$ consisting of all $x \in L$ such that $x \leqslant k$ satisfies the modular law

$$a \leqslant c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c),$$

as well as the orthomodular law

$$a \leqslant b \Rightarrow b = a \wedge (b \vee a^{\perp}).$$

In the case of classical as well as quantum systems, observables can be represented as σ-homomorphisms from $B(R)$, the Boolean σ-algebra of measurable subsets of the real line, into L. If O is a maximal observable for the system in question, the range of O is a maximal Boolean sub-σ-algebra of L. In the classical case, this is L itself. In the quantum case, it is a proper subalgebra of L.

3. LOGICAL PRODUCTS OF PROPOSITION SYSTEMS

The characterization given in section 1 of the event space of a pair of quantum systems was extrinsic, involving explicit reference to the tensor product of Hilbert spaces. In what follows, I offer a characterization of the product event space which embodies all of the mathematical features that one could reasonably demand. Specifically, it is proved that the *logical product* of Hilbert space lattices is itself a Hilbert space lattice, without assuming anything in the definition concerning vector spaces.

Let L_1 and L_2 be proposition systems and let (i_1, i_2, L) be such that

(i) L is a proposition system,

(ii) $i_1 : L_1 \rightarrow L$ and $i_2 : L_2 \rightarrow L$ are orthoisomorphisms.

(iii) if B_1 is a maximal Boolean subalgebra of L_1 and B_2 is a maximal Boolean subalgebra of L_2, then $i_1(B_1) \cup i_2(B_2)$ generates a maximal Boolean subalgebra B of L such that

$(i_{1/B_1}, i_{2/B_2}, B)$ is a Boolean σ-product of B_1 and B_2.

(iv) $i_1(L_1) \cup i_2(L_2)$ σ-generates L.

Conditions (i) and (ii) require no comment. Condition (iii) makes it clear that the object being defined is a natural generalization of the product of classical event spaces. Indeed the motivation for (iii) is to make the product of general proposition systems as like the classical product as is compatible with the recognition that we are concerned with a larger class of structures. Condition (iv) is just the counterpart of condition (iv) in the definition of the Boolean product. These conditions, then, represent very minimal and natural assumptions. Nonetheless, they enable us to prove some rather striking theorems. For notational convenience, if $a \in L_1$ and $b \in L_2$, then denote $i_1(a)$, $i_2(b)$ by $^*a$, b^*. If $a \in L_1$ and $b \in L_2$ are atoms, then (iii) guarantees that $^*a \wedge b^*$ is an atom of L. We have the following theorem.

THEOREM 1: Let H_1 and H_2 be Hilbert spaces of dimension $\geqslant 2$. If (i_1, i_2, L) satisfies (i)–(iv), then L contains atoms which are not of the form $^*a \wedge b^*$, where a is an atom of $L(H_1)$ and b is an atom of $L(H_2)$.

Proof: The proof proceeds by choosing arbitrary atoms $a, a' \in L(H_1)$ and $b, b' \in L(H_2)$ such that $a \perp a'$, $b \perp b'$ and showing that $^*a \wedge b^*$ and $^*a' \wedge b'^*$ are strongly perspective. It is then shown that any atom $x \in L$ such that x is an axis of perspectivity for $^*a \wedge b^*$ and $^*a' \wedge b'^*$ [i.e., such that $((^*a \wedge b^*) \vee (^*a' \wedge b'^*)) = ((^*a \wedge b^*) \vee x) = ((^*a' \wedge b'^*) \vee x)$] cannot satisfy $x = (^*y \wedge z^*)$, y and z atoms. Details are in the appendix.

Thus, the minimal conditions for constructing the event space of a pair of quantum systems require the existence of events corresponding to type-II states. With the help of the following lemmas we can prove another important theorem.

LEMMA 1: Let (i_1, i_2, L) be as in theorem 1. If $B_1 \subseteq L(H_1)$ and $B_2 \subseteq L(H_2)$ are maximal Boolean subalgebras, then every pair of atoms in the maximal Boolean subalgebra B generated by $i_1(B_1) \cup i_2(B_2)$ is strongly perspective.

Proof: If $\{a_i\}$ is the set of atoms of B_1 and $\{b_j\}$ is the set of atoms of B_2, then $\{^*a_i \wedge b_j^*\}$ is the set of atoms of B. For $i \neq k$, $j \neq l$, the proof of theorem 1 assures us of the existence of an appropriate perspective element for $^*a_i \wedge b_j^*$, $^*a_k \wedge b_l^*$. For pairs of elements $^*a_i \wedge b_j^*$, $^*a_i \wedge b_l^*$, any element $^*a \wedge c^*$ with c an atom which is an axis of perspectivity for

b_j, b_l will suffice. (The structure of $L(H_2)$ guarantees the existence of such atoms.) Similar remarks apply for pairs $*a_i \wedge b_j^*$, $*a_k \wedge b_j^*$.

LEMMA 2: Let L be a proposition system and B a maximal Boolean subalgebra of L such that every pair of atoms a, b of B is strongly perspective. It follows that L is irreducible.

Proof: Assume the contrary. Then there are elements A, $A \neq \emptyset$, 1 in the center of L. Let $\{a_i\}$ be the set of atoms of B. Since A is central, we have $A = A \wedge \bigvee_i a_i = \bigvee_j a_j$ for some set $\{a_j\} \subset \{a_i\}$. Similarly, for an appropriately chosen set $\{a_k\} \subset \{a_i\}$, we have $A^\perp = \bigvee_k a_k$. Let $x \in \{a_j\}$ and $y \in \{a_k\}$. By hypothesis x and y are strongly perspective. Let z be an atom which is an axis of perspectivity for x and y. We have $(x \vee y) = (x \vee z) = (y \vee z)$. Clearly $\{a_j\} \cap \{a_k\} = \emptyset$. Since A is central we have $(A \wedge x) \vee (A \wedge y) = (A \wedge x) \vee (A \wedge z) = A \wedge x = x$. Therefore, $A \wedge z = \emptyset$. Similarly, $A^\perp \wedge z = \emptyset$. But since z commutes with A, we have $z = (z \wedge A) \vee (z \wedge A^\perp) = \emptyset$, which contradicts the fact that z is an atom. Therefore, there are no central elements in L distinct from $\emptyset$, 1.

THEOREM 2: Let $L(H_1)$, $L(H_2)$, (i_1, i_2, L) be as in theorem 1. L is irreducible.

Proof: This is an immediate consequence of lemmas 1 and 2.

Remark: Since L in theorem 2 is of dimension $\geqslant 4$, it can be shown that L is a complete projective logic in the sense of Varadarajan, p. 176. (Varadarajan relies on this fact in his proof of Piron's theorem.) It is therefore a consequence of theorem 7.40 in Varadarajan that L is isomorphic to the lattice of closed subspaces of a vector space over a division ring D. Moreover, if H_1 and H_2 are over the same division ring K, then $L(H_1 \otimes H_2)$ together with the canonical mappings from $L(H_1)$, $L(H_2)$ into $L(H_1 \otimes H_2)$ satisfies (i)–(iv).[4]

We may now ask to what extent (i)–(iv) determine L completely. At present, the answer is not clear. However, by adding one additional constraint (which may prove to be eliminable) we define an object for which a fully determinate representation theorem exists in the most interesting case. The additional axiom is

(v) Let $a \in L_1$, $b \in L_2$ be nonzero. Define $i_1' : [a] \to [*a \wedge b^*]$ by $i_1'(x) = *x \wedge b^*$ and $i_2' : [b] \to [*a \wedge b^*]$ by $i_2'(y) = *a \wedge y^*$. The lattices $[a]$, $[b]$ and the triple $(i_1, i_2, [*a \vee b^*])$ satisfy (i)–(iv) when orthocomplement in $[a]$ is understood as relative orthocomplement in L_1 mod a and similarly for $[b]$, $[{}^*a \wedge b^*]$.

Some comments on (v). First, in the case where L_1 and L_2 are Boolean algebras, it is easily verified that (v) is redundant. Further, (v) can be provided with a certain intuitive motivation. It is often the case that a physical system, due for instance to considerations of conservation or symmetry, is restricted to some subset of the events in its event space. From the point of view of the system, as it were, the effective event space is the lattice under some event $e \neq 1$. If S_1 and S_2 each occupy such "invariant subspaces", (v) simply requires that all of the events open to $S_1 + S_2$ be determined by the events in the appropriate sublattices $[e_1]$, $[e_2]$ of L_1 and L_2. Thus, (v) can be provided with a certain plausibility independent of its technical role. The important mathematical consequence of (v), however, is embodied in the following lemma.

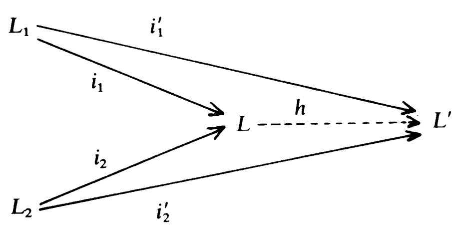
LEMMA 3: Let $a \in L_1$ be nonzero and let $b \in L_2$ be an atom. The morphism $i_1' : [a] \to [{}^*a \wedge b^*]$ defined as in (v) is surjective.

Proof: By (v), we know that every element of $[{}^*a \wedge b^*]$ is expressible in terms of elements from $i_1'[a] \cup i_2'[b]$. We note that $i_2'[b] = \{{}^*a \wedge b^*, \emptyset\} = \{{}^*a \wedge b^*, {}^*\emptyset \wedge b^*\}$. Further, if $\{a_i\}$ is the set of elements of $[a]$ then $i_1'[a] = \{{}^*a_i \wedge b^*\} \supseteq i_1'[b]$. Thus $[{}^*a \wedge b^*]$ is generated by $i_1'[a]$. It therefore follows that $[a^* \wedge b^*]$ is isomorphic to $[a]$. If $e = \{\phi^* a_i \wedge b^*\}$ where ϕ is a (possibly infinite) lattice polynomial, then $e = i_1'(\phi\{a_i\})$.[5] This lemma leads immediately to the following important theorem.

THEOREM 3: Let $L(H_1)$, $L(H_2)$, (i_1, i_2, L) satisfy (i)–(v). Suppose that $\dim(H_1), \dim(H_2) \geqslant 3$. Suppose further that H_1 and H_2 are over the same division ring D. Then $L = L(H)$ for some Hilbert space H.

Proof: By theorem 2, L is irreducible and clearly $\dim(L) \geqslant 4$. It can therefore be shown that L is a complete projective logic in the sense of Varadarajan (p. 176). It follows from Varadarajan's theorem 7.40 that there exists a division ring D, a vector space V over D, an involutive anti-automorphism θ of D and a definite, symmetric, θ-bilinear form $\langle .,. \rangle$ on $V \times V$ such that L is isomorphic to $L(V, \langle .,. \rangle)$, where $\langle .,. \rangle$ induces the orthocomplementation. Let $a \in L(H_1)$ be finite-dimensional with $\dim(a) \geqslant 3$ and let b be any atom of $L(H_2)$. We know from lemma 3 that $[{}^*a \wedge b^*]$ is isomorphic to $[a]$, which is the lattice of subspaces of finite-dimensional Hilbert space. It is clear from the proof of Varadarajan's theorem 7.40 (pp. 180–81) that θ is the conjugation and hence L is *associated* with D (p. 181). From Varadarajan's theorem 7.42 and the remarks following theorem 7.40 it is clear that $(V, \langle .\,, .\rangle)$ is a Hilbert space. Moreover, if H_1 and H_2 are separable, then by (iii) there exists a *countable* maximal orthogonal set in L. (If B_1 and B_2 are atomic σ-algebras with countably many atoms, then their Boolean σ-product has countably many atoms.) Therefore, in this case V is separable. (Halmos, section 16, Varadarajan, theorem 7.44).

If L_1 and L_2 are proposition systems let us call a triple satisfying (i)–(v) a *logical product* of L_1 and L_2. We can now ask, in the case where L_1 and L_2 are lattices of subspaces of Hilbert spaces, whether the logical product is unique. That is, suppose that (i_1, i_2, L) and (i_1', i_2', L') are two logical products of L_1 and L_2. Does there exist an isomorphism $h: L \to L'$ such that the following diagram commutes?



If we restrict ourselves to the cases of real and complex Hilbert space, theorem 3 leads to an answer via a result proved by Matolcsi. As noted earlier, Matolcsi considers for Hilbert space lattices the problem considered in this paper for the more general category of proposition systems. (Since Matolcsi does not have to prove that the product of Hilbert space lattices, as he defines it, is itself a Hilbert space lattice, the results of this paper provide a considerable strengthening of his results.) Matolcsi's definition is as follows (with notation altered to correspond to our own).

Let H_1, H_2, H be Hilbert spaces, all complex or all real. $(i_1, i_2, L(H))$ is called a tensor product of $L(H_1)$, $L(H_2)$ if

(i) $i_j : L(H_j) \to L(H)$ is a σ-orthoisomorphism $(j = 1, 2)$

(ii) $\bigvee_{n=1}^{\infty} \bigvee_{m=1}^{\infty} (i_1(M_1^n) \wedge i_2(M_2^m)) = \bigvee_{n=1}^{\infty} i_1(M_1^n) \wedge (\bigvee_{m=1}^{\infty} i_2(M_2^m))$ for any pairwise orthogonal elements M_1^n of $L(H_1)$ and pairwise orthogonal elements of M_2^m of $L(H_2)$

(iii) $i_1(L(H_1))$ and $i_2(L(H_2))$ generate $L(H)$.

Let us compare this with the definition in this paper. As a consequence of theorem 3 the logical product of $L(H_1)$ and $L(H_2)$ is a Hilbert space lattice together with a pair of isomorphisms. Thus, Matolcsi's (i) is satisfied. Since $\{M_1^n\}$ and $\{M_2^m\}$ belong to maximal Boolean subalgebras, our condition (iii) guarantees the fulfillment of Matolcsi's condition (ii). Finally, Matolcsi's (iii) corresponds to our (iv). To the three conditions just enumerated, Matolcsi adds a fourth, which he calls a condition of fullness

(iv) Let $a \in L(H_1)$ and $b \in L(H_2)$ be atoms. The maps $i_1' : L(H_1) \to [i_2(b)]$ and $i_2' : L(H_2) \to [i_1(a)]$ defined by

$$i_1'(M_1) = i_1(M_1) \wedge i_2(b)$$
$$i_2'(M_2) = i_1(a) \wedge i_2(M_2)$$

are surjective.

The proof of lemma 3 is symmetric in $L(H_1)$ and $L(H_2)$ and so this condition is satisfied by the logical product.

Matolcsi proves the following theorem. If H is a Hilbert space, let $\bar{H}$ denote its conjugate Hilbert space, the elements of which correspond canonically to the elements of H such that if $\bar{x}$ and $\bar{y}$ in $\bar{H}$ correspond to x and y in H then $\bar{x} + \bar{y}$ corresponds to $x + y$, $\lambda\bar{x}$ to λx (λ a number) and $\langle \bar{x}, \bar{y} \rangle$ to $\langle y, x \rangle$ where ., . is the inner product in H and H. If H is real then $H = \bar{H}$.

THEOREM: Let H_1 and H_2 be Hilbert spaces, dimH_1, $H_2 \geqslant 3$. If H_1 and H_2 are complex there are exactly two nonequivalent tensor products of $L(H_1)$ and $L(H_2)$ satisfying the condition of fullness. They are given

(i) $H = H_1 \otimes H_2,\quad i_1(M_1) = M \otimes H_2$
$\qquad\qquad\qquad\qquad i_2(M_2) = H_1 \otimes M_2$

(ii) $H = \bar{H}_1 \otimes H_2,\quad i_1(M_1) = \bar{M}_1 \otimes H_2$
$\qquad\qquad\qquad\qquad i_2(M_2) = \bar{H} \otimes M_2$

If H_1 and H_2 are real then (i) characterizes the unique tensor product.

The above theorem results from an application of Wigner's proof that the symmetries of a Hilbert space are all induced either by unitary or anti-unitary transformations. It is clear from what has been said that Matolcsi's theorem also characterizes the logical products of $L(H_1)$ and $L(H_2)$.

As was noted at the outset, this paper is intended to provide a mathematical preliminary to a discussion of the logical features of pairs and larger families of quantum systems. It is to be hoped that the logical approach will help to clarify the issues of holism and locality. In addition, the results discussed here give rise to an interesting question, viz., what is the physical significance of the existence of nonequivalent logical products? The answer, I suspect, will come from considerations of time-reversal and particle–anti-particle pairs.

APPENDIX

Proof of Theorem 1:
Choose distinct $a, a', c, c' \in L_1$ such that $a \perp a'$, $c \perp c'$ and $a \vee a' = c \vee c'$. Similarly, choose distinct b, b', d, $d' \in L_2$ such that $b \perp b'$, $d \perp d'$ and $b \vee b' = d \vee d'$. Let P denote $a \vee a'$ and Q denote $b \vee b'$. Note that there exist maximal Boolean subalgebras B_1, B_3 of L_1 such that a, $a' \in B_1$, c, $c' \in B_3$ and maximal Boolean subalgebras B_2, B_4 of L_2 such that b, $b' \in B_2$, d, $d' \in B_4$. It is therefore easily seen that

$$\begin{aligned}(^*a \wedge b^*) \vee (^*a \vee b'^*) \vee (^*a' \wedge b^*) \vee (^*a' \wedge b'^*) &= {}^*P \wedge Q^* \\ &= (^*c \wedge d^*) \vee (^*c \wedge d'^*) \vee (^*c' \wedge d'^*) \vee (^*c' \wedge d'^*).\end{aligned}$$

It follows that $\dim(^*P \wedge Q^*) = 4$ and that $[^*P \wedge Q^*] = \{x \in L: x \leqslant {}^*P \wedge Q^*\}$ is modular. Denote $[^*P \wedge Q^*]$ by L'. We first show that $^*a \wedge b^*$ and $^*a' \wedge b'^*$ are perspective in L' (i.e., have a common complement). Since L' is modular, it follows that $^*a \wedge b^*$ and $^*a' \wedge b'^*$ are strongly perspective (i.e., have a common complement in $[(^*a \wedge b^*) \vee (^*a' \wedge b'^*)]$, which complement is an atom of L)We show this by proving that each of them has

(1) $\qquad (^*c \wedge d^*) \vee (^*c \wedge d'^*) \vee (^*c' \wedge d^*)$

as a complement in L'. In the proof, we shall appeal frequently to the well-known result of Holland and Foulis that in an orthomodular lattice, if x and y both commute with z, then (x, y, z) is a distributive triple.

We have that (1) is equal to

(2) $\qquad (^*c \wedge Q^*) \vee (^*c' \wedge d^*)$.

We must show that

(3) $\qquad (^*a \wedge b^*) \wedge ((^*c \wedge Q^*) \vee (^*c' \wedge d^*)) = \phi$

and

(4) $\qquad (^*a \wedge b^*) \vee ((^*c \wedge Q^*) \vee (^*c \wedge d^*)) = {}^*P \wedge Q^*$.

Suppose (3) is false. We then have

(5) $({}^*a \wedge b^*) \leqslant (({}^*c \wedge Q^*) \vee ({}^*c' \wedge d^*))$.

The right side of (5) belongs to $[{}^*P \wedge Q^*]$. We may therefore invoke modularity and infer that

(6) $({}^*a \wedge b^*) \vee \{({}^*c \wedge b^*) \wedge (({}^*c \wedge Q^*) \vee ({}^*c' \wedge d^*))\}$

is identical with

(7) $\{({}^*a \wedge b^*) \vee ({}^*c \wedge b^*)\} \wedge \{({}^*c \wedge Q^*) \vee ({}^*c' \wedge d^*)\}$.

But (6) reduces to

(8) $({}^*a \wedge b^*) \vee ({}^*c \wedge b^*)$

which, by the Holland and Foulis result, is

(9) $({}^*a \vee {}^*c) \wedge b^*$

which, given our choice of a, c, is ${}^*P \wedge b^*$.

By the identity of (6) and (7), then,

(10) ${}^*P \wedge b^* = \{({}^*a \wedge b^*) \vee ({}^*c \wedge b^*)\} \wedge \{({}^*c \wedge Q^*) \vee ({}^*c' \wedge d^*)\}$

and by two applications of the Holland and Foulis result,

(11) ${}^*P \wedge b^* = \{({}^*P \wedge b^*) \wedge (({}^*c \wedge Q^*) \vee ({}^*c' \wedge d^*))\}$

(12) ${}^*P \wedge b^* = ({}^*c \wedge b^*) \vee ({}^*P \wedge b^* \wedge {}^*c' \wedge d^*)$.

However, $b^* \wedge d^* = \emptyset$. Therefore, we have

(13) ${}^*P \wedge b^* = ({}^*c \wedge b^*)$.

But this is impossible, since $\dim({}^*P \wedge b^*) = 2$ and $\dim({}^*c \wedge b^*) = 1$. Therefore

(14) $({}^*a \wedge b^*) \wedge \{(c \wedge d^*) \vee ({}^*c \wedge d'^*) \vee ({}^*c' \wedge d^*)\} = \emptyset$.

Further, since $({}^*a \wedge b^*)$ is an atom not under the three dimensional element $({}^*c \wedge d^*) \vee ({}^*c \wedge d'^*) \vee ({}^*c' \wedge d^*)$ and both are under ${}^*P \wedge Q^*$, we have

$$({}^*a \wedge b^*) \vee \{({}^*c \wedge d^*) \vee ({}^*c \wedge d'^*) \vee ({}^*c' \wedge d^*)\} = {}^*P \wedge Q^*.$$

Thus, these elements are complements in $[{}^*P \wedge Q^*]$. A similar argument shows that $({}^*a' \wedge b'^*)$ and $\{({}^*c \wedge d^*) \vee ({}^*c \wedge d^*) \vee ({}^*c' \wedge d^*)\}$ are complements in $[{}^*P \wedge Q^*]$. Thus, $({}^*a \wedge b^*)$ and $({}^*a' \wedge b'^*)$ are strongly perspective.

We must now show that any element x of L which satisfies

(i) $\qquad ({}^*a \wedge b^*) \wedge x = \emptyset = ({}^*a' \wedge b'^*) \wedge x$

and

(ii) $\qquad ({}^*a \wedge b^*) \vee x = ({}^*a \wedge b^*) \vee ({}^*a' \wedge b'^*) = ({}^*a' \wedge b'^*) \vee x$

cannot be of the form ${}^*y \wedge z^*$ (y and z atoms). First, suppose that y is an atom of L_1 not under P. If (ii) is to be satisfied, we must have

$$ {}^*y \wedge z^* \leq {}^*P \wedge Q^* $$

and hence

$$ {}^*y \wedge z^* \leq {}^*P. $$

But $y \wedge P = \emptyset$, hence ${}^*y \wedge z^* \nleq {}^*P$. Similar remarks apply if $z \nleq Q^*$. Thus, if $x = {}^*y \wedge z^*$, then $y \leq P$ and $z \leq Q$.

It is clear that $x \neq ({}^*a \wedge b^*)$, $({}^*a \wedge b'^*)$. Further, x cannot be any of $({}^*a \wedge z^*)$, $({}^*a' \wedge z^*)$, $({}^*y \wedge b^*)$, $({}^*y \wedge b'^*)$. The case of $({}^*a \wedge z^*)$ is representative. We have

(15) $$ \begin{aligned} ({}^*a \wedge z^*) \vee ({}^*a \wedge b^*) &= {}^*a \wedge Q^* \\ &\neq ({}^*a \wedge b^*) \vee ({}^*a' \wedge b'^*). \end{aligned} $$

It now remains to consider $y \neq a, a'$, $z \neq b, b'$. Suppose

(16) $\qquad ({}^*y \wedge z^*) \leq ({}^*a \wedge b^*) \vee ({}^*a' \wedge b'^*)$.

We then have

(17) $$ \begin{aligned} ({}^*y \wedge z^*) &\leq ({}^*a \wedge b^*) \vee ({}^*a' \wedge b'^*) \vee ({}^*a \wedge b'^*) \\ &= ({}^*a \wedge Q^*) \vee ({}^*a' \wedge b'^*). \end{aligned} $$

By modularity,

(18) $\qquad ({}^*y \wedge z^*) \vee \{({}^*a \wedge z^*) \wedge (({}^*a \wedge Q^*) \vee ({}^*a' \wedge b'^*))\}$

is identical with

(19) $\qquad \{({}^*y \wedge z^*) \vee ({}^*a \wedge z^*)\} \wedge \{({}^*a \wedge Q^*) \vee ({}^*a' \wedge b'^*)\}$.

Using Holland and Foulis's result and the fact that ${}^*a \wedge z^* \leq {}^*a \wedge Q^*$, (18) becomes

(20) $\qquad {}^*P \wedge z^*$.

We can similarly reduce (19) to

(21) $({}^*P \wedge z^*) \wedge \{({}^*a \wedge Q^*) \vee ({}^*a' \wedge b'^*)\}$

and since we can distribute, (21) is

(22) ${}^*a \wedge z^*$.

We thus get ${}^*a \wedge z^* = {}^*P \wedge z^*$, which is clearly false.

Therefore, there exists an atom $x \leqslant {}^*P \wedge Q^*$ which is not of the form ${}^*y \wedge z^*$.

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NOTES

* The author gratefully acknowledges the support of the general research board of the University of Maryland for financial assistance during the preparation of this paper.

[1] I suspect that Zecca's definition of a product is equivalent to (i)–(iv) in this paper, but have not confirmed this at the time of writing. I wish to thank Professors David Foulis and Charles Randall for bringing the papers by Matolcsi and Zecca to my attention.

[2] This definition is from R. Sikorski, *Boolean Algebras*.

[3] See, J. Jauch, *Foundations of Quantum Mechanics* for a discussion of proposition systems.

[4] If H_1 and H_2 are *not* over the same division ring, then one would expect that their product would not exist. I believe that this is true, but have not yet shown it. I suspect that if it were proved, it could be shown that condition (v) is eliminable.

[5] Recall that we are considering *complete* lattices.

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