

Probability current in zero-spin relativistic quantum mechanics

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Abstract

We show that the antisymmetric spinor tensor representation of spin-0 relativistic quantum mechanics provides a conserved current with positive definite timelike component, interpretable as probability density. The construction runs in complete analogy to the spin-1/2 case, and provides an analogously natural one-particle Hilbert space description for spin 0. Except for the free particle, the obtained formulation proves to be inequivalent to the one based on the Klein–Gordon equation. The second quantized version may lead to new field theoretical interaction terms for zero-spin particles.¹

1 Introduction

The Klein–Gordon equation for a scalar wave function,

$$[(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) - m^2]\phi = 0 \quad (1)$$

(see notation conventions below) is very plausible relativistic quantum mechanical model for a spin-0 particle in an external four-potential field but the corresponding conserved four current

$$\frac{1}{2im}(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi) \quad (2)$$

cannot be interpreted as a probability current since its timelike component is not positive definite. While the so-called Feshbach–Villars formalism [1] rewrites (1) as a Schrödinger-type – first-order in time derivative – equation on the two-component wave function

$$\begin{pmatrix} \phi + \frac{i}{m}(\partial_0 + ieA_0)\phi \\ \phi - \frac{i}{m}(\partial_0 + ieA_0)\phi \end{pmatrix}, \quad (3)$$

and can restrict solutions to those with positive integral of the timelike component of (2), a Hilbert space structure cannot be established.

For the free particle, the ‘positive’ solutions of

$$(\partial^\mu\partial_\mu - m^2)\phi = 0 \quad (4)$$

satisfy [2]

$$i\partial_0\phi = \sqrt{-\Delta + m^2}\phi \quad (5)$$

¹Material presented at the Zimányi School, December 7-11, 2015, Budapest, Hungary.

and admit a time independent Hilbert space scalar product

$$\langle \phi_1, \phi_2 \rangle = \int \phi_1^\dagger \phi_2 d^3 \mathbf{x} \quad (6)$$

but its generalization

$$i(\partial_0 + ieA_0)\phi = \sqrt{(\partial_j + ieA_j)(\partial_j + ieA_j) + m^2} \phi \quad (7)$$

to $A_\mu \neq 0$ is not equivalent to (1),

$$[(\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) - m^2] \phi \neq 0 \quad (8)$$

since ∂_μ and A_ν do not commute.

The situation is in sharp contrast with the spin-1/2 case where the Dirac equation automatically provides ‘positive’ solutions, a conserved current with positive definite timelike component, and a Hilbert space structure, all for $A_\mu \neq 0$ as well.

In this writing, we show that the antisymmetric spinor tensor representation of free spin-0 particles (see, e.g., [3]) can be generalized to $A_\mu \neq 0$ with a conserved current with positive definite timelike component, and a corresponding Hilbert space. In [3], only the free system is presented and only in momentum space – here, we perform the transformation to coordinate space, and carry out the generalization $A_\mu \neq 0$.

The construction can be established in complete analogy to the spin-1/2 case, and provides an analogously natural consistent one-particle theory for spin 0.

The free particle case is shown, via the adaptation of the spin-1/2 Foldy–Wouthuysen transformation, to be equivalent to the (5) version of scalar Klein–Gordon quantum mechanics, while for nonzero external field the equivalence is broken. Accordingly, we expect that, e.g., the Coulomb problem admits a spectrum different from the scalar Klein–Gordon one.

We start with revisiting the case of spin 1/2. This review is intentionally detailed and pedagogical – the spin-0 version can then be presented in a straightforward step-by-step way, due to the strong analogy between the two situations.

2 Basics

2.1 Notations

In most respects, our notations follow those of [4]. We work in the convention $\hbar \equiv c \equiv 1$, consider the Lorentz metric g with signature $(+---)$, and use spacetime four-indices $\mu, \nu = 0, 1, 2, 3$ and three-indices $j, k = 1, 2, 3$. Repeated indices involve summation. Complex conjugate is denoted by $*$, and its combination with transposition T is indicated by † . With the Pauli matrices and the 2×2 unit matrix,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (9)$$

we introduce the 4×4 matrices

$$\beta = \gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \alpha_j = -\beta\gamma_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad (10)$$

where 0 denotes the 2×2 zero matrix as well. These possess the properties

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I_4, \quad \gamma_0^\dagger = \gamma_0, \quad \gamma_j^\dagger = -\gamma_j, \quad \gamma_\mu^\dagger \beta = \beta \gamma_\mu, \quad \alpha_j^\dagger = \alpha_j. \quad (11)$$

For elements θ of \mathbb{C}^4 , as well as for complex 4×4 matrices Θ , we put

$$\bar{\theta} = \theta^\dagger \beta, \quad \bar{\Theta} = \Theta^\dagger \beta. \quad (12)$$

We discuss the case of positive particle mass, $m > 0$ only. Four-momenta p with

$$p_0 = \sqrt{p_j p_j + m^2} = \sqrt{\mathbf{p}^2 + m^2} \quad (13)$$

form the set \mathcal{P} – the positive mass shell –, on which the Lorentz invariant integration measure is $\frac{m}{p_0} d^3 \mathbf{p}$ (up to proportionality).

2.2 Geometric ingredients

For any p from \mathcal{P} , the eigenvalues of the matrix

$$p_\mu \gamma^\mu \tag{14}$$

are m and $-m$, following from that $(p_\mu \gamma^\mu)^2 = m^2 I_4$; and the corresponding two eigensubspaces $N_+(p)$, $N_-(p)$ are both two-dimensional. For example, especially simple is the case of the four-momentum that is at rest with respect to the inertial reference frame used:

$$\check{p} = \begin{pmatrix} m \\ \mathbf{0} \end{pmatrix}, \quad \check{p}_\mu \gamma^\mu = m \gamma_0 = m \beta : \tag{15}$$

then

$$N_+(\check{p}) \text{ is spanned by } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad N_-(\check{p}) \text{ is spanned by } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{16}$$

For other ps , upper and lower components become mixed.

One finds, analogously, that the same $N_+(p)$ and $N_-(p)$ are the eigensubspaces of

$$\alpha_j p_j + \beta m, \tag{17}$$

with eigenvalues p_0 and $-p_0$, respectively.

3 Spin 1/2 quantum mechanics

3.1 Free particle, momentum space

The momentum space version of the Dirac equation,

$$\gamma^\mu p_\mu \psi(p) = m \psi(p) \tag{18}$$

has, in the light of the previous section, the simple geometric interpretation that, at each p , $\psi(p)$ has to be an element of $N_+(p)$,

$$\gamma^\mu p_\mu \psi(p) = m \psi(p) \iff \psi(p) \in N_+(p). \tag{19}$$

For example, at $\check{p} = \begin{pmatrix} m \\ \mathbf{0} \end{pmatrix}$, a solution $\psi(\check{p})$ can have only upper nonzero components,

$$\psi(\check{p}) = \begin{pmatrix} \psi_1(\check{p}) \\ \psi_2(\check{p}) \\ 0 \\ 0 \end{pmatrix}. \tag{20}$$

For other ps , upper and lower components become mixed but there are still only two degrees of freedom – when we expand $\psi(p)$ with respect to basis vectors n_1, n_2, n_3, n_4 in \mathbb{C}^4 where n_1 and n_2 are in $N_+(p)$ and n_3, n_4 are in $N_-(p)$,

$$\psi(p) = c_1 n_1 + c_2 n_2 + c_3 n_3 + c_4 n_4, \quad n_1, n_2 \in N_+(p), \quad n_3, n_4 \in N_-(p), \tag{21}$$

then

$$(\gamma^\mu p_\mu - m) \psi(p) = (\gamma^\mu p_\mu - m) (c_1 n_1 + \dots + c_4 n_4) = 0 \implies c_3 = c_4 = 0, \tag{22}$$

and only the two coefficients c_1, c_2 can be nonzero.

Being on the positive mass shell (13) involves

$$p_0 \psi = \sqrt{p_j p_j + m^2} \psi \tag{23}$$

from which

$$p_0^2 \psi = (p_j p_j + m^2) \psi \tag{24}$$

follows so we obtain the momentum space version of the Klein–Gordon equation,

$$(p^\mu p_\mu - m^2) \psi = 0. \quad (25)$$

The same conclusion can be derived via acting by $\gamma^\mu p_\mu + m$ on $(\gamma^\mu p_\mu \psi - m) \psi = 0$ [see (18)] from the left.

To each proper Lorentz transformation L there exists (see, *e.g.*, [5]) – uniquely up to a unit multiplier – a 4×4 matrix D_L such that

$$\beta D_L^\dagger \beta = D_L^{-1}, \quad D_L (\gamma^\mu p_\mu) D_L^{-1} = \gamma^\mu (Lp)_\mu. \quad (26)$$

As a consequence, $\overline{D_L \theta} D_L \theta' = \overline{\theta} \theta'$. Proper Lorentz transformations map four-momenta of \mathcal{P} to four-momenta still within \mathcal{P} , and the formula

$$(U_{(a,L)} \psi)(p) = e^{ip_\mu a^\mu} D_L \psi(L^{-1}p) \quad (27)$$

proves [3, 5] to give the spin-1/2 irreducible unitary ray representation of the proper Poincaré group (a : a translation, L : a Lorentz transformation) on the Hilbert space $\mathcal{H}_p^{4,+}$ of (measurable) \mathbb{C}^4 valued functions ψ defined on \mathcal{P} with $\psi(p)$ being in $N_+(p)$ for each p , and with finite integral

$$\int \overline{\psi(p)} \psi(p) \frac{m d^3 \mathbf{p}}{p_0} = \int \psi(p)^\dagger \psi(p) \frac{m^2 d^3 \mathbf{p}}{p_0^2}. \quad (28)$$

Why these two integrals are equal follows from that, for $\psi(p)$ in $N_+(p)$,

$$(\alpha_j p_j + \beta m) \psi(p) = p_0 \psi(p), \quad (29)$$

$$\sigma_j p_j \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (p_0 + m) \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \quad \sigma_j p_j \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = (p_0 - m) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (30)$$

$$\overline{\psi} \psi = \frac{2m}{p_0 + m} (|\psi_1|^2 + |\psi_2|^2), \quad \psi^\dagger \psi = \frac{2p_0}{p_0 + m} (|\psi_1|^2 + |\psi_2|^2), \quad (31)$$

$$\overline{\psi(p)} \psi(p) = \frac{m}{p_0} \psi(p)^\dagger \psi(p). \quad (32)$$

The first form of the integral – the lhs of (28) – makes Lorentz invariance apparent while the second [the rhs of (28)] emphasizes positive definiteness of the integrand. It is important to bear in mind that, in the integrals, ψ is not an arbitrary square integrable \mathbb{C}^4 valued function – that would mean a larger Hilbert space \mathcal{H}_p^4 – but has to be in the subspace $\mathcal{H}_p^{4,+}$ of that larger Hilbert space \mathcal{H}_p^4 , defined by the condition $\psi(p) \in N_+(p)$. It is the solution space of (19) that becomes a Hilbert space by the integral (28).

In a rephrased form, the multiplier operator $\alpha_j p_j + \beta m$ is self-adjoint in \mathcal{H}_p^4 , and has the spectrum $(-\infty, -m) \cup [m, \infty)$. The proper Hilbert space $\mathcal{H}_p^{4,+}$ is the subspace corresponding to the positive half of this spectrum.

The physical meaning of being a representation of the Poincaré group is to be a free system on special relativistic spacetime (not to be connected to anything distinguished). The physical meaning of being an irreducible representation is that the system is elementary, not some composite one (not some decomposable one).

Except for $p = \check{p}$ [recall (20)], elements of $N_+(p)$ have some nonzero third and/or fourth component. If L_p denotes the Lorentz boost that brings p to \check{p} then $W(p) \equiv D_{L_p}$, the so-called Foldy–Wouthuysen transformation [6, 7, 8, 9], transforms elements of $N_+(p)$ to elements of $N_+(\check{p})$, in other words, to upper components only:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi^W(p) = W(p) \psi(p), \quad \psi^W = \begin{pmatrix} \psi_1^W \\ \psi_2^W \\ 0 \\ 0 \end{pmatrix}. \quad (33)$$

As D_L is unique only up to unitary equivalence, so is $W(p)$. One possible choice is

$$W(p) = \frac{1}{\sqrt{2m(p_0 + m)}} [(p_0 + m)I_4 - \alpha_j p_j]. \quad (34)$$

Since $(p^\mu p_\mu - m^2) I_4$ commutes with $W(p)$, from (25) we find that

$$(p^\mu p_\mu - m^2) \psi^W = 0 \quad (35)$$

also holds. In parallel, utilizing (33), (28) can be further expressed as

$$\int \bar{\psi} \psi \frac{m d^3 \mathbf{p}}{p_0} = \int \psi^\dagger \psi \frac{m^2 d^3 \mathbf{p}}{p_0^2} = \int (|\psi_1^W|^2 + |\psi_2^W|^2) \frac{m^2 d^3 \mathbf{p}}{p_0^2}. \quad (36)$$

3.2 Free particle, coordinate space

Fourier transformation transports elements of $\mathcal{H}_p^{4,+}$ to coordinate space:

$$\frac{1}{(2\pi)^2} \int e^{-ip_\mu x^\mu} \psi(p) \frac{m d^3 \mathbf{p}}{p_0} = \frac{1}{(2\pi)^2} \int e^{-ip_0 x_0} e^{ip_j x_j} \psi(p) \frac{m d^3 \mathbf{p}}{p_0} = \frac{1}{\sqrt{2\pi}} \psi(t, \mathbf{x}) \quad (37)$$

[$t \equiv x^0$]. The momentum space condition $\psi(p) \in N_+(p)$, which can also be expressed in the two other forms

$$\gamma^\mu p_\mu \psi(p) = m \psi(p), \quad (\alpha_j p_j + \beta m) \psi(p) = p_0 \psi(p), \quad (38)$$

is transformed to

$$\gamma^\mu (i\partial_\mu) \psi(x) = m \psi(x), \quad i\partial_t \psi(t, \mathbf{x}) = [\alpha_j (-i\partial_j) + \beta m] \psi(t, \mathbf{x}). \quad (39)$$

Because of the unitary three-Fourier transform inside (37), the coordinate space integral

$$\int (\psi^\dagger \psi)(t, \mathbf{x}) d^3 \mathbf{x} \quad (40)$$

is time independent along the solutions of (39), and corresponds to the squared norm of ψ in the momentum space Hilbert space $\mathcal{H}_p^{4,+}$. [This time independence can also be seen from the conserved probability current to be introduced in (42).] The coordinate space scalar product related to (40),

$$\langle \psi_1, \psi_2 \rangle = \int \psi_1^\dagger \psi_2 d^3 \mathbf{x}, \quad (41)$$

is also time independent along solutions. Accordingly, spatially integrable solutions of (39), endowed with (41), form the Hilbert space $\mathcal{H}_x^{4,+}$ of the free spin-1/2 particle. $\mathcal{H}_x^{4,+}$ is the coordinate space equivalent of the momentum space Hilbert space $\mathcal{H}_p^{4,+}$.

Physical quantities of the particle can be realized on $\mathcal{H}_x^{4,+}$. One interesting example is that of position [10].

For solutions of (39), the four-current

$$j^\mu = \bar{\psi} \gamma^\mu \psi \quad (42)$$

is conserved, that is, its four-divergence $\partial_\mu j^\mu$ is zero. The timelike component,

$$j^0 = \bar{\psi} \gamma^0 \psi = \bar{\psi} \beta \psi = \psi^\dagger \psi, \quad (43)$$

is positive definite and is the probability density in the integral (40), i.e., j^μ is the conserved probability current.

Actually, (39) is the Euler–Lagrange equation stemming from the Lagrangian

$$\mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu - m] \psi, \quad (44)$$

and j is the conserved Noether current derivable from \mathcal{L} corresponding to the global gauge transformations $\psi \mapsto e^{i\chi} \psi$.

Solutions of (39), i.e., of

$$(i\gamma^\mu \partial_\mu - m) \psi = 0, \quad (45)$$

satisfy the free Klein–Gordon equation componentwise,

$$(\partial^\mu \partial_\mu - m^2) \psi = 0, \quad (46)$$

as follows by acting on (45) by $i\gamma^\mu \partial_\mu + m$ from the left – as well as from (25).

Should one start with (46), selecting the ‘positive’ solutions [solutions of (39)] is not straightforward – instead, it is the momentum space equivalent (25) that is advantageous for this purpose.

Transforming (23) to coordinate space leads to

$$i\partial_0 \psi = \sqrt{-\Delta + m^2} \psi, \quad (47)$$

where the square root of the positive operator is well-defined. Nevertheless, (47) is technically inconvenient, and is not suitable for generalization of this free particle theory to nonfree ones.

3.3 Particle in external field

The generalization to nonfree cases, more specifically, to motion under the action of a four-potential external field A , can be done via minimal coupling in (39), that is, via the substitution

$$\partial_\mu \rightsquigarrow \partial_\mu + ieA_\mu, \quad (48)$$

yielding

$$\gamma^\mu (i\partial_\mu - eA_\mu) \psi = m\psi, \quad i\partial_0\psi = [\alpha_j (-i\partial_j - eA_j) + \beta m + eA_0] \psi. \quad (49)$$

The four-current $j^\mu = \bar{\psi}\gamma^\mu\psi$ remains conserved under the more general equation (49), and its positive definite timelike component ensures a Hilbert space structure for the solutions of (49) like in the free case. Similarly, (49) is the Euler–Lagrange equation corresponding to the Lagrangian

$$\mathcal{L} = \bar{\psi} [\gamma^\mu (i\partial_\mu - eA_\mu) - m] \psi, \quad (50)$$

and j is still the conserved Noether current corresponding to global gauge transformations.

On the other side, the minimally coupled Klein–Gordon equation is not satisfied,

$$[(\partial^\mu + ieA^\mu) (\partial_\mu + ieA_\mu) - m^2] \psi \neq 0. \quad (51)$$

In fact, acting on $[\gamma^\nu (i\partial_\nu - eA_\nu) - m] \psi = 0$ by $[\gamma^\mu (i\partial_\mu - eA_\mu) + m]$ gives a second order equation that differs from the minimally coupled Klein–Gordon one by

$$\begin{aligned} & [\gamma^\mu (i\partial_\mu - eA_\mu) + m] [\gamma^\nu (i\partial_\nu - eA_\nu) - m] \psi \\ & - g^{\mu\nu} [(\partial_\mu + ieA_\mu) (\partial_\nu + ieA_\nu) - m^2] \psi \\ & = ie (g^{\mu\nu} I_4 - \gamma^\mu \gamma^\nu) (\partial_\mu A_\nu) \psi \neq 0. \end{aligned} \quad (52)$$

4 Spin 0 quantum mechanics

The antisymmetric spinor tensor formulation of the spin 0 case can be done in close analogy to the spin-1/2 situation. The analogy is actually so strong that it is enough to compare the two cases in form of a table that lists the steps in brief form.

4.1 Free particle, momentum space

SPIN 1/2	SPIN 0
\mathbb{C}^4 valued ψ	$\mathbb{C}^4 \wedge \mathbb{C}^4$ valued ζ (antisymmetric 4×4 complex matrix valued)
$\gamma^\mu p_\mu \psi = m\psi$	$\gamma^\mu p_\mu \zeta = m\zeta$
$\psi(p) \in N_+(p)$	$\zeta(p) \in N_+(p) \wedge N_+(p)$
$\psi(\vec{p}) = \begin{pmatrix} \psi_1(\vec{p}) \\ \psi_2(\vec{p}) \\ 0 \\ 0 \end{pmatrix}$	$\zeta(\vec{p}) = \begin{pmatrix} 0 & \zeta_{12}(\vec{p}) & 0 & 0 \\ -\zeta_{12}(\vec{p}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
2 degrees of freedom: $(\gamma^\mu p_\mu - m)(c_1 n_1 + \dots + c_4 n_4) = 0$ \implies only c_1, c_2 can be nonzero	$2 \wedge 2 = 1$ degree of freedom: $(\gamma^\mu p_\mu - m)(c_{12} n_1 \wedge n_2 + c_{13} n_1 \wedge n_3 + \dots + c_{34} n_3 \wedge n_4) = 0 \implies$ only c_{12} can be nonzero
$(U_{(a,L)}\psi)(p) = e^{ip_\mu a^\mu} D_L \psi(L^{-1}p)$	$(U_{(a,L)}\zeta)(p) = e^{ip_\mu a^\mu} (D_L \otimes D_L) \zeta(L^{-1}p)$ $= e^{ip_\mu a^\mu} D_L \zeta(p) D_L^T$
irreducible unitary ray representation of the Poincaré group (free elementary object)	irreducible unitary ray representation of the Poincaré group (free elementary object)
$\psi^W = W\psi = \begin{pmatrix} \psi_1^W \\ \psi_2^W \\ 0 \\ 0 \end{pmatrix}$	$\zeta^W = W\zeta W^T = \begin{pmatrix} 0 & \zeta_{12}^W & 0 & 0 \\ -\zeta_{12}^W & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$p_0 \psi = \sqrt{p_j p_j + m^2} \psi$	$p_0 \zeta = \sqrt{p_j p_j + m^2} \zeta$
$(p^\mu p_\mu - m^2) \psi = 0$	$(p^\mu p_\mu - m^2) \zeta = 0$
$(p^\mu p_\mu - m^2) \psi^W = 0$	$(p^\mu p_\mu - m^2) \zeta^W = 0$
$\int \bar{\psi} \psi \frac{m d^3 \mathbf{p}}{p_0} = \int \psi^\dagger \psi \frac{m^2 d^3 \mathbf{p}}{p_0^2}$ $= \int (\psi_1^W ^2 + \psi_2^W ^2) \frac{m^2 d^3 \mathbf{p}}{p_0^2}$	$\int \frac{1}{2} \text{Tr}(\bar{\zeta} \zeta) \frac{m d^3 \mathbf{p}}{p_0} = \int \frac{1}{2} \text{Tr}(\zeta^\dagger \zeta) \frac{m^3 d^3 \mathbf{p}}{p_0^3}$ $= \int \zeta_{12}^W ^2 \frac{m^3 d^3 \mathbf{p}}{p_0^3}$
$\mathcal{H}_p^{4,+}$	$\mathcal{H}_p^{4 \wedge 4,+}$

4.2 Free particle, coordinate space

SPIN 1/2	SPIN 0
$\gamma^\mu (i\partial_\mu) \psi = m\psi$	$\gamma^\mu (i\partial_\mu) \zeta = m\zeta$
$i\partial_t \psi = [\alpha_j (-i\partial_j) + \beta m] \psi$	$i\partial_t \zeta = [\alpha_j (-i\partial_j) + \beta m] \zeta$
$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi$	$\mathcal{L} = \frac{1}{2} \text{Tr} [\bar{\zeta} (i\gamma^\mu \partial_\mu - m) \zeta]$
$j^\mu = \bar{\psi} \gamma^\mu \psi$ conserved Noether current	$j^\mu = \frac{1}{2} \text{Tr} (\bar{\zeta} \gamma^\mu \zeta)$ conserved Noether current
$j^0 = \psi^\dagger \psi = \psi_1^W ^2 + \psi_2^W ^2 \geq 0$	$j^0 = \frac{1}{2} \text{Tr} (\zeta^\dagger \zeta) = \zeta_{12}^W ^2 \geq 0$
$\langle \psi_1, \psi_2 \rangle = \int \psi_1^\dagger \psi_2 d^3\mathbf{x}$	$\langle \zeta_1, \zeta_2 \rangle = \int \frac{1}{2} \text{Tr} (\zeta_1^\dagger \zeta_2) d^3\mathbf{x}$
$\mathcal{H}_x^{A,+}$	$\mathcal{H}_x^{A \wedge A,+}$
$(\partial^\mu \partial_\mu - m^2) \psi = 0$	$(\partial^\mu \partial_\mu - m^2) \zeta = 0$
$(\partial^\mu \partial_\mu - m^2) \psi^W = 0$	$(\partial^\mu \partial_\mu - m^2) \zeta^W = 0$
$i\partial_0 \psi = \sqrt{-\Delta + m^2} \psi$	$i\partial_0 \zeta = \sqrt{-\Delta + m^2} \zeta$
$i\partial_0 \psi^W = \sqrt{-\Delta + m^2} \psi^W$	$i\partial_0 \zeta^W = \sqrt{-\Delta + m^2} \zeta^W$
$i\partial_0 \psi_1^W = \sqrt{-\Delta + m^2} \psi_1^W$	$i\partial_0 \zeta_{12}^W = \sqrt{-\Delta + m^2} \zeta_{12}^W$
$i\partial_0 \psi_2^W = \sqrt{-\Delta + m^2} \psi_2^W$	equivalence to scalar Klein–Gordon theory via $\zeta_{12}^W \equiv \phi$

4.3 Particle in external field

SPIN 1/2	SPIN 0
$\gamma^\mu (i\partial_\mu - eA_\mu) \psi = m\psi$	$\gamma^\mu (i\partial_\mu - eA_\mu) \zeta = m\zeta$
$i\partial_0 \psi = [\alpha_j (\frac{1}{i}\partial_j - eA_j) + \beta m + eA_0] \psi$	$i\partial_0 \zeta = [\alpha_j (\frac{1}{i}\partial_j - eA_j) + \beta m + eA_0] \zeta$
$\mathcal{L} = \bar{\psi} [\gamma^\mu (i\partial_\mu - eA_\mu) - m] \psi$	$\mathcal{L} = \frac{1}{2} \text{Tr} \{ \bar{\zeta} [\gamma^\mu (i\partial_\mu - eA_\mu) - m] \zeta \}$
$j^\mu = \bar{\psi} \gamma^\mu \psi$ conserved Noether current	$j^\mu = \frac{1}{2} \text{Tr} (\bar{\zeta} \gamma^\mu \zeta)$ conserved Noether current
$j^0 = \psi^\dagger \psi \geq 0$	$j^0 = \frac{1}{2} \text{Tr} (\zeta^\dagger \zeta) \geq 0$
$\langle \psi_1, \psi_2 \rangle = \int \psi_1^\dagger \psi_2 d^3\mathbf{x}$	$\langle \zeta_1, \zeta_2 \rangle = \int \frac{1}{2} \text{Tr} (\zeta_1^\dagger \zeta_2) d^3\mathbf{x}$
$[(\partial^\mu + ieA^\mu) (\partial_\mu + ieA_\mu) - m^2] \psi \neq 0$ [see (52)]	$[(\partial^\mu + ieA^\mu) (\partial_\mu + ieA_\mu) - m^2] \zeta \neq 0$ [see (52), with replacement $\psi \rightsquigarrow \zeta$]

5 Outlook – a list of tasks for the future

Having established the relationship between the spinor tensor formalism and the Klein–Gordon scalar one in the free case, some further investigation between the two could give some further insight. For example, the Klein–Gordon conserved current could be expressed in terms of the spinor tensor formalism, while the spinor tensor conserved current could be transformed to the Klein–Gordon scalar language.

The relationship between the known Klein–Gordon scalar version of the Foldy–Wouthuysen transformation [11] (formulated in terms of the Feshbach–Villars form) and the one shown here (adaptation of the spin-1/2 version) would be interesting to investigate.

One of the most exciting questions is the spectrum of the Coulomb problem in the spinor tensor theory.

Second quantization of the free case would be important and seems straightforward.

Concerning interacting quantum field theories in general, we expect some new interaction and self-interaction terms (recall that the Foldy–Wouthuysen relationship between the spinor tensor and the Klein–Gordon scalar is ‘nonlocal’ in the free case, and even this type of equivalence is broken for nonfree situations). Renormalization properties of the spinor tensor field might be better than what superficial power counting says because of possible cancellations due to antisymmetry of the field.

As one concrete example, the spinor tensor QED may behave differently from the Klein–Gordon based ‘scalar QED’.

Another interesting idea is to realize interaction terms through which the Higgs particle can give mass to fermions, and to explore Higgs self-interaction potentials.

As another direction of outlook, similarly to the version of zero-spin quantum mechanics presented here, the treatment of higher spin particles is also worth revisiting. For example, the spin-1 particle is known to admit a symmetric spinor tensor description (see a historical overview in [4]). Issues like ‘the wave function of the photon’ and ‘position operator of the photon’ as well as possible quantum field theoretical benefits motivate such a line of investigation.

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