RADIATION REACTION: CHARGE DISTRIBUTIONS OR POINT CHARGES?

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Abstract. A point charge is frequently approximated by charge distributions deriving the Lorentz-Abraham-Dirac (LAD) equation. Here it is shown that the field of a uniformly charged rigid spherical shell cannot be substituted by the field of a central effective point charge. The related calculations are short and transparent due to a reference frame free representation of spacetime and the mathematical theory of generalised functions.

1. Introduction

The Lorentz-Abraham-Dirac(LAD) equation, [1, 2, 3, 4], is the simplest possible model of field-matter interaction, coupling Maxwell equations to Newton equation, electrodynamics to mechanics and mechanics to electrodynamics. LAD equation, however, is a wrong model of any phenomena, because of unacceptable mathematical and physical properties. There are numerous attempts to remedy the related problems, with various and different strategies to remove the mathematical and physical inconsistencies. These strategies are the following:

- Nonlocal modifications of the electromagnetic part, the Maxwell equations, [5, 6, 7].
- Dissipative modifications of the mechanical part, the Newton equation [8, 9, 10, 11].
- A suitable interpretation of the LAD equation, e.g. excluding particular solutions, [4, 12, 13].
- Application of continuum charge distributions instead of point charges. This is the method of the classical papers of Lorentz, Abraham and Dirac, too, [1, 2, 3, 4]. There are two main aspects of this strategy:
  - One may improve the classical theory, with the identification and elimination of the mathematical problems, [14, 15],
  - One may modify the point charge model with the help of quantum mechanics, or with various renormalization procedures, [16, 17, 18, 19, 20, 21].

The related literature is vast, the interested reader may look into the monographs of the field, [22, 23]. Worth mentioning, that recent experiments of radiation reaction related phenomena, see, e.g. [24, 25, 26], open the way toward the verification of the mentioned theories and also toward less rigorous, but applicable approaches, [27, 28, 29].

The mathematical side of the problem is that the Maxwell equations are singular along the worldline of a point particle. A related issue is the infinite electromagnetic energy of a point charge or a line current. That is why continuum charge distributions are promising, where one derives the corresponding point charge equation of motion by a limiting procedure [30, 31]. Here the simplest possible model is probably a rigid, uniformly charged spherical shell. However, it is not simple, various pitfalls of the related calculations were discovered step by step, [4, 14].

The theory of generalised functions, [32], is the mathematical tool to deal with simple singularities. It is easy to show, that electromagnetic energies of a point charge or a line current are not singular at all if they are represented by distributions [33]. Moreover, in this way we can obtain the radiation reaction force without approximations, limiting procedures and subtraction of infinite quantities [34].

Recently Bild, Decker and Ruhl [15] analysed the problem of the LAD equation with the help of a continuous charge distribution. They pointed to that the equation of motion for a point charge obtained from truncated series requires the proof of the convergence. Moreover, they have observed that the definition of the world tube and the Gauss-Ostrogradskii theorem must be correctly applied, when contrasted to the derivation of Dirac [15]. They have obtained a delayed equation of motion, with promising properties.

There are two problems, however, with their analysis. First of all, their definition of the world tube breaks down for high accelerations. In an actual case, it is hard to guarantee that the acceleration is less than a given value. The second, more problematic condition in their derivation is the assumption that outside a uniformly charged spherical shell the field is identical with a field of a point charge in the centre. This condition is valid for inertial charged spheres but for accelerating ones it is not evident because the different points of the spherical shell are not equivalent: their accelerations, as well as the corresponding retarded proper times, are different. In this paper, we show that, indeed, the field of a uniformly accelerated spherical shell with uniform charge distribution does not equal the field of a point charge at the centre of the sphere. This is a strong argument that the problematic aspects of the LAD equations may not require any limiting procedure or modification, but rather a proper interpretation like in [13].

Our calculation is based on distribution theory, and a reference frame free model of special relativistic spacetime ensures the precise mathematical treatment. That shortens the analysis of [15] considerably. The necessary mathematical concepts are collected in the Appendix.

2. About our formalism

The mathematical formulae in the article [15], though being correct, are extremely large and it is quite impossible to survey them. A more condensed discussion is possible using a coordinate and reference-free treatment of spacetime which can be found in the book [35].

A brief summary of that (with the speed of the light $c = 1$):

Spacetime points and spacetime vectors (which are often confused in coordinates) are distinguished:

- M, mathematically a four dimensional affine space, is the set of *spacetime points*,
- $\mathbf{M}$, mathematically a four dimensional vector space, is the set of *spacetime vectors*,
- the set of *time periods* is $\mathbb{T}$, a one dimensional oriented vector space,
- the exact treatment requires the tensorial quotients of $\mathbf{M}$ by $\mathbb{T}$, here we do not refer to them explicitely,

- the Lorentz product of the spacetime vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\boldsymbol{x} \cdot \boldsymbol{y}$ (which is $x_k y^k$ in coordinates); $\boldsymbol{x}$ is timelike if $\boldsymbol{x} \cdot \boldsymbol{x} < 0$ (which means in coordinates the signature $(-1, 1, 1, 1)$ of the Lorentz form),
- an *absolute velocity* is a futurelike vector $\boldsymbol{u}$ for which $\boldsymbol{u} \cdot \boldsymbol{u} = -1$,
- for an absolute velocity $\boldsymbol{u}$, $\mathbf{S}_{\boldsymbol{u}} := \{\boldsymbol{x} \mid \boldsymbol{u} \cdot \boldsymbol{x} = 0\}$ is the set of $\boldsymbol{u}$-*spacelike vectors*, a three dimensional Euclidean subspace of $\mathbf{M}$
- the action of linear and bilinear maps is denoted by a dot, too; e.g the action of a linear map $\boldsymbol{L}$ on a vector $\boldsymbol{x}$ is $\boldsymbol{L} \cdot \boldsymbol{x}$ (which is $L^i{}_k x^k$ in coordinates),
- the adjoint of a linear map $\boldsymbol{L}$ is the linear map $\boldsymbol{L}^*$ defined by $(\boldsymbol{L}^* \cdot \boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{L} \cdot \boldsymbol{y}$ for all vectors $\boldsymbol{x}, \boldsymbol{y}$; shortly, $\boldsymbol{L}^* \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \boldsymbol{L}$; similarly, $\boldsymbol{y} \cdot \boldsymbol{L}^* = \boldsymbol{L} \cdot \boldsymbol{y}$,
- $\boldsymbol{L}$ is a *Lorentz transformation* if and only if $\boldsymbol{L}^* = \boldsymbol{L}^{-1}$,
- $\mathbf{1}$ denotes the identity map of vectors (which is $g^i{}_k = \delta^i{}_k$ in coordinates).

Some other required consequent mathematical background is given in the Appendix.

3. Uniformly accelerated sphere – a world tube

Let us consider the rigid uniformly accelerated observer treated in subsection II.6.4 of [35]. Its space points are uniformly accelerated world lines.

Pick up such a world line, as the *center of the sphere* being the range of the world line function,

$$r_c(s_c) = x_c + \boldsymbol{u}_c \frac{\sinh(a_c s_c)}{a_c} + \boldsymbol{n}_c \frac{\cosh(a_c s_c) - 1}{a_c}.$$

where

- x_c is an arbitrarily chosen spacetime point of the world line,
- s_c is the proper time of the world line,
- a_c is the magnitude of the acceleration of the world line,
- $\boldsymbol{u}_c$ is an absolute velocity and $\boldsymbol{n}_c$ is an $\boldsymbol{u}_c$-spacelike unit vector, that is $\boldsymbol{u}_c \cdot \boldsymbol{n}_c = 0$ and $\boldsymbol{n}_c \cdot \boldsymbol{n}_c = 1$

The corresponding absolute velocity function is

$$\dot{r}_c(s_c) = \boldsymbol{u}_c \cosh(a_c s_c) + \boldsymbol{n}_c \sinh(a_c s_c). \tag{1}$$

Further, we take other space points (world lines in spacetime) of the observer in question in order to represent a *uniformly accelerated sphere* as follows.

Let us take the set of $\boldsymbol{u}_c$-spacelike unit vectors,

$$S_c(1) := \{\boldsymbol{n} \mid \boldsymbol{u}_c \cdot \boldsymbol{n} = 0,\ \ \boldsymbol{n} \cdot \boldsymbol{n} = 1\}; \tag{2}$$

for all of its elements and for an $\epsilon > 0$ we consider the uniformly accelerated world line function

$$r_{\epsilon\boldsymbol{n}}(s_{\epsilon\boldsymbol{n}}) = x_c + \epsilon\boldsymbol{n} + \boldsymbol{u}_c \frac{\sinh(a_{\epsilon\boldsymbol{n}} s_{\epsilon\boldsymbol{n}})}{a_{\epsilon\boldsymbol{n}}} + \boldsymbol{n}_c \frac{\cosh(a_{\epsilon\boldsymbol{n}} s_{\epsilon\boldsymbol{n}}) - 1}{a_{\epsilon\boldsymbol{n}}}.$$

for which

$$\dot{r}_{\epsilon\boldsymbol{n}}(s_{\epsilon\boldsymbol{n}}) = \boldsymbol{u}_c \cosh(a_{\epsilon\boldsymbol{n}} s_{\epsilon\boldsymbol{n}}) + \boldsymbol{n}_c \sinh(a_{\epsilon\boldsymbol{n}} s_{\epsilon\boldsymbol{n}}) \tag{3}$$

where, of course, $\dot{}$ denotes the differentiation by the proper time $s_{\epsilon\boldsymbol{n}}$.

The observer applies the synchronization according to which the proper times of r_c and $r_{\epsilon\boldsymbol{n}}$ are simultaneous if and only if

$$\dot{r}_c(s_c) = \dot{r}_{\epsilon\boldsymbol{n}}(s_{\epsilon\boldsymbol{n}}) \qquad \text{(simultaneity condition)} \tag{4}$$

holds from which it follows that

$$a_c s_c = a_{\epsilon\boldsymbol{n}} s_{\epsilon\boldsymbol{n}}.$$

Then let us take the Lorentz boost (see (24)) from $\boldsymbol{u}_c$ to $\dot{r}_c(s_c)$,

$$\boldsymbol{L}(s_c) := \mathbf{1} + \frac{(\dot{r}_c(s_c) + \boldsymbol{u}_c) \otimes (\dot{r}_c(s_c) + \boldsymbol{u}_c)}{1 - \dot{r}_c(s_c) \cdot \boldsymbol{u}_c} - 2\dot{r}_c(s_c) \otimes \boldsymbol{u}_c. \tag{5}$$

Using

$$\dot{r}_c(s_c) + \boldsymbol{u}_c = \boldsymbol{u}_c\big(\cosh(a_c s_c) + 1\big) \quad \text{and} \quad 1 - \dot{r}_c(a_c s_c) = 1 + \cosh(a_c s_c), \tag{6}$$

a short calculation results in

$$L(s_c) \cdot \boldsymbol{n} = \boldsymbol{n} + (\boldsymbol{n}_c \cdot \boldsymbol{n})\big(\boldsymbol{u}_c \sinh(a_c s_c) + \boldsymbol{n}_c(\cosh(a_c s_c - 1)\big) \tag{7}$$

and

$$\dot{\boldsymbol{L}}(s_c) \cdot \boldsymbol{n} = (\boldsymbol{n}_c \cdot \boldsymbol{n}) a_c \big(\boldsymbol{u}_c \cosh(a_c s_c) + \boldsymbol{n}_c \sinh(a_c s_c)\big). \tag{8}$$

All these imply that the range of

$$r_c(s_c) + \epsilon \boldsymbol{L}(s_c) \cdot \boldsymbol{n} = x_c + \epsilon \boldsymbol{n} + \big(\boldsymbol{u}_c \sinh(a_c s_c) + \boldsymbol{n}_c(\cosh(a_c s_c) - 1)\big) \left(\frac{1}{a_c} + \epsilon \boldsymbol{n}_c \cdot \boldsymbol{n}\right) \tag{9}$$

equals the range of $r_{\epsilon\boldsymbol{n}}$; therefore we have

$$a_{\epsilon\boldsymbol{n}} = \frac{a_c}{1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})}, \quad s_c = \frac{s_{\epsilon\boldsymbol{n}}}{1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})}.$$

Moreover, we obtain

$$\dot{r}_c(s_c) + \epsilon \dot{\boldsymbol{L}}(s_c) \cdot \boldsymbol{n} = (1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})) \dot{r}_{\epsilon\boldsymbol{n}} \left((1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n}) s_c)\right). \tag{10}$$

The sphere of radius ϵ with centre $\mathrm{Ran}(r_c)$ in spacetime is the subset

$$T_\epsilon := \bigcup_{\boldsymbol{n} \in S_c(1)} \mathrm{Ran}(r_{\epsilon\boldsymbol{n}}),$$

which is a world tube of type treated in [15]: the distance between $\mathrm{Ran}(r_{\epsilon\boldsymbol{n}})$ and $\mathrm{Ran}(r_c)$ equals ϵ at every synchronization instant.

4. Lebesgue measure on the world tube

For the notions and formulae appearing in this section, we refer to the subsections 8.3 and 8.4 of the Appendix.

The world tube T_ϵ is a three dimenional submanifold in spacetime, its very definition gives the map

$$p : \mathbb{T} \times S_c(1) \to \mathrm{M}, \quad (s_c, \boldsymbol{n}) \mapsto r_c(s_c) + \epsilon \boldsymbol{L}(s_c) \cdot \boldsymbol{n}; \tag{11}$$

putting here $\boldsymbol{n} := p_c(\vartheta, \varphi)$, we get a parametrization of T_ϵ; accordingly, (11) is called a generalised parametrization and its use admits a concise formulation in the following. The derivative of p is

$$\mathrm{D}p[s_c, \boldsymbol{n}] = \big(\partial_{s_c} p(s_c \boldsymbol{n}) \quad \partial_{\boldsymbol{n}} p(s_c, \boldsymbol{n})\big) = \big(\dot{r}_c(s_c) + \epsilon \dot{\boldsymbol{L}}(s_c) \cdot \boldsymbol{n} \quad \epsilon \boldsymbol{L}(s_c)|_{\mathbf{E}_{\mathbf{u}_c \boldsymbol{n}}}\big) \tag{12}$$

where the last symbol denotes the restriction of the Lorentz boost to the linear subspace in question, $\mathbf{E}_{\mathbf{u}_c \boldsymbol{n}}$.

For the sake of brevity, using the notation

$$\boldsymbol{z}(s_c, \boldsymbol{n}) := \dot{r}_c(s_c) + \epsilon \dot{\boldsymbol{L}}(s_c) \cdot \boldsymbol{n} \tag{13}$$

and then omitting the variables and subscripts, we have

$$(\mathrm{D}p)^* \cdot \mathrm{D}p = \begin{pmatrix} \boldsymbol{z} \\ \epsilon\,(\boldsymbol{L}|_{\mathbf{E}})^* \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{z} & \epsilon \boldsymbol{L}|_{\mathbf{E}} \end{pmatrix} \qquad = \begin{pmatrix} \boldsymbol{z}\cdot\boldsymbol{z} & \boldsymbol{z}\cdot\epsilon\boldsymbol{L}|_{\mathbf{E}} \\ \epsilon\,(\boldsymbol{L}|_{\mathbf{E}})^*\cdot\boldsymbol{z} & \epsilon^2 \mathbf{1}_{\mathbf{E}} \end{pmatrix}. \tag{14}$$

Note that the adjoint of $\boldsymbol{L}|_{\mathbf{E}} : \mathbf{E} \to \mathbf{M}$ is the linear map $(\boldsymbol{L}|_{\mathbf{E}})^* : \mathbf{M} \to \mathbf{E}$ defined by $((\boldsymbol{L}|_{\mathbf{E}})^* \cdot \boldsymbol{x}) \cdot \boldsymbol{q} = \boldsymbol{x} \cdot \boldsymbol{L}|_{\mathbf{E}} \cdot \boldsymbol{q} = \boldsymbol{x} \cdot \boldsymbol{L} \cdot \boldsymbol{q}$ for all $\boldsymbol{x} \in \mathbf{M}$, $\boldsymbol{q} \in \mathbf{E}$.

Thus, recalling the projection $\boldsymbol{P}$ onto $\mathbf{E}$, for all vectors $\boldsymbol{y}$ we have,

$$((\boldsymbol{L}|_{\mathbf{E}})^* \cdot \boldsymbol{z}) \cdot (\boldsymbol{P} \cdot \boldsymbol{y}) = \boldsymbol{z} \cdot \boldsymbol{L} \cdot (\boldsymbol{P} \cdot \boldsymbol{y}) = (\boldsymbol{L}^* \cdot \boldsymbol{z}) \cdot (\boldsymbol{P} \cdot \boldsymbol{y}) = (\boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z}) \cdot \boldsymbol{y}$$

which means that the block matrix form of $(\mathrm{D}p)^* \cdot \mathrm{D}p$ is symmetric: $\boldsymbol{z} \cdot \boldsymbol{L}|_{\mathbf{E}} = (\boldsymbol{L}|_{\mathbf{E}})^* \cdot \boldsymbol{z} = \boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z} \in \mathbf{E}$.

As a consequence, since $\mathbf{E}$ is two dimensional[1]

$$\det((\mathrm{D}p)^* \cdot \mathrm{D}p) = \epsilon^4 \boldsymbol{z} \cdot \boldsymbol{z} - \epsilon^4 (\boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z}) \cdot (\boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z}). \tag{15}$$

Here

$$\begin{aligned}(\boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z}) \cdot (\boldsymbol{P} \cdot \boldsymbol{L}^* \cdot \boldsymbol{z}) &= (\boldsymbol{L}^* \cdot \boldsymbol{z}) \cdot (\mathbf{1} + \boldsymbol{u}_c \otimes \boldsymbol{u}_c - \boldsymbol{n} \otimes \boldsymbol{n}) \cdot \boldsymbol{L}^* \cdot \boldsymbol{z} = \\ &= \boldsymbol{z} \cdot \boldsymbol{z} + (\boldsymbol{z} \cdot \boldsymbol{L} \cdot \boldsymbol{u}_c)^2 - (\boldsymbol{z} \cdot \boldsymbol{L} \cdot \boldsymbol{n})^2.\end{aligned}$$

Equalities $\dot r_c \cdot \boldsymbol{L} \cdot \boldsymbol{n} = 0$ and $(\dot{\boldsymbol{L}} \cdot \boldsymbol{n}) \cdot (\boldsymbol{L} \cdot \boldsymbol{n}) = 0$, together with the definition (13) imply that $\boldsymbol{z} \cdot \boldsymbol{L} \cdot \boldsymbol{n} = 0$.

Further, (1) and (8) imply $(\dot{\boldsymbol{L}} \cdot \boldsymbol{n}) \cdot \dot r_c = -a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})$, thus from (13) and the property $\boldsymbol{L} \cdot \boldsymbol{u}_c = \dot r_c$ of the Lorentz boost we have that $\boldsymbol{z} \cdot \boldsymbol{L} \cdot \boldsymbol{u}_c = -(1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n}))$.

Finally, according to (10), $\boldsymbol{z} \cdot \boldsymbol{z} = -\big(1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})\big)^2$.

Summing up, the second term of (15) is zero and

$$\sqrt{|\det \mathrm{D}p^*[s_c, \boldsymbol{n}] \cdot \mathrm{D}p[s_c, \boldsymbol{n}]|} = \epsilon^2 \big(1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})\big),$$

and the Lebesgue measure λ_{T_ϵ} on the tube is given by the integration formula

$$\int f\ d\lambda_{T_\epsilon} = \epsilon^2 \int_{\mathbb{T}} \int_{S_c(1)} f\big(r_c(s_c) + \epsilon L(s_c)\boldsymbol{n})\big)\big(1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})\big)\ d\boldsymbol{n}\ ds_c. \tag{16}$$

5. Uniform charge density on the sphere – world current

According to (4) and (8), the set of spacetime points simultaneous with $r_c(s_c)$ is the hyperplane (in fact only a convenient part of it) which contains $r_c(s_c)$ and is Lorentz-orthogonal to $\dot r_c(s_c)$. Thus, at every synchronization instant the world tube T_ϵ (i.e. the intersection of the tube and the instant hyperplane) is a sphere of radius ϵ. The world current of a uniform charge density σ on the sphere is the vector measure

$$\boldsymbol{j}_\epsilon := \sigma \boldsymbol{u}_\epsilon \lambda_{T_\epsilon}$$

where $\boldsymbol{u}_\epsilon$ is the function (a vector field) on the world tube which assigns the corresponding absolute velocities to the points, i.e. $\boldsymbol{u}_\epsilon(r_{\epsilon\boldsymbol{n}}) := \dot r_{\epsilon,\boldsymbol{n}}$, in other words, by (10),

$$\boldsymbol{u}_\epsilon\big(r_c(s_c) + \epsilon \boldsymbol{L}(s_c) \cdot \boldsymbol{n}\big) = \frac{\dot r_c(s_c) + \epsilon \dot{\boldsymbol{L}}(s_c) \cdot \boldsymbol{n}}{1 + \epsilon a_c(\boldsymbol{n}_c \cdot \boldsymbol{n})}. \tag{17}$$

[1]The matrix (14) in spherical coordinates has the form $\begin{pmatrix} \alpha & \epsilon\beta_1 & \epsilon\beta_2 \\ \epsilon\beta_1 & \epsilon^2 & 0 \\ \epsilon\beta_2 & 0 & \epsilon^2 \end{pmatrix}$

6. Electromagnetic potentials

For the notions and formulae appearing in this section, we refer to the subsection 8.5 of the Appendix.

From now on, for the sake of brevity, we write s and a instead of s_c and a_c.

Now we are ready to compare the electromagnetic potentials of a point charge and the one of a charged spherical shell. In electrostatics, they are equal outside the shell; however, we will show that they are different in case of uniform acceleration.

The electromagnetic potential produced by the uniformly accelarated charged sphere is the distribution $\boldsymbol{j}_\epsilon * \lambda_{\mathrm{L}^\rightarrow}$ which, according to (16) and (17), acts on a test function Φ as follows:

$$(\boldsymbol{j}_\epsilon * \lambda_{\mathrm{L}^\rightarrow} \mid \Phi) = \int\int \Phi(x+\mathbf{x})\ \sigma \boldsymbol{u}_\epsilon(x)\ d\lambda_{T_\epsilon}(x)\ d\lambda_{\mathrm{L}}^{\rightarrow}(\mathbf{x}) =$$
$$= \sigma\epsilon^2 \int_{\mathbb{T}}\int_{S_c(1)}\int_{\mathrm{L}^\rightarrow} \Phi(r_c(s)+\epsilon\boldsymbol{L}(s)\cdot\boldsymbol{n}+\mathbf{x})\big(\dot r_c(s)+\epsilon\dot{\boldsymbol{L}}(s)\cdot\boldsymbol{n}\big)\ d\boldsymbol{n}\ ds\ d\lambda_{\mathrm{L}^\rightarrow}(\mathbf{x}). \tag{18}$$

Let us consider now a point charge $\sigma 4\pi\epsilon^2$ existing on the centre of the world tube. The corresponding world current is

$$\boldsymbol{j}_c := \sigma 4\pi\epsilon^2 \dot r_c \lambda_{\mathrm{Ran}\, r_c}$$

producing the electromagnetic potential $\boldsymbol{j}_c * \lambda_{\mathrm{L}^\rightarrow}$ for which[2]

$$(\boldsymbol{j}_c * \lambda_{\mathrm{L}^\rightarrow} \mid \Phi) = \sigma 4\pi\epsilon^2 \int_{\mathrm{L}^\rightarrow}\int_{\mathbb{T}} \Phi(r_c(s)+\mathbf{x})\dot r_c(s)\ ds\ d\lambda_{\mathrm{L}^\rightarrow}(\mathbf{x}). \tag{19}$$

The electromagnetic potentials produced by the uniformly accelerated charged sphere and by the point charge would be equal outside the world tube if and only if the integrals (18) *and* (19) *were equal for all* Φ *having support outside the world tube.*

Taking such a Φ and rewriting (19) in the form

$$\sigma\epsilon^2 \int_{\mathrm{L}^\rightarrow}\int_{\mathbb{T}}\int_{S_c(1)} \Phi(r_c(s)+\mathbf{x})\dot r_c(s)\ d\boldsymbol{n}\ ds\ d\lambda_{\mathrm{L}^\rightarrow}(\mathbf{x}), \tag{20}$$

let us examine the difference of the integrals.

According to the equality

$$\Phi\big(r_c(s)+\epsilon\boldsymbol{L}(s)\cdot\boldsymbol{n}+\mathbf{x}\big) = \Phi\big(r_c(s)+\mathbf{x}\big)+$$
$$+ (\epsilon\boldsymbol{L}(s)\cdot\boldsymbol{n})\cdot \mathrm{D}\Phi[r_c(s)+\mathbf{x}] + (\epsilon\boldsymbol{L}(s)\cdot\boldsymbol{n})\cdot \mathrm{D}^2\Phi[r_c(s)+\mathbf{x}]\cdot(\epsilon\boldsymbol{L}(s)\cdot\boldsymbol{n}) + \mathrm{ordo}(\epsilon^2),$$

the difference of the integrands, omitting the variables for the sake of perspicuity, becomes

$$\Big(\Phi + (\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\cdot\mathrm{D}\Phi + (\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\cdot\mathrm{D}^2\Phi\cdot(\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\Big)\epsilon\dot{\boldsymbol{L}}\cdot\boldsymbol{n}+$$
$$\Big((\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\cdot\mathrm{D}\Phi + (\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\cdot\mathrm{D}^2\Phi\cdot(\epsilon\boldsymbol{L}\cdot\boldsymbol{n})\Big)\dot r_c + \mathrm{ordo}(\epsilon^2). \tag{21}$$

The integration by $\boldsymbol{n}$ (see (26)) yields zero for the terms linear and trilinear in $\boldsymbol{n}$. As concerns the bilinear terms, $\big((\boldsymbol{L}\cdot\boldsymbol{n})\cdot\mathrm{D}\Phi\big)\dot{\boldsymbol{L}}\cdot\boldsymbol{n} = \dot{\boldsymbol{L}}\cdot(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{L}^*\cdot\mathrm{D}\Phi$ and

[2]The known actual form of the potential is obtained by the substitution $x := r_c(s)+\mathbf{x}$ from which the retarded proper time $s_{\mathrm{ret}}(x)$ is determined in such a way that $x - r(s_{\mathrm{ret}}(x))$ be lightlike

$(\boldsymbol{L}\cdot\boldsymbol{n})\cdot \mathrm{D}^2\Phi\cdot(\boldsymbol{L}\cdot\boldsymbol{n}) = \boldsymbol{n}\cdot\big(\boldsymbol{L}^*\cdot \mathrm{D}\Phi\boldsymbol{L}\big)\boldsymbol{n}$; consequntly, the integration by $\boldsymbol{n}$ yields (recall that $\boldsymbol{L}^* = \boldsymbol{L}^{-1}$ and a cyclic permutation can be made under a trace)

$$\epsilon^2\frac{4\pi}{3}\Big(\dot{\boldsymbol{L}}\cdot(1+\boldsymbol{u}_c\otimes\boldsymbol{u}_c)\cdot\boldsymbol{L}^*\cdot \mathrm{D}\Phi + \mathrm{Tr}\left(\nabla^2_{\mathbf{u}_c}\Phi\right)\dot{r}_c\Big). \tag{22}$$

To find the properties of the first term, we proceed as follows. Since $\boldsymbol{u}_c\cdot\boldsymbol{L}^* = \boldsymbol{L}\cdot\boldsymbol{u}_c = \dot{r}_c$, we have

$$\dot{\boldsymbol{L}}\cdot(1+\boldsymbol{u}_c\otimes\boldsymbol{u}_c)\cdot\boldsymbol{L}^* = \dot{\boldsymbol{L}}\cdot\boldsymbol{L}^* + \ddot{r}_c\otimes\dot{r}_c. \tag{23}$$

$\dot{\boldsymbol{L}}\cdot\boldsymbol{L}^*$ is evidently the linear combination of $\boldsymbol{u}_c\otimes\boldsymbol{u}_c$, $\boldsymbol{u}_c\otimes\boldsymbol{n}_c$, $\boldsymbol{n}_c\otimes\boldsymbol{u}_c$ and $\boldsymbol{n}_c\otimes\boldsymbol{n}_c$; the coefficients are obtained by $\boldsymbol{u}_c\cdot\dot{\boldsymbol{L}}\cdot\boldsymbol{L}^*\cdot\boldsymbol{u}_c = (\boldsymbol{u}_c\cdot\dot{\boldsymbol{L}})\cdot(\boldsymbol{u}_c\cdot\boldsymbol{L})$ etc.

A simple calculation based on (5) and (6) yields

$$\boldsymbol{u}_c\cdot\boldsymbol{L}(s) = \boldsymbol{u}_c\cosh(as) - \boldsymbol{n}_c\sinh(as) \qquad \boldsymbol{u}_c\cdot\dot{\boldsymbol{L}}(s) = a\big(\boldsymbol{u}_c\sinh(as) - \boldsymbol{n}_c\cosh(as)\big),$$

$$\boldsymbol{n}_c\cdot\boldsymbol{L}(s) = -\boldsymbol{u}_c\sinh(as) + \boldsymbol{n}_c\cosh(as) \qquad \boldsymbol{n}_c\cdot\dot{\boldsymbol{L}}(s) = a\big(-\boldsymbol{u}_c\cosh(as) + \boldsymbol{n}_c\sinh(as)\big)$$

from which it follows that $\dot{\boldsymbol{L}}\cdot\boldsymbol{L}^* = a(\boldsymbol{u}_c\otimes\boldsymbol{n}_c - \boldsymbol{n}_c\otimes\boldsymbol{u}_c)$. At last,

$$a(\boldsymbol{u}_c\otimes\boldsymbol{n}_c - \boldsymbol{n}_c\otimes\boldsymbol{u}_c) + a(\boldsymbol{u}_c\sinh + \boldsymbol{n}_c\cosh)\otimes(\boldsymbol{u}_c\cosh + \boldsymbol{n}_c\sinh) =$$
$$a(\boldsymbol{u}_c\cosh + \boldsymbol{n}_c\sinh)\otimes(\boldsymbol{u}_c\sinh + \boldsymbol{n}_c\cosh) = \dot{r}_c\otimes\ddot{r}_c$$

and we have that the first term in (22) is

$$\Big(\ddot{r}_c(s)\cdot \mathrm{D}\Phi[r_c(s)+\mathbf{x}]\Big)\dot{r}_c(s).$$

As a consequence, we can state that the two electromagnetic potentials would be equal if and only if

$$\int_{\mathbb{T}}\int_{\mathrm{L}^{\rightarrow}}\Big(\ddot{r}_c(s)\cdot \mathrm{D}\Phi[r_c(s)+\mathbf{x}] + \mathrm{Tr}\left(\nabla^2_{\mathbf{u}_c}\Phi[r_c(s)+\mathbf{x}]\right)\Big)\dot{r}_c(s)\; ds\; d\lambda_{\mathrm{L}}^{\rightarrow}(\mathbf{x})$$

were zero for all Φ having support outside the world tube. The integrand has timelike values or zero, it is evident then that the integral cannot be zero for all Φ in question.

As a consequence, *the electromagnetic potentials produced by the uniformly accelarated charged sphere and by the point charge at the centre are not equal outside the world tube.*

7. Conclusions

The physical and mathematical paradoxes of LAD equation challenge the foundations of physics since more than a century. It is widely believed that the problem is not physical and the instabilities are due to the oversimplified mathematical model of point charge. Therefore a continuum theory, a charge distribution is considered a better model that eventually can remove the paradoxes.

The simplest possible continuum model is the uniformly charged rigid spherical shell, a model that is continuously kept analysed and improved since the seminal work of Lorentz [1]. The latest elegant and profound analysis of Bild, Deckert and Ruhl reveals further problems and improves the previous calculations. Then they obtain a delayed equation with promising properties. They assume, however, that outside the accelerating charged medium the electromagnetic field is the same as the field of a point charge in the centre of the sphere. Here we have shown that this assumption is wrong. Our mathematical tools ensured that the result is exact and also transparent due to the short, simple calculations.

The assumption of uniform charge distribution on the spherical shell seems to be too strong. Perhaps weaker assumptions such as non-uniform charge distribution or non-spherical shell can work, but we do not think so. Anyway, according to our analysis, a continuum charge distribution requires delicate care.

8. Appendix

8.1. **Tensor products.** 1. The tensor product $\boldsymbol{a}\otimes\boldsymbol{b}$ of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be considered either a linear or a bilinear map; its action on the vector $\boldsymbol{x}$ or the vectors $\boldsymbol{y}$ and $\boldsymbol{x}$ is $(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{x} := \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{x})$ or, $\boldsymbol{y}\cdot(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{x} := (\boldsymbol{y}\cdot\boldsymbol{a})\boldsymbol{b}\cdot\boldsymbol{x})$.

2. For an absolute velocity $\boldsymbol{u}$, $\mathbf{1}+\boldsymbol{u}\otimes\boldsymbol{u}$ is the projection onto the linear subspace $\mathbf{S}_{\boldsymbol{u}}$ of $\boldsymbol{u}$-spacelike vectors.

3. For two absolute velocities $\boldsymbol{u}$ and $\boldsymbol{u}'$, the Lorentz boost from $\boldsymbol{u}$ to $\boldsymbol{u}'$ is

$$\mathbf{1}+\frac{(\boldsymbol{u}'+\boldsymbol{u})\otimes(\boldsymbol{u}'+\boldsymbol{u})}{1-\boldsymbol{u}'\cdot\boldsymbol{u}}-2\boldsymbol{u}'\otimes\boldsymbol{u}, \tag{24}$$

which sends $\boldsymbol{u}$ to $\boldsymbol{u}'$ and maps the Euclidean subspace $\mathbf{S}_{\boldsymbol{u}}$ onto the Euclidean subspace $\mathbf{S}_{\boldsymbol{u}'}$ in a rotation-free way.

8.2. **Derivatives.** For more details, see section VI.3 of [35].

The symbol ordo means a function $\mathbb{R}^+\to\mathbb{R}$, defined in a neighbourhood of zero, such that $\lim_{\alpha\to 0}\frac{\mathrm{ordo}(\alpha)}{\alpha}=0$.

The derivative of a differentiable function $\Phi : \mathrm{M}\to\mathbb{R}$ at x is the linear map $\mathrm{D}\Phi[x] : \mathbf{M}\to\mathbb{R}$ for which

$$\Phi(x+\mathbf{x}) = \Phi(x)+\mathbf{x}\cdot\mathrm{D}\Phi[x]+\mathrm{ordo}(\|\mathbf{x}\|) \qquad (\mathbf{x}\in\mathbf{M})$$

holds for an arbitrary norm $\|\ \|$ on $\mathbf{M}$.

The second derivative of a twice differentiable function $\Phi : \mathrm{M}\to\mathbb{R}$ at x is the bilinear $\mathrm{D}^2\Phi[x]$ map for which

$$\Phi(x+\mathbf{x}) = \Phi(x)+\mathbf{x}\cdot\mathrm{D}\Phi[x]+\mathbf{x}\cdot\mathrm{D}^2\Phi[x]\cdot\mathbf{x}+\mathrm{ordo}(\|\mathbf{x}\|^2) \qquad (\mathbf{x}\in\mathbf{M})$$

holds.

Similar formulae are valid for functions defined in an arbitrary affine space and having values in M.

For an absolute velocity $\boldsymbol{u}$, the $\boldsymbol{u}$-spacelike derivative of Φ is the restriction of $\mathrm{D}\Phi[x]$ onto the linear subspace of $\boldsymbol{u}$-spacelike vectors,

$$\nabla_{\boldsymbol{u}}\mathrm{D}\Phi[x] := \mathrm{D}\Phi[x]|_{\mathbf{S}_{\boldsymbol{u}}} = (\mathbf{1}+\boldsymbol{u}\otimes\boldsymbol{u})\cdot\mathrm{D}\Phi[x].$$

8.3. **Submanifolds.** For more details, see section VI.4 of [35].

1. For $d\in\{0,1,2,3,4\}$ a submanifold H of dimension d in M ($\mathbf{M}$) is a subset for which there is a smooth map $p:\mathbb{R}^d\to\mathrm{M}$ ($\mathbf{M}$), called a parametrization, such that

- $\mathrm{Ran}(p)=H$,
- p is injective and p^{-1} is continuous,
- $\mathrm{D}p[\xi]:\mathbb{R}^d\to\mathbf{M}$ is injective for all ξ in the domain of p.

More generally, the domain of a parametrization can be a d dimensional affine space instead of $\mathbb{R}^d$.

The tangent space of H over $p(\xi)$ is the range of $\mathrm{D}p[\xi]$, a linear subspace of $\mathbf{M}$.

2. For an absolute velocity $\boldsymbol{u}_c$, the unit sphere $S_c(1)$ of $\boldsymbol{u}_c$-spacelike vectors is a two dimensional submanifold which can be parametrized by the usual angles ϑ and φ: taking two unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, orthogonal to each other and to $\boldsymbol{n}_c$,

$$p_c(\vartheta,\varphi) := \boldsymbol{u}_c + \boldsymbol{n}_c \cos\vartheta + \boldsymbol{a}\sin\vartheta\cos\varphi + \boldsymbol{b}\sin\vartheta\sin\varphi. \tag{25}$$

The tangent space of $S_c(1)$ over $\boldsymbol{n}$ is the linear subspace $\mathbf{E}_{\boldsymbol{u}_c\boldsymbol{n}}$ consisting of vectors Lorentz orthogonal to both $\boldsymbol{u}_c$ and $\boldsymbol{n}$; it is the range of the linear projection $\boldsymbol{P}_{\boldsymbol{u}_c\boldsymbol{n}} := (\mathbf{1} + \boldsymbol{u}_c \otimes \boldsymbol{u}_c - \boldsymbol{n}\otimes\boldsymbol{n})$.

3. The futurelike light cone

$$\mathrm{L}^{\to} := \{\boldsymbol{x} \mid \boldsymbol{x}\cdot\boldsymbol{x} = 0,\ \boldsymbol{u}\cdot\boldsymbol{x} < 0 \text{ for all absolute velocities } \boldsymbol{u}\}$$

is a three dimensional submanifold in $\mathbf{M}$ which can be parametrized with the aid of an arbitrary absolute velocity $\boldsymbol{u}$:

$$p(\boldsymbol{q}) := \boldsymbol{u}|\boldsymbol{q}| + \boldsymbol{q} \qquad (\boldsymbol{u}\cdot\boldsymbol{q} = 0) \qquad (\boldsymbol{q}\in\mathbf{S}_{\boldsymbol{u}}).$$

8.4. **Lebesgue measures.** For measure theory we refer to [36, 37].

1. The Lebesgue measure on M (on $\mathbf{M}$) is the translation invariant measure λ_{M} ($\lambda_{\mathbf{M}}$) which assigns the value $|\boldsymbol{a}|\,|\boldsymbol{b}|\,|\boldsymbol{c}|\,|\boldsymbol{d}|$ to a prism defined by Lorentz orthogonal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, where $|\ |$ denotes the pseudo-length.

For the integrals of functions f defined in M, we use the notations

$$\int f\ d\lambda_{\mathrm{M}} = \int_{\mathrm{M}} f(x)\ dx.$$

2. Let H be a d dimensional submanifold in M (in $\mathbf{M}$). Then for a parametrization p of H, $(\mathrm{D}p[\xi])^*$ is a linear map $\mathbf{M}\to\mathbb{R}^d$, thus $(\mathrm{D}p[\xi])^*\cdot\mathrm{D}p[\xi]$ is a linear map $\mathbb{R}^d\to\mathbb{R}^d$, i.e. it is a d times d matrix.

The Lebesgue measure λ_H on the submanifold H is defined in such a way that for a function f defined in H,

$$\int f\ d\lambda_H := \int_{\mathbb{R}^d} f(p(\xi))\sqrt{|\det\big((\mathrm{D}p[\xi])^*\cdot\mathrm{D}p[\xi]\big)}\ d\xi$$

for an arbitrary parametrization p (thus, the integral is the same for all p).

Of course, also a function defined in M ($\mathbf{M}$) can be integrated by λ_H, taking the restriction of f onto H; in this way we can – and we do – consider λ_H a measure on M ($\mathbf{M}$), too.

If $\boldsymbol{h}: H\to\mathbf{M}$ is a continuous function then $\boldsymbol{h}\lambda_H$ is a vector measure, for which the integrals are defined by $d(\boldsymbol{h}\lambda_H) := \boldsymbol{h}\ d\lambda_H$.

3. The Lebesgue measure on the unit sphere, $S_c(1)$, is given by the well known formula

$$\int f\ d\lambda_{S_c(1)} = \int_{S_c(1)} f(\boldsymbol{n})\ d\boldsymbol{n} = \int_0^{2\pi}\int_0^{\pi} f(p(\vartheta,\varphi))\ \sin\vartheta\ d\vartheta\ d\varphi.$$

Then it is easy to find that

$$\int_{S_c(1)} \boldsymbol{n}\,d\boldsymbol{n} = 0, \quad \int_{S_c(1)} \boldsymbol{n}\otimes\boldsymbol{n}\,d\boldsymbol{n} = \frac{4\pi}{3}(\mathbf{1}+\boldsymbol{u}_c\otimes\boldsymbol{u}_c), \quad \int_{S_c(1)} \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\,d\boldsymbol{n} = 0, \tag{26}$$

which go over tensors, too; for instance,

$$\int_{S_c(1)} \boldsymbol{n}\cdot(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{n}\; d\boldsymbol{n} = \int_{S_c(1)} \boldsymbol{a}\cdot(\boldsymbol{n}\otimes\boldsymbol{n})\cdot\boldsymbol{b}\; d\boldsymbol{n} =$$
$$=\frac{4\pi}{3}\mathbf{a}\cdot(\mathbf{1}+\boldsymbol{u}_c\otimes\boldsymbol{u}_c)\cdot\boldsymbol{b} =$$
$$=\frac{4\pi}{3}\big((\mathbf{1}+\boldsymbol{u}\otimes\boldsymbol{u}_c)\cdot\mathbf{a}\big)\big(\boldsymbol{b}\cdot(\mathbf{1}+\boldsymbol{u}\otimes\boldsymbol{u})\big) =$$
$$=\frac{4\pi}{3}\mathrm{Tr}\Big((\mathbf{1}+\boldsymbol{u}_c\otimes\boldsymbol{u}_c)\cdot(\boldsymbol{a}\otimes\boldsymbol{b})\cdot(\mathbf{1}+\boldsymbol{u}_c\otimes\boldsymbol{u}_c)\Big),$$

and the last formula holds for arbitrary bilinear maps instead of $\boldsymbol{a}\otimes\boldsymbol{b}$.

4. According to the general definition, the Lebesgue measure on the futurelike light cone, $\mathrm{L}^{\rightarrow}$, is zero. Nevertheless, we can define on it a nonzero measure quite naturally, which we consider to be the Lebesgue measure. Namely, for all $a>0$

$$V(a) := \{\mathbf{x}\in\mathbf{M} \mid \boldsymbol{x}\cdot\boldsymbol{x} = -a^2,\ \boldsymbol{x} \text{ is futurelike}\}$$

is a three dimensional submanifold which can be parametrized with the aid of an arbitrary absolute velocity $\boldsymbol{u}$:

$$p(\boldsymbol{q}) := \boldsymbol{u}\sqrt{|\boldsymbol{q}|^2+a^2}+\boldsymbol{q} \qquad (\boldsymbol{q}\in\mathbf{S}_{\boldsymbol{u}}).$$

It is simply obtained that $\sqrt{|\det(\mathrm{D}p[\boldsymbol{q}]^*\cdot\mathrm{D}p[\boldsymbol{q}])|} = \frac{a}{\sqrt{|\boldsymbol{q}|^2+a^2}}$. Then the Lebesgue measure on the light cone is defined to be

$$\lambda_{\mathrm{L}^{\rightarrow}} := \lim_{a\to 0}\frac{1}{a}\lambda_{V(a)}$$

in an appropriate sense of the limit procedure; then we have the integration formula

$$\int f\; d\lambda_{\mathrm{L}^{\rightarrow}} = \int_{\mathrm{L}^{\rightarrow}} f(\mathbf{x})\; d\mathbf{x} = \int_{\mathbf{S}_{\boldsymbol{u}}} f(\mathbf{u}|\boldsymbol{q}|+\boldsymbol{q})\;\frac{d\boldsymbol{q}}{|\boldsymbol{q}|}$$

for an arbitrary absolute velocity $\boldsymbol{u}$.

8.5. **Distributions.** For distribution theory we refer to [38, 32].

The usual setting of distribution theory is based on $\mathbb{R}^n$. It is a quite simple generalisation that we consider the affine space M instead of $\mathbb{R}^n$. Thus, the space of our test functions, $\mathcal{D}(\mathrm{M})$, consists of smooth functions $\Phi:\mathrm{M}\to\mathbb{R}$, with compact support. A distribution is a continuous linear map $\mathcal{D}(\mathrm{M})\to\mathbb{R}$.

Another simple generalisation that we consider vector distributions, too, i.e. continuous linear maps $\mathcal{D}(\mathrm{M})\to\mathbf{M}$. The action of a (vector) distribution T on the test function Φ is denoted by $(T\mid\Phi)$.

The present article needs only the following facts from distribution theory.

1. The derivative of a (vector) distribution T is the distribution $\mathrm{D}T$ defined by

$$(\mathbf{a}\cdot\mathrm{D}T\mid\Phi) := -(T\mid\mathbf{a}\cdot\mathrm{D}\Phi) \qquad (\mathbf{a}\in\mathbf{M}).$$

2. A Lebesgue measure of a submanifold H is a distribution by the definition

$$(\lambda_H\mid\Phi) := \int\Phi\; d\lambda_H = \int_H \Phi(x)\; dx.$$

3. If H is a submanifold in M, f is a continuous function defined in H, and K is a submanifold in $\mathbf{M}$, the convolution $f\lambda_H * \lambda_K$ is the distribution defined by

$$(f\lambda_H*\lambda_K\mid\Phi) := \int_H\int_K f(x)\Phi(x+\mathbf{x})\; dx\; d\mathbf{x}.$$

4. For the d'Alembert operator $\Box := \mathrm{D} \cdot \mathrm{D} := \mathrm{Tr}(\mathrm{D}^2)$ (in coordinates $\partial_k \partial^k$)
$$\Box(T * \lambda_{\mathrm{L}\to}) = T.$$

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References

[1] H.A. Lorentz. La théorie électromagnétique de Maxwell et son application aux corps mouvants. *Archives Neerlandaises des Sciences Exactes et Naturelles*, 25:363–552, 1892.

[2] Max Abraham. *Theorie der Elektrizität, Vol. II: Elektromagnetische Theorie der Strahlung.* Teubner, Leipzig, 1905.

[3] H.A. Lorentz. *The Theory of Electrons.* Teubner, Leipzig, 1909.

[4] P. A. M. Dirac. Classical theory of radiating electrons. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 167(929):148–169, 1938.

[5] M. Born and L. Infeld. Foundations of the new field theory. *Proceedings of the Royal Society of London. Series A*, 144(852):425–451, 1934.

[6] F. Bopp. Eine lineare theorie des elektrons. *Annalen der Physik*, 430(5):345–384, 1940.

[7] M. K.-H. Kiessling. Force on a point charge source of the classical electromagnetic field. *Physical Review D*, 100(6):065012, 2019. arXiv:1907.11239v4, with erratum.

[8] G.W. Ford and O'Connell. Radiation reaction in electrodynamics and the elimination of runaway solutions. *Physics Letters A*, 157(4-5):217–220, 1991.

[9] J. Verhás. White areas on the map of applying non-equilibrium thermodynamics: on the self accelerating electron. *Atti dell'Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali, Suppl. I.*, 86:1–7, 2008.

[10] J. Polonyi. Effective dynamics of a classical point charge. *Annals of Physics*, 342:239–263, 2014.

[11] M. Formanek, A. Steinmetz, and J. Rafelski. Radiation reaction friction: I. Resistive material medium. *arXiv:2004.09634*, 2020.

[12] H. Spohn. The critical manifold of the Lorentz-Dirac equation. *EPL (Europhysics Letters)*, 50(3):287, 2000.

[13] Matolcsi T., F ul op T., and Weiner M. Second order equation of motion for electromagnetic radiation back-reaction. *Journal of Modern Physics*, 32(27):1750147(18), 2017. arXiv:1207.0428.

[14] S. Parrott. *Relativistic electrodynamics and differential geometry.* Springer, 2012.

[15] C. Bild, D.-A. Deckert, and H. Ruhl. Radiation reaction in classical electrodynamics. *Physical Review D*, 99(9):096001, 2019.

[16] E. J. Moniz and D.H. Sharp. Absence of runaways and divergent self-mass in nonrelativistic quantum electrodynamics. *Physical Review D*, 10(4):1133, 1974.

[17] E. J. Moniz and D.H. Sharp. Radiation reaction in nonrelativistic quantum electrodynamics. *Physical Review D*, 15(10):2850, 1977.

[18] C. R. Galley, A. K. Leibovich, and I. Z. Rothstein. Finite size corrections to the radiation reaction force in classical electrodynamics. *Physical review letters*, 105(9):094802, 2010.

[19] P. Forgács, T. Herpay, and P. Kovács. Comment on "Finite Size Corrections to the Radiation Reaction Force in Classical Electrodynamics". *Physical review letters*, 109(2):029501, 2012.

[20] C. R. Galley, A.K. Leibovich, and I.Z. Rothstein. Galley, leibovich, and rothstein reply. *Physical Review Letters*, 109(2):029502, 2012.

[21] J. Polonyi. The Abraham–Lorentz force and electrodynamics at the classical electron radius. *International Journal of Modern Physics A*, 34(15):1950077, 2019.

[22] Arthur Yaghjian. *Relativistic dynamics of a charged sphere: updating the Lorentz-Abraham model*. Springer, 2006. 2nd edition, first: 1992.

[23] Fritz Rohrlich. *Classical charged particles*. World Scientific Publishing Company, 2007. 3rd edition, original: 1965.

[24] T.N. Wistisen, A. Di Piazza, H. V. Knudsen, and U. I. Uggerhøj. Experimental evidence of quantum radiation reaction in aligned crystals. *Nature communications*, 9(1):1–6, 2018.

[25] K. Poder, M. Tamburini, G. Sarri, A. Di Piazza, S. Kuschel, C.D. Baird, K. Behm, S. Bohlen, J.M. Cole, D.J. Corvan, and +other 14 authors. Experimental signatures of the quantum nature of radiation reaction in the field of an ultraintense laser. *Physical Review X*, 8(3):031004, 2018.

[26] T.G. Blackburn. Radiation reaction in electron–beam interactions with high-intensity lasers. *Reviews of Modern Plasma Physics*, 4(1):1–37, 2020.

[27] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields (Course of Theoretical Physics, vol. 2)*. Pergamon Press, Oxford, 2th edition, 1959.

[28] I.V. Sokolov. Renormalization of the Lorentz-Abraham-Dirac equation for radiation reaction force in classical electrodynamics. *Journal of Experimental and Theoretical Physics*, 109(2):207–212, 2009.

[29] D.B. Zot'ev. Critical remarks on sokolov's equation of the dynamics of a radiating electron. *Physics of Plasmas*, 23(9):093302, 2016.

[30] R. Geroch. Partial differential equations of physics. arXiv:gr-qc/9602055.

[31] S.E. Gralla, A.I. Harte, and R.M. Wald. Rigorous derivation of electromagnetic self-force. *Physical Review D*, 80(2):024031, 2009.

[32] G. van Dijk. *Distribution theory: convolution, Fourier transform, and Laplace transform*. Walter de Gruyter, 2013.

[33] T. Matolcsi. Classical electrodynamics. *Preprints of the Department of Applied Analysis of ELTE TTK*, (4), 1977. http://szofi.elte.hu/∼szaboa/MatolcsiKonyvek/djvu/jegyzet/classeldin.djvu.

[34] T. Matolcsi. A rigorous derivation of the Lorentz-Dirac equation for point charges, without approximations. manuscript.

[35] T. Matolcsi. *Spacetime Without Reference Frames*. Minkowski Institute Press, 4 edition, 2020.

[36] N. Dinculeanu. *Vector measures*. Pergamon Press, 1967.

[37] H. Amann and J. Escher. *Analysis III*. Birkhäuser, 2009.

[38] Horváth J. *Topological vector spaces and distributions*. Dover, 2012.

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