Abstract mathematical treatment of relativistic phenomena

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Abstract

This preprint concerns a mathematically rigorous treatment of an interesting physical phenomenon in relativity theory. We would like to draw the reader's attention particularly to the abstract mathematical formalism of relativity (which was developed in full detail in [7]). This treatment allows all mathematically oriented readers to understand relativity without feeling the awkward ambiguities that are so common after reading a standard text on relativity.

In the extensive literature dealing with the relativistic phenomenon of Thomas rotation several methods have been developed for calculating the Thomas rotation angle of a gyroscope along a circular world line. One of the most appealing methods [14], however, subsequently led to a contradiction in [4] when three different Thomas rotation angles were obtained for the same circular world line. In this paper we resolve this contradiction by rigorously examining the theoretical background and the limitations of the principle of [14].

1 Introduction

The relativistic phenomena of Thomas rotation and Thomas precession have been treated in relativity theory, both special and general, from various points of view (see e.g. [1], [2], [5], [10], [11], [12], [13], [14], [16], [18], [19]). As this preprint is aimed at a mathematical readership, we will include a short description of the appearing physical phenomena. Also, the mathematics being used does not go beyond standard facts from linear algebra, differential geometry and calculus. We firmly believe that this abstract formalism is the appropriate language of relativity, where paradoxes and confusion simply vanish, and the

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physical concepts become clear. *Readers unfamiliar with special relativity (or, this abstract form of it) should not be discouraged; it may be helpful to study [10] where the formulae are less involved.*

Unfortunately, there seems to be no standard agreement in the literature as to the usage of the terminologies 'Thomas precession' and 'Thomas rotation'; we will adhere to the terminologies used in [10]. We remark, that these notions also provide a possible way to put relativity to the test in practice (Thomas rotation is one of the relativistic effects contributing to the gyroscopic precession currently being measured in the Gravity Probe B experiment). While the results of this paper stay exclusively in the realm of special relativity, the appearing concepts can also be generalized to general relativistic spacetime models, to which the authors hope to return in a subsequent publication.

To grasp the essence of Thomas rotation, let us briefly describe this intriguing phenomenon as an analogue of the well known twin paradox. Consider two twins in an inertial frame. One of them remains in that frame for all times, while the other goes for a trip in spacetime, and later returns to his brother. It is well-known that different times have passed for the two twins: the traveller is younger than his brother. What may be surprising is that the space of the traveller when he arrives back, even if he experienced no torque during his journey (i.e. he thinks that his gyroscope kept its direction throughout the journey; the meaning of this must, of course, be formulated in precise terms), will be rotated compared to the space of his brother; this is, in fact, the Thomas rotation. This analogy is illuminating in one more respect: until the traveller returns to the original frame of reference it makes no sense to ask 'how much younger is the traveller compared to his brother?' and 'by what angle is the traveller's gyroscope rotated compared to that of his brother?' Different observers may give different answers. When the traveller returns to his brother, these questions suddenly make perfect sense, and there is an absolute answer (independent of who the observer is) as to how much younger and how much rotated the traveller is.

Of course, an arbitrtary inertial frame can observe the brothers continuously, and can tell at each frame-instant, what difference *he sees* between the ages of the brothers. More explicitly, as it is well known, given a world line, an arbitrary inertial frame can tell the relation between the frame's time and the proper time of the world line. This relation depends on the inertial frame: different inertial frames establish different relations.

Similarly, an arbitrary inertial frame, observing the two brothers, can tell at each frame-instant what difference *he sees* between the directions of the gyroscopes of the brothers. (That is, an inertial observer sees the gyroscope of the traveller 'wobble'; this is Thomas precession.) Different inertial frames establish different relations.

This philsophy makes a clear distinction between Thomas rotation and Thomas precession connected to a given world line. On the one hand, Thomas rotation

- makes sense only for 'returning' gyroscopes,
- is a discrete phenomenon (i.e. it makes sense only for the (usually) discrete

set of proper time instances when the gyroscope happens to be in its initial frame of reference),

– is an *absolute* notion, i.e. independent of who observes it (the same angle of Thomas rotation will be measured by all inertial frames observing the gyroscope).

On the other hand, *Thomas precession* refers to the instantenous angular velocity, *with respect to a particular inertial frame*, of a gyroscope moving along an arbitrary world line. Thus, Thomas precession

– makes sense for arbitrary gyroscopes with respect to arbitrary inertial frames,

– is a continuous phenomenon,

– is a *relative* notion, i.e. the same gyroscope may show different instantaneous precessions with respect to different inertial frames.

In terms of any particular inertial frame one can think of Thomas rotation as the time-integral of Thomas precessions (and while Thomas precession, as a function of time, will differ from one inertial frame to another, its integral will always give the same angle: the Thomas rotation).

Evaluating the Thomas rotation angle, even for a gyroscope moving along a circular orbit, can lead to lengthy calculations. In order to arrive at the result in the shortest possible way, diverse approaches have been developed in the literature (see e.g. [10], [12], [13], [14]).

One of the simplest and most appealing concepts, introduced in [14], is to relate the Thomas precession of the gyroscope to the angular velocity of an observer co-moving with the gyroscope (and then calculate the Thomas rotation angle from the precession). The principle in that paper is that, heuristically, if the gyroscope keeps direction in itself and the co-moving rotating observer has instantenous angular velocity $\mathbf{\Omega}$ then it will see the gyroscope precess with angular velocity $-\mathbf{\Omega}$, and when the gyroscope returns to its initial local rest frame one can evaluate the Thomas rotation angle from the knowledge of instantenous precessions $-\mathbf{\Omega}(t)$ along the way.

However, this principle was applied subsequently in [4] to three different rotating observers co-moving with the gyroscope (the existence of such different observers is not unexpected), and three different Thomas rotation angles were derived; an obvious contradiction. In fact, the correct angle was obtained for the conventional rotating observer, while the 'Trocheris-Takeno' and the 'modified-Trocheris-Takeno' rotating observers gave erroneous angles. It is therefore desirable to examine the theoretical background of the above heuristic principle and see where its limitations are, i.e. what observers it can justifiably be applied to. In doing so, we will also introduce the natural concept of Foucault precession and examine its relation to the angular velocity of the observer and the Thomas rotation of the gyroscope.

Along the way we obtain a mathematical criterium for the existence of the Foucault precession in the space point of a noninertial observer (Section 3.2). Then we check this criterium for different rotating observers to see whether the Foucault precession with repect to them is meaningful or not (Section 4.4). It turns out that the Foucault precession is only meaningful in the case of the conventional rotating observer, in which case its angular velocity is indeed the negative of the angular velocity of the observer (in accordance with the principle in [14]). We also conjecture, more generally, that whenever the Foucault precession makes sense for an observer, it is equal to the negative of the angular velocity of the observer. Finally, we examine the relation of the Foucault precession and the Thomas rotation angle (Section 5.2). We find that even if the Foucault precession in the space point of an observer makes sense, a further property of the observer is necessary so that the Thomas rotation angle be evaluated from the knowledge of all instantenous Foucault precessions.

Throughout the paper we shall use an abstract formalism of special relativity (see [6], [7]). Our basic concept is that special relativistic spacetime has a four-dimensional affine structure, and the customary coordinatization (relative to some reference frame) is, in many cases, unnecessary in the description of physical phenomena. Besides yielding mathematically rigorous formulae, this coordinate-free treatment of relativity also allows us to make clear conceptual distinction of the appearing concepts.

2 Fundamental notions

In this section the necessary notions and results of the special relativistic spacetime model as a mathematical structure ([6], [7]) will be recapitulated. As the formalism slightly differs from the usual textbook treatments of special relativity (but only the formalism: our treatment is *mathematically equivalent* to the usual treatments), we will point out several relations between textbook formulae and those of our formalism, and also advise the reader to consult [7] for a more detailed account. A concise summary of the appearing notions is also contained in [10]. The advantage of this abstract formalism is that tacit assumptions and intuitive notions (that can go wrong so easily) are ruled out; each appearing concept (starting from the very notion of observers and synchronizations) are mathematically defined.

2.1 Observers and synchronizations

Special relativistic spacetime is a four dimensional affine space M over the vector space $\mathbf{M}$; the spacetime distances form an oriented one dimensional vector space $\mathbf{I}$, and an arrow oriented Lorentz form $\mathbf{M}\times\mathbf{M}\to\mathbf{I}\otimes\mathbf{I}$, $(\mathbf{x},\mathbf{y})\mapsto\mathbf{x}\cdot\mathbf{y}$ is given.

An *absolute velocity* $\mathbf{u}$ is a future directed element of $\frac{\mathbf{M}}{\mathbf{I}}$ for which $\mathbf{u}\cdot\mathbf{u}=-1$ holds (absolute velocity corresponds to four-velocity in usual terminology).

For an absolute velocity $\mathbf{u}$, we define the three-dimensional spacelike linear subspace

$$\mathbf{E_u} := \{\mathbf{x}\in\mathbf{M} \mid \mathbf{u}\cdot\mathbf{x}=0\}; \tag{1}$$

then

$$1+\mathbf{u}\otimes\mathbf{u} : \mathbf{M}\to\mathbf{E_u}, \quad \mathbf{x}\mapsto\mathbf{x}+\mathbf{u}(\mathbf{u}\cdot\mathbf{x}) \tag{2}$$

is the projection onto $\mathbf{E_u}$ along $\mathbf{u}$. The restriction of the Lorentz form onto $\mathbf{E_u}$ is positive definite, so $\mathbf{E_u}$ is a Euclidean vector space (this will correspond to the space vectors of an inertial observer with velocity $\mathbf{u}$).

The history of a classical material point is described by a differentiable *world line function* $r : \mathbf{I} \to M$ such that $\dot{r}(\mathbf{s})$ is an absolute velocity for all proper time values $\mathbf{s}$. The range of a world line function – a one dimensional submanifold – is called a *world line*.

An *observer* $\mathbf{U}$ is an absolute velocity valued smooth map defined in a connected open subset of M. (This is just a mathematical definition; it may sound unfamiliar at first, but considering that something that an observer calls a 'fixed space-point' is, in fact, a world line in spacetime, this definition will make perfect 'physical' sense). A maximal integral curve of $\mathbf{U}$ – a world line – is a *space point* of the observer, briefly a $\mathbf{U}$-*space point*; the set of the maximal integral curves of $\mathbf{U}$ is the *space* of the observer, briefly the $\mathbf{U}$-*space*.

For every spacetime point x in the domain of $\mathbf{U}$ there is a unique $\mathbf{U}$-space point $C_{\mathbf{U}}(x)$ containing x.

A *synchronization* or *simultaneity* is a smooth equivalence relation on a connected open subset of M such that the equivalence classes are connected three-dimensional smooth submanifolds (hypersurfaces) whose tangent spaces are spacelike (a vector $\mathbf{x} \in \mathbf{M}$ is spacelike if $\mathbf{x} \cdot \mathbf{x} > 0$).

Given a synchronization S, an equivalence class is called an S-*instant*; the set I_S of S-instants is called S-*time*.

For every world point x in the domain of S there is a unique S-instant $\tau_S(x)$ containing x; moreover, there is a unique absolute velocity value $\mathbf{U}_S(x)$ such that $\mathbf{E}_{\mathbf{U}_S(x)}$ is the tangent space of $\tau_S(x)$ at x. The smoothness of the synchronization means that the velocity field $x \mapsto \mathbf{U}_S(x)$ is smooth. (Thus an observer $\mathbf{U}_S$ corespsonds to every synchronization S; it is worth mentioning that there are observers which do not correspond to any synchronization.)

A *reference frame* is a pair $(S, \mathbf{U})$, where S is a synchronization and $\mathbf{U}$ is an observer. We remark that there is no a priori relation between $\mathbf{U}_S$ (the velocity field corresponding to S) and $\mathbf{U}$ (an arbitrary observer). Let us also mention that a reference frame makes it possible to 'coordinatize' spacetime by S-instants and U-space points.

An observer having consant value is called *inertial*. The space points of an inertial observer are parallel straight lines. The inertial observer with absolute velocity $\mathbf{u}$ establishes its standard synchronization in which the instants are hyperplanes over the vector space $\mathbf{E_u}$. An inertial observer together with its standard synchronization is called a *standard inertial frame*.

2.2 Nearly standard local synchronizations

A non-inertial observer $\mathbf{U}$ has no standard synchronization. However, for every $\mathbf{U}$-space point we can give a *nearly standard local synchronization*.

More generally, if r is a smooth world line function, then we define the nearly standard local synchronization due to r in a neighbourhood of the range of r

(the world line determined by r) in such a way that the instants of that synchronization are subsets of spacelike hyperplanes in such a way that the hyperplane at an arbitrary point is Lorentz orthogonal to the tangent vector of the world line in question. In other words, the synchronization instant corresponding to $r(\mathbf{s})$ is a part of the hyperplane $r(\mathbf{s}) + \mathbf{E}_{\dot r(\mathbf{s})}$. The implicit function theorem assures that such a synchronization is well defined: for fixed $\mathbf{s}_0 \in \mathbf{I}$ and for x in a neighbourhood of $r(\mathbf{s}_0)$, the relation $(x - r(\mathbf{s})) \cdot \dot r(\mathbf{s}) = 0$ can be solved for $\mathbf{s}$ and the implicit function $x \mapsto \mathbf{s}(x)$ satisfies

$$\frac{d\mathbf{s}(x)}{dx} = -\frac{\dot r(\mathbf{s}(x))}{1 + (x - r(\mathbf{s}(x))) \cdot \ddot r(\mathbf{s}(x))}. \tag{3}$$

Note that

$$\left.\frac{d\mathbf{s}(x)}{dx}\right|_{x=r(\mathbf{s})} = -\dot r(\mathbf{s}). \tag{4}$$

As usual, the standard inertial frame with absolute velocity value $\dot r(\mathbf{s})$ is called *the local rest frame* corresponding to $r(\mathbf{s})$. Roughly speaking, attaching the local rest frame to every world point in the range of r, we get the above described nearly standard local synchronization due to r.

The time instants of this nearly standard local synchronization can be identified with the proper time values of the world line function r.

2.3 Splitting of spacetime

A reference frame $(S, \mathbf{U})$ *splits* spacetime into S-time and $\mathbf{U}$-space which means that the corresponding S-instants and $\mathbf{U}$-space points are assigned to spacetime points:

$$M \to I_S \times E_{\mathbf{U}}, \qquad x \mapsto (\tau_S(x), C_{\mathbf{U}}(x)). \tag{5}$$

It is well known from the theory of manifolds that both S-time I_S and $\mathbf{U}$-space $E_{\mathbf{U}}$ can be endowed with a distinguished smooth structure, according to which both τ_S and $C_{\mathbf{U}}$, and consequently, the splitting will be smooth. The smooth structure of $\mathbf{U}$-space is defined in such a way that given an S-instant t – a hypersurface in spacetime –, every $\mathbf{U}$-space point – a world line in spacetime – has a neighbourhood in $\mathbf{U}$-space which is diffeomorphic with an open subset of the hypersurface t via the correspondence $q \mapsto t \cap q$; the tangent map of this diffeomorphism sends the tangent space of $E_{\mathbf{U}}$ at q into $\mathbf{E}_{\mathbf{U}_S(t\cap q)}$, the tangent space of the hypersurface t at the meeting point of t and q.

The derivative of $C_{\mathbf{U}}$, depending on the world points, establishes a mapping from the spacetime vectors to the tangent space of $E_{\mathbf{U}}$:

$$\mathrm{D}C_{\mathbf{U}}(x) : \mathbf{M} \to T_{C_{\mathbf{U}}(x)}(E_{\mathbf{U}}), \tag{6}$$

where T denotes tangent space.

2.4 Representation of an observer space by a synchronization instant

We shall apply the previous considerations to the nearly standard local synchronization S due to a $\mathbf{U}$-line function r. Then

– the S-time instants are labelled by the proper time values $\mathbf{s}$ of r,

– the S-instant corresponding to $\mathbf{s}$ is a subset of the hyperplane $r(\mathbf{s})+\mathbf{E}_{\dot r(\mathbf{s})}$,

– $\mathbf{U}_S(x) = \mathbf{U}(r(\mathbf{s}(x))) = \dot r(\mathbf{s}(x))$, where where $\mathbf{s}(x)$ is defined in Subsection 2.2.

For further investigations, let us introduce the mapping (the 'flow' defined by the observer)

$$\mathbf{I}\times M \to M, \quad (\mathbf{t}, x) \mapsto R(\mathbf{t}, x) \tag{7}$$

where $\mathbf{t} \mapsto R(\mathbf{t}, x)$ is the world line function of $\mathbf{U}$ passing through the world point x, i.e. $\mathbf{R}(0, x) = x$ and

$$\frac{\partial R(\mathbf{t}, x)}{\partial \mathbf{t}} = \mathbf{U}(R(\mathbf{t}, x)). \tag{8}$$

It follows from the uniqueness of the solutions of the differential equation (8) that $R(\mathbf{t}, R(\mathbf{t}', x)) = R(\mathbf{t}+\mathbf{t}', x)$; differentiating it with respect to $\mathbf{t}'$ and then putting $\mathbf{t}' = 0$, we have

$$\frac{\partial R(\mathbf{t}, x)}{\partial x}\cdot \mathbf{U}(x) = \mathbf{U}(R(\mathbf{t}, x)). \tag{9}$$

For a given x let $\mathbf{t}(x)$ be the proper time value of the $\mathbf{U}$-line passing through x for which the $\mathbf{U}$-line meets the hyperplane $r(0) + \mathbf{E}_{\dot r(0)}$, i.e. $\mathbf{t}(x)$ is defined implicitly by $\dot r(0)\cdot\big(R(\mathbf{t}, x) - r(0)\big) = 0$. Then the implicit function theorem gives us

$$\frac{d\mathbf{t}(x)}{dx} = -\frac{\dot r(0)\cdot \frac{\partial R(\mathbf{t},x)}{\partial x}}{\dot r(0)\cdot \mathbf{U}(R(\mathbf{t}, x))}\bigg|_{\mathbf{t}=\mathbf{t}(x)}. \tag{10}$$

Note that we have $\mathbf{t}(r(\mathbf{s})) = -\mathbf{s}$, $R(-\mathbf{s}, r(\mathbf{s})) = r(0)$, so with the notation

$$\mathbf{R}(\mathbf{s}) := \frac{\partial R(-\mathbf{s}, x)}{\partial x}\bigg|_{x=r(\mathbf{s})} \tag{11}$$

we obtain

$$\frac{d\mathbf{t}(x)}{dx}\bigg|_{x=r(\mathbf{s})} = \dot r(0)\cdot \mathbf{R}(\mathbf{s}). \tag{12}$$

It is well known from the theory of differential equations that $\mathbf{R}(\mathbf{s}) : \mathbf{M} \to \mathbf{M}$ is a linear bijection. Moreover, (9) implies that

$$\mathbf{R}(\mathbf{s})\dot r(\mathbf{s}) = \dot r(0). \tag{13}$$

Now we find it convenient to introduce the notation

$$\mathbf{P}(\mathbf{s}) := 1 + \dot r(\mathbf{s})\otimes \dot r(\mathbf{s}) \tag{14}$$

for the Lorentz orthogonal projection onto $\mathbf{E}_{\dot r(\mathbf{s})}$.

We infer from (13) that $\mathbf{P}(0)\mathbf{R}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(0)\mathbf{R}(\mathbf{s})$, thus

$$\mathbf{A}(\mathbf{s}) := \mathbf{P}(0)\mathbf{R}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(0)\mathbf{R}(\mathbf{s}) \tag{15}$$

establishes a linear bijection from $\mathbf{E}_{\dot r(\mathbf{s})}$ onto $\mathbf{E}_{\dot r(0)}$ and similarly,

$$\mathbf{A}(\mathbf{s})^{-1} := \mathbf{P}(\mathbf{s})\mathbf{R}(\mathbf{s})^{-1}\mathbf{P}(0) = \mathbf{P}(\mathbf{s})\mathbf{R}(\mathbf{s})^{-1} \tag{16}$$

establishes a linear bijection from $\mathbf{E}_{\dot r(0)}$ onto $\mathbf{E}_{\dot r(\mathbf{s})}$ and

$$\mathbf{A}(\mathbf{s})^{-1}\mathbf{A}(\mathbf{s}) = \mathbf{P}(\mathbf{s}), \qquad \mathbf{A}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1} = \mathbf{P}(0). \tag{17}$$

According to the definition of the smooth structure of $\mathbf{U}$-space, a neighbourhood of the $\mathbf{U}$-space point given by r will be represented by an open subset of the S-instant (hyperplane) corresponding to $\mathbf{s} = 0$ via the diffeomorphism

– $E_{\mathbf{U}} \to r(0) + \mathbf{E}_{\dot r(0)}$, $q \mapsto \big(r(0) + \mathbf{E}_{\dot r(0)}\big) \cap q$.

Then it follows from our previous considerations that

– the map $M \to E_{\mathbf{U}}$, $x \mapsto C_{\mathbf{U}}(x)$ is represented by $x \mapsto R(\mathbf{t}(x), x)$,

– the tangent space of $E_{\mathbf{U}}$ at an arbitrary point is represented by $\mathbf{E}_{\dot r(0)}$ and $\mathrm{D}C_{\mathbf{U}}(x)$ is represented by $\frac{dR(\mathbf{t}(x),x)}{dx}$.

In particular, $\mathrm{D}C_{\mathbf{U}}(r(\mathbf{s}))$ is represented by

$$\frac{dR(\mathbf{t}(x),x)}{dx}\Big|_{x=r(\mathbf{s})} = \left(\frac{\partial R(\mathbf{t},x)}{\partial \mathbf{t}} \otimes \frac{d\mathbf{t}(x)}{dx} + \frac{\partial R(\mathbf{t},x)}{\partial x}\right)\Big|_{\mathbf{t}=-\mathbf{s},x=r(\mathbf{s})} =$$
$$= (1 + \dot r(0) \otimes \dot r(0))\mathbf{R}(\mathbf{s}) = \mathbf{A}(\mathbf{s}). \tag{18}$$

2.5 Spatial metric of an observer

The usual 'spatial metric' described in coordinates by $\gamma_{ik} := g_{ik} + \frac{g_{i0}g_{0k}}{-g_{00}}$ is obtained in the previous framework as follows: we give Euclidean forms for all $\mathbf{s}$ (representing the synchronization instants) on the tangent space of all $\mathbf{U}$-space points; the collection of these Euclidean forms define a Riemannian metric – depending on $\mathbf{s}$ – on the space of the observer (see [7], Section II.9.4.).

We will only need the local Euclidean form $\gamma_{\mathbf{s}}$ corresponding to the $\mathbf{U}$-space point described by r. According to our representation of tangent spaces it is given on $\mathbf{E}_{\dot r(0)}$ and has the following expression:

$$\gamma_{\mathbf{s}}(\mathbf{q},\mathbf{q}) = |\mathbf{A}(\mathbf{s})^{-1}\mathbf{q}|^2 \qquad (\mathbf{q} \in \mathbf{E}_{\dot r(0)}). \tag{19}$$

2.6 Angular velocity of an observer

According to the usual definition, *the angular velocity of an observer* $\mathbf{U}$ is

$$\mathbf{\Omega}_{\mathbf{U}} := -\frac{1}{2}(1 + \mathbf{U} \otimes \mathbf{U})(\mathrm{D} \wedge \mathbf{U})(1 + \mathbf{U} \otimes \mathbf{U}) \tag{20}$$

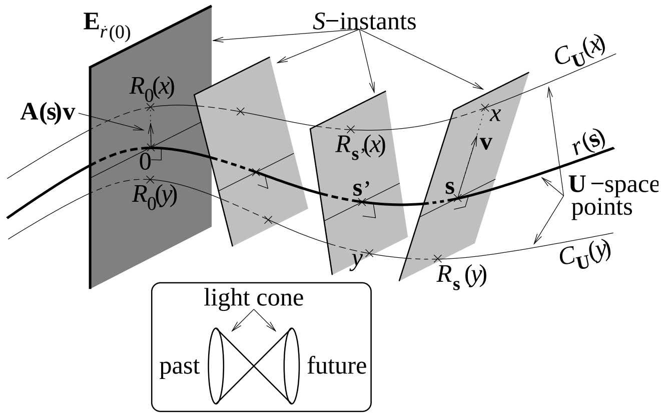

Figure 1: Representation of an observer space. For a spacetime point $x \in M$, $R(\mathbf{t}(x), x) = \big(r(0) + \mathbf{E}_{\dot r(0)}\big) \cap C_{\mathbf{U}}(x)$ is its projection along the $\mathbf{U}$-lines onto the S-instant $r(0)+\mathbf{E}_{\dot r(0)}$. The mapping $\mathbf{A}(\mathbf{s})$ pulls back the vector $\mathbf{v} \in \mathbf{E}_{\dot r(\mathbf{s})}$ 'along the flow' defined by $\mathbf{U}$ to $\mathbf{E}_{\dot r(0)}$.

where D denotes differentiation, $\mathrm{D} \wedge \mathbf{U} := (\mathrm{D}\mathbf{U})^* - \mathrm{D}\mathbf{U}$ is the antisymmetric (exterior) derivative of $\mathbf{U}$ (in usual coordinates: $\mathrm{D} \wedge \mathbf{U} \sim \partial_i U_k - \partial_k U_i$). $\mathbf{\Omega}_{\mathbf{U}}(x)$ is an antisymmetric linear map $\mathbf{E}_{\mathbf{U}(x)} \to \frac{\mathbf{E}_{\mathbf{U}(x)}}{\mathbf{I}}$ (in the literature, mostly the unique vector in $\frac{\mathbf{E}_{\mathbf{U}(x)}}{\mathbf{I}}$ assigned to $\mathbf{\Omega}_{\mathbf{U}}(x)$ by the Levi-Civita tensor is called the angular velocity).

The angular velocity of an observer $\mathbf{U}$ refers to the change of the mutual spacetime position of neighbouring $\mathbf{U}$-space points (which are world lines, maximal integral curves of the velocity field $\mathbf{U}$). We call attention to the fact that *the angular velocity of a single world line cannot be defined* (see Subsection 5.1)

3 Gyroscopes

3.1 Thomas rotation and Thomas precession

We would like to give a precise mathematical meaning to the intuitive concept of a travelling gyroscope keeping its direction. We have to express two facts: first, the gyroscope is always spacelike according to the local rest frame and, second, the gyroscope 'keeps its direction'. We need to refer to [10] for the detailed justification (in this abstract formalism) of the following standard definition:

A gyroscopic vector on a world line r is a pair of functions $(r, \mathbf{z}) : \mathbf{I} \to M \times \mathbf{M}$, where r is a world line function (the centre of the gyroscopic vector), $\dot r \cdot \mathbf{z} = 0$ (the vector $\mathbf{z}$ is always spacelike according to the local rest frame), moreover, the Fermi-Walker differential equation

$$\dot{\mathbf{z}} = (\dot r \wedge \ddot r)\mathbf{z} = \dot r(\ddot r \cdot \mathbf{z}) \tag{21}$$

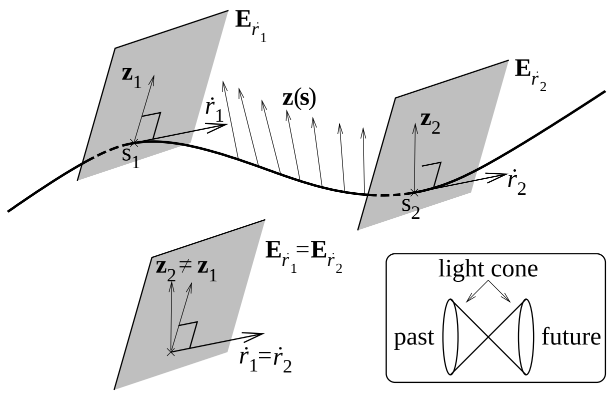

Figure 2: Thomas rotation. At two different proper time values $\mathbf{s}_1$ and $\mathbf{s}_2$ the absolute velocities $\dot{r}_1 = \dot{r}(\mathbf{s}_1)$ and $\dot{r}_2 = \dot{r}(\mathbf{s}_2)$ are equal, so $\mathbf{E}_{\dot{r}_1} = \mathbf{E}_{\dot{r}_2}$, but the initial and final gyroscopic vectors $\mathbf{z}_1 = \mathbf{z}(\mathbf{s}_1) \in \mathbf{E}_{\dot{r}_1}$ and $\mathbf{z}_2 = \mathbf{z}(\mathbf{s}_2) \in \mathbf{E}_{\dot{r}_2}$ are different.

is satisfied (which expresses the fact that $\mathbf{z}$ 'keeps direction, does not rotate in itself').

For proper time values $\mathbf{s}_1$ and $\mathbf{s}_2$, the vectors $\mathbf{z}(\mathbf{s}_1)$ and $\mathbf{z}(\mathbf{s}_2)$ are in different three-dimensional Euclidean vector spaces unless $\dot{r}(\mathbf{s}_2) = \dot{r}(\mathbf{s}_1)$. Even if so, $\mathbf{z}(\mathbf{s}_2) \neq \mathbf{z}(s_1)$, in general: the gyroscopic vector starts at $\mathbf{s}_1$, tramps over diverse Euclidean spaces 'keeping its direction' in the above sense, and at $\mathbf{s}_2$ it arrives back to the starting Euclidean space and its final direction differs from its initial direction (it arrives rotated, see Figure 2). This phenomenon, called *Thomas rotation*, is an absolute notion (independent of reference frames) and makes sense only if at least two absolute velocity values of the world line in question are equal.

The Thomas precession, on the other hand, is a relative notion: a standard inertial frame with velocity value $\mathbf{u}$ boosts $\mathbf{z}$ continuously to its own space, obtaining the function $\mathbf{z}_\mathbf{u} : I_\mathbf{u} \to \mathbf{E}_\mathbf{u}$ ($I_\mathbf{u}$ is the standard synchronization time of $\mathbf{u}$) which satisfies

$$\mathbf{z}'_\mathbf{u} = \boldsymbol{\Omega}_\mathbf{u}\mathbf{z}_\mathbf{u}, \tag{22}$$

where the prime denotes derivation with respect to the $\mathbf{u}$-time, and

$$\boldsymbol{\Omega}_\mathbf{u} := \frac{\gamma_\mathbf{u}^2}{1+\gamma_\mathbf{u}}\mathbf{v}_\mathbf{u} \wedge \mathbf{a}_\mathbf{u} \tag{23}$$

is the angular velocity of the precession, expressed in terms of the relative ve-

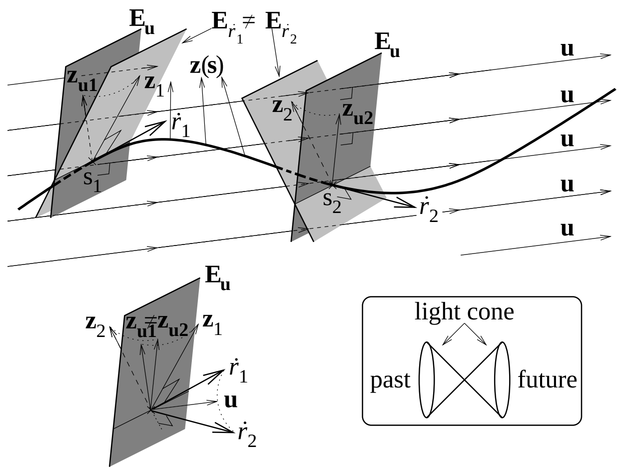

Figure 3: Thomas precession. At every instant $\mathbf{s}$ the inertial observer $\mathbf{u}$ boosts the gyroscopic vector $\mathbf{z}(\mathbf{s}) \in \mathbf{E}_{\dot r(\mathbf{s})}$ to its own space $\mathbf{E_u}$, and observes that the initial vector $\mathbf{z_{u1}}$ and the final vector $\mathbf{z_{u2}}$ are different. In $\mathbf{E_u}$ the vector $\mathbf{z_u}$ performs a precession at an angular velocity $\mathbf{\Omega_u}$.

locity $\mathbf{v_u}$ and the relative acceleration $\mathbf{a_u}$

$$\mathbf{v_u}(t) := \frac{\dot r(\mathbf{s}(t))}{-\mathbf{u}\cdot\dot r(\mathbf{s}(t))} - \mathbf{u} \tag{24}$$

$$\mathbf{a_u}(t) := \frac{1}{\big(-\mathbf{u}\cdot\dot r(\mathbf{s}(t))\big)^2}\left(\ddot r(\mathbf{s}(t)) + \frac{\dot r(\mathbf{s}(t))\big(\mathbf{u}\cdot\ddot r(\mathbf{s}(t))\big)}{-\mathbf{u}\cdot\dot r(\mathbf{s}(t))}\right) \tag{25}$$

of the world line. The proper time $\mathbf{s}(t)$ of r as a function of $\mathbf{u}$-time is determined by

$$\frac{d\mathbf{s}(t)}{dt} = -\mathbf{u}\cdot\dot r(\mathbf{s}(t)) = \frac{1}{\sqrt{1-|\mathbf{v_u}(t)|^2}} =: \gamma_{\mathbf{u}}(t). \tag{26}$$

(For the details of the derivation of these formulae see [10].)

The standard inertial frame sees the gyroscopic vector – which keeps direction in itself – precessing. Note that the same gyroscopic vector shows different Thomas precessions with respect to different standard inertial frames (i.e. $\mathbf{\Omega_u}$ really depends on $\mathbf{u}$). Note also, that different gyroscopic vectors on the same world line precess with the same angular velocity $\mathbf{\Omega_u}$ with respect to the inertial frame $\mathbf{u}$, i.e. in (23) $\mathbf{\Omega_u}$ does not depend on $\mathbf{z}$.

3.2 Foucault precession

As we have seen, Thomas precession is defined with respect to *inertial* reference frames. Although the notion of Thomas precession does not seem possible to generalize to make sense with respect to non-inertial frames, we can introduce the natural notion of Foucault precession with respect to observers co-moving with the gyroscope, i.e. those having the centre of the gyroscope as a space point. With the help of this notion we can investigate the validity of the principle described in the Introduction.

The history of a material point is perceived by a reference frame $(S, \mathbf{U})$ as a *motion* which is a function assigning $\mathbf{U}$-space points to S-instants as follows. Let r be the world line function of the material point; then the corresponding world line meets every hypersurface $t \in I_S$ at most in one point, thus we can give a function $I_S \to \mathbf{I}$, $t \mapsto \mathbf{s}(t)$ such that $r(\mathbf{s}(t))$ is the meeting point of the world line and the hypersurface t, i.e. $\mathbf{s}(t)$ is the proper time of r as a function of S-time t. The unique $\mathbf{U}$-space point passing through the meeting point of the world line and the hypersurface t is assigned to t, i.e. the motion in question is described by the function

$$r_{S,\mathbf{U}} : I_S \to E_{\mathbf{U}}, \qquad t \mapsto C_{\mathbf{U}}\big(r(\mathbf{s}(t))\big). \tag{27}$$

Then, according to the well known formulae of manifolds, the motion of a gyroscopic vector $(r, \mathbf{z})$ with respect to the reference frame $(S, \mathbf{U})$ is described by the function

$$(r_{S,\mathbf{U}}, \mathbf{z}_{S,\mathbf{U}}) : I_S \to T(E_{\mathbf{U}}) \tag{28}$$

where $T(E_{\mathbf{U}})$ is the tangent bundle of $E_{\mathbf{U}}$ and

$$\mathbf{z}_{S,\mathbf{U}}(t) := \mathrm{D}C_{\mathbf{U}}\big(r(\mathbf{s}(t))\big)\mathbf{z}(\mathbf{s}(t)) \tag{29}$$

(see Subsection 2.3).

We consider the exceptional case when the gyroscope rests in the space of a non-inertial observer; i.e. the world line of the gyroscope is a space point of the observer. This will lead us back to giving a precise meaning to the precession principle described in the Introduction.

The gyroscope rests in the space of a noninertial observer, keeping direction in itself, and the observer sees the gyroscope precessing: this is exactly the famous Foucault experiment, therefore we will introduce the terminology Foucault precession for this case. (See Figure 4.) The Foucault precession, a natural fact, is conceptually different from the Thomas precession which is a counterintuitive relativistic phenomenon: an inertial reference frame, when observes a moving gyroscope which keeps direction in itself, sees a precession.

The Foucault precession in a space point of an observer $\mathbf{U}$ is formally defined as follows.

Let us consider a gyroscopic vector $\mathbf{z}$ on the world line function r, where $\dot{r}(\mathbf{s}) = \mathbf{U}(r(\mathbf{s}))$. In this case $r_{S,\mathbf{U}}$ is constant for an arbitrary synchronization. Applying the nearly standard local synchronization S due to r, we use the formulae of Subsection 2.4.

Then (see (29), (18) and (15)) $\mathbf{z}_{S,\mathbf{U}}(\mathbf{s})$ is represented by

$$\mathbf{z}_0(\mathbf{s}) := \mathbf{A}(\mathbf{s})\mathbf{z}(\mathbf{s}) \tag{30}$$

and we infer from (19) that the length of $\mathbf{z}_0(\mathbf{s})$, calculated with respect to the metrics $\gamma_{\mathbf{s}}(r(\mathbf{s}))$, equals the Lorentz length of $\mathbf{z}(\mathbf{s})$ which does not depend on $\mathbf{s}$.

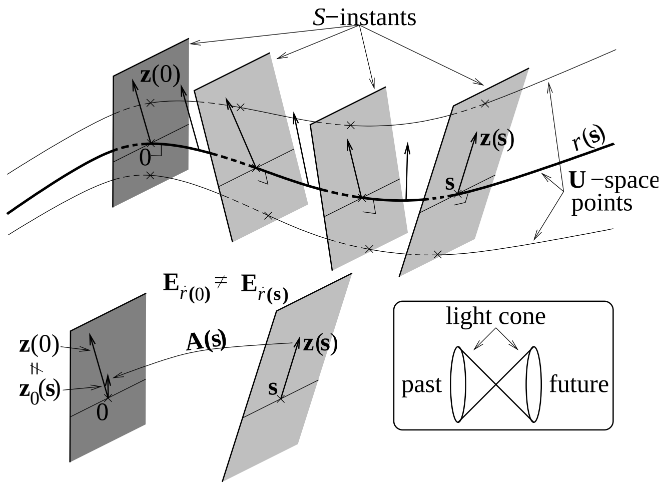

Figure 4: Foucault precession. A noninertial 'co-moving' observer $\mathbf{U}$ perceives a precession of a gyroscopic vector $\mathbf{z}(\mathbf{s})$ whose footpoint "rests" at the space point r of the observer.

Since $\mathbf{A}(\mathbf{s})\dot{r}(\mathbf{s}) = 0$ and $\dot{\mathbf{z}}$ is parallel to $\dot{r}$ (see (21)), we have

$$\dot{\mathbf{z}}_0(\mathbf{s}) = \dot{\mathbf{A}}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1}\mathbf{z}_0(\mathbf{s}). \tag{31}$$

Therefore our candidate for the instantenous angular velocity of the precession is $\dot{\mathbf{A}}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1}$. Thus, the Foucault precession in the $\mathbf{U}$-space point given by r is *meaningful* if and only if for all proper time values $\mathbf{s}$ of r, the restriction of $\dot{\mathbf{A}}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1}$ onto $\mathbf{E}_{\dot{r}(0)}$ (and so mapping into $\frac{\mathbf{E}_{\dot{r}(0)}}{\mathbf{I}}$) is antisymmetric with respect to the Euclidean form $\gamma_{\mathbf{s}}(r(\mathbf{s}))$, i.e. $\big(\mathbf{A}(\mathbf{s})^{-1}\mathbf{q}\big)\cdot\big(\mathbf{A}(\mathbf{s})^{-1}(\dot{\mathbf{A}}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1}\mathbf{q})\big) = 0$ for all $\mathbf{q} \in \mathbf{E}_{\dot{r}(0)}$. Equivalently, $\mathbf{h}\cdot\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{h} = 0$ for all $\mathbf{h} \in \mathbf{E}_{\dot{r}(\mathbf{s})}$ which, in turn, holds if and only if $\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{P}(\mathbf{s})$ is antisymmetric with respect to the Lorentz form.

Note that according to (15) and (16)), we have

$$\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(\mathbf{s})\mathbf{R}(\mathbf{s})^{-1}\dot{\mathbf{R}}(\mathbf{s})\mathbf{P}(\mathbf{s}). \tag{32}$$

In section 4.4 we shall examine whether the Foucault precession in the space of rotating observers is meaningful or not.

Having the notion of Foucault rotation at hand we can furhter investigate the principle described in the introduction. Namely, we will examine the validity and limitations of the following assertions: "the angular velocity of the Foucault precession is always the negative of the angular velocity of the observer" and "having access to the instantenous Foucault angular velocities one can determine the Thomas rotation angle".

4 Rotating observers

4.1 Properties of rotating observers

Heuristically a (uniformly) rotating observer is characterized by the property that its space points are rotating around an inertial centre which is the world line $o + \mathbf{u}\mathbf{I}$, described by a specific point $o \in M$, and an absolute velocity $\mathbf{u} \in \frac{\mathbf{M}}{\mathbf{I}}$. The rotation around the centre, i.e. in the spacelike hyperplane $\mathbf{E_u}$, is characterized by the angular velocity of the rotation, an antisymmetric linear map $0 \neq \mathbf{\Omega} : \mathbf{E_u} \to \frac{\mathbf{E_u}}{\mathbf{I}}$ which is conveniently extended to the whole $\mathbf{M}$ in such a way that $\mathbf{\Omega u} = 0$. Then at an arbitrary point $x \in M$ the velocity of the rotation relative to the centre is proportional to $\mathbf{\Omega}(x-o)$, so $\mathbf{U}(x)$ is the linear combination of $\mathbf{u}$ and $\mathbf{\Omega}(x-o)$. We restrict our attention to the case when the coefficients in the linear combination depend only on $|\mathbf{\Omega}(x-o)|^2$ (and not on x). Thus, we accept that given positive real valued smooth functions $\alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\alpha(|\mathbf{\Omega}(x-o)|^2)^2 - \beta(|\mathbf{\Omega}(x-o)|^2)^2|\mathbf{\Omega}(x-o)|^2 = 1, \tag{33}$$

a corresponding rotating observer is defined as

$$\mathbf{U}(x) = \alpha(|\mathbf{\Omega}(x-o)|^2)\mathbf{u} + \beta(|\mathbf{\Omega}(x-o)|^2))\mathbf{\Omega}(x-o). \tag{34}$$

(The normalization condition (33) ensures that $\mathbf{U}$ does indeed map to the set of absolute velocities.)

For the sake of brevity, let us introduce the notation

$$k(x) := |\mathbf{\Omega}(x-o)|^2. \tag{35}$$

Note the following special cases

1. $\alpha(k(x)) = \beta(k(x)) = \dfrac{1}{\sqrt{1-|\mathbf{\Omega}(x-o)|^2}}$ which is the conventional rotating observer ([11], [7]),

2. $\alpha(k(x)) = \cosh|\mathbf{\Omega}(x-o)|$, $\beta(k(x)) = \dfrac{\sinh|\mathbf{\Omega}(x-o)|}{|\mathbf{\Omega}(x-o)|}$ (the Trocheris-Takeno (TT) observer [17], [15]),

3. $\alpha(k(x)) = \sqrt{1+|\mathbf{\Omega}(x-o)|^2}$, $\beta = 1$ ([7]),

4. $\alpha = \text{const} > 1$, $\beta(k(x)) = \dfrac{\sqrt{\alpha^2-1}}{|\mathbf{\Omega}(x-o)|}$.

It is worth mentioning that rotating observers usually appear in the literature as coordinate transformations, i.e. the observers together with a synchronization. It is interesting to note that the Trocheris-Takeno transformation and the 'modified Trocheris-Takeno transformation' (MTT) ([3]) concern the same observer with different synchronizations. It is also remarkable that finding the 'right' coordinate system describing the reference frame of a rotating observer has been a minor but long-standing problem in special relativity theory (that is why the TT and MTT were introduced, after theorists were not satisfied with the conventional observer). All introduced systems have some undesireable properties. The solution to this problem is simply that there is no 'right' coordinate system: one can list the desired properties of such coordinatization, and prove that they cannot be satisfied simultaneously.

All the **U**-space points are circular world lines. In particular, the one passing through the world point x is given by the function

$$\mathbf{t} \mapsto o + \mathbf{t}\alpha(k(x))\mathbf{u} + e^{\mathbf{t}\beta((k(x))\mathbf{\Omega}}(x - o) =: R(\mathbf{t}, x). \tag{36}$$

We shall use the following formulae:

$$\frac{dk(x)}{dx} = \frac{d|\mathbf{\Omega}(x-o)|^2}{dx} = -2\mathbf{\Omega}^2(x-o), \tag{37}$$

$$\frac{d\alpha(k(x))}{dx} = -2\alpha'(k(x))\mathbf{\Omega}^2(x-o), \tag{38}$$

$$\frac{d\beta(k(x))}{dx} = -2\beta'(k(x))\mathbf{\Omega}^2(x-o), \tag{39}$$

where the prime denotes differentiation with respect to the real variable of the functions. Moreover, we infer from (33) that

$$2\alpha(k)\alpha'(k) - 2\beta(k)\beta'(k)k = \beta^2(k). \tag{40}$$

4.2 Angular velocity of a rotating obvserver

As a consequence of the previous formulae,

$$\mathrm{D}\mathbf{U}(x) = -2\big(\alpha'(k(x))\mathbf{u} + \beta'(k(x))\mathbf{\Omega}(x-o)\big) \otimes \mathbf{\Omega}^2(x-o) + \beta(k(x))\mathbf{\Omega}, \tag{41}$$

so

$$-\frac{1}{2}\mathrm{D}\wedge\mathbf{U}(x) = -\big(\alpha'(k(x))\mathbf{u} + \beta'(k(x))\mathbf{\Omega}(x-o)\big) \wedge \mathbf{\Omega}^2(x-o) + \beta(k(x))\mathbf{\Omega}. \tag{42}$$

Then, taking into account that $\mathbf{\Omega}\mathbf{U}(x) = -\mathbf{U}(x)\mathbf{\Omega} = \beta(k(x))\mathbf{\Omega}^2(x-o)$ and $\mathbf{U}(x) \cdot \mathbf{\Omega}^2(x-o) = 0$, we can calculate the angular velocity of the rotating observer according to (20):

$$\mathbf{\Omega}_{\mathbf{U}}(x) = \beta\mathbf{\Omega} - \left(\left(\frac{\alpha\beta^2}{2} + \alpha'\right)\mathbf{u} + \left(\frac{\beta^3}{2} + \beta'\right)\mathbf{\Omega}(x-o)\right) \wedge \mathbf{\Omega}^2(x-o) \tag{43}$$

where, for the sake of brevity, we have written α instead of $\alpha(k(x))$ etc.

4.3 Representation of the space of a rotating observer

Now we apply the formulae of Subsection 2.4 to a rotating observer by considering a fixed integral curve

$$r(\mathbf{s}) = o + \mathbf{s}\alpha_0\mathbf{u} + e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{d}, \qquad \alpha_0 := \alpha(|\mathbf{\Omega}\mathbf{d}|^2), \qquad \beta_0 := \beta(|\mathbf{\Omega}\mathbf{d}|^2) \tag{44}$$

of the uniformly rotating observer (34), determined by the initial point $o+\mathbf{d}\in M$, where $\mathbf{d}\in\mathbf{E_u}$. Then

$$\dot r(\mathbf{s})=\alpha_0\mathbf{u}+\beta_0 e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{\Omega}\mathbf{d},\qquad \text{so}\qquad \dot r(0)=\alpha_0\mathbf{u}+\beta_0\mathbf{\Omega}\mathbf{d},\tag{45}$$

$$\ddot r(\mathbf{s})=\beta_0^2 e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{\Omega}^2\mathbf{d},\qquad \text{so}\qquad \ddot r(0)=\beta_0^2\mathbf{\Omega}^2\mathbf{d}.\tag{46}$$

Next we calculate the actual form of the linear maps defined in (11) and (15). The flow of the rotating observer is given in (36), therefore we find that

$$\frac{\partial R(-\mathbf{s},x)}{\partial x}=2\mathbf{s}\left(\alpha'\mathbf{u}+\beta' e^{-\mathbf{s}\beta\mathbf{\Omega}}\mathbf{\Omega}(x-o)\right)\otimes\mathbf{\Omega}^2(x-o)+e^{-\mathbf{s}\beta\mathbf{\Omega}},\tag{47}$$

where again α means $\alpha(k(x))$ etc. For $x=r(\mathbf{s})$ we have $\alpha(k(x))=\alpha_0$, $\alpha'(k(x))=\alpha'(|\mathbf{\Omega}\mathbf{d}|^2)=:\alpha'_0$ etc, $\mathbf{\Omega}(x-o)=e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{\Omega}\mathbf{d}$. Therefore, (see (11))

$$\mathbf{R}(\mathbf{s})=2\mathbf{s}\left(\alpha'_0\mathbf{u}+\beta'_0\mathbf{\Omega}\mathbf{d}\right)\otimes e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{\Omega}^2\mathbf{d}+e^{-\mathbf{s}\beta_0\mathbf{\Omega}}.\tag{48}$$

Now using (15) and (40) leads to

$$\mathbf{A}(\mathbf{s})=\mathbf{P}(0)e^{-\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{P}(\mathbf{s})+\mathbf{s}\big((2\alpha'_0-\alpha_0\beta_0^2)\mathbf{u}+(2\beta'_0-\beta_0^3)\mathbf{\Omega}\mathbf{d}\big)\otimes e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{\Omega}^2\mathbf{d}.\tag{49}$$

4.4 Foucault precession in a space point of a rotating observer

Now let us examine whether the Foucault precession in a space point of a rotating observer $\mathbf{U}$ is meaningful or not. We shall apply the formulae of Subsection 3.2 for the world line function r given by (44). The meaningfulness of the Foucault precession requires that $\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{P}(\mathbf{s})=\mathbf{P}(\mathbf{s})\mathbf{R}(\mathbf{s})^{-1}\dot{\mathbf{R}}(\mathbf{s})\mathbf{P}(\mathbf{s})$ be antisymmetric with respect to the Lorentz form.

First we consider the case $\mathbf{s}=0$; then

$$\dot{\mathbf{R}}(0)=-\beta_0\mathbf{\Omega}+2(\alpha'_0\mathbf{u}+\beta'_0\mathbf{\Omega}\mathbf{d})\otimes\mathbf{\Omega}^2\mathbf{d}.\tag{50}$$

Consequently, using (40) and $\mathbf{R}(0)=1$, we obtain

$$\begin{aligned}&\mathbf{P}(0)\mathbf{R}(0)^{-1}\dot{\mathbf{R}}(0)\mathbf{P}(0)=\\&\qquad=-\beta_0\mathbf{P}(0)\mathbf{\Omega}\mathbf{P}(0)+((2\alpha'_0-\alpha_0\beta_0^2)\mathbf{u}+(2\beta'_0-\beta_0^3)\mathbf{\Omega}\mathbf{d})\otimes\mathbf{\Omega}^2\mathbf{d}.\end{aligned}\tag{51}$$

The expression on the right hand side is antisymmetric if and only if the second term is antisymmetric, i.e. with an abbreviated notation $(a\mathbf{u}+b\mathbf{\Omega}\mathbf{d})\otimes\mathbf{\Omega}^2\mathbf{d}=-\mathbf{\Omega}^2\mathbf{d}\otimes(a\mathbf{u}+b\mathbf{\Omega}\mathbf{d})$. Applying both sides to the vectors $\mathbf{u}$ and $\mathbf{\Omega}\mathbf{d}$, we see that a and b are necessarily zero:

$$2\alpha'_0=\alpha_0\beta_0^2\qquad 2\beta'_0=\beta_0^3.\tag{52}$$

If this holds, then

$$\mathbf{A}(\mathbf{s})=\mathbf{P}(0)e^{-\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{P}(\mathbf{s})=\mathbf{P}(0)e^{-\mathbf{s}\beta_0\mathbf{\Omega}}=e^{-\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{P}(\mathbf{s}),\tag{53}$$

$$\mathbf{A}(\mathbf{s})^{-1} = \mathbf{P}(\mathbf{s})e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{P}(0) = e^{\mathbf{s}\beta_0\mathbf{\Omega}}\mathbf{P}(0) = \mathbf{P}(\mathbf{s})e^{\mathbf{s}\beta_0\mathbf{\Omega}} \tag{54}$$

and $\dot{\mathbf{A}}(\mathbf{s}) = -\beta_0\mathbf{P}(0)\mathbf{\Omega}e^{-\mathbf{s}\beta_0\mathbf{\Omega}}$. Therefore,

$$\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{P}(\mathbf{s}) = -\beta_0\mathbf{P}(\mathbf{s})\mathbf{\Omega}\mathbf{P}(\mathbf{s}) \tag{55}$$

which is evidently Lorentz antisymmetric for all $\mathbf{s}$. Thus, the Foucault precession in a spacepoint of a rotating observer is meaningful if and only if (52) is satisfied.

Since $\alpha_0 = \alpha(|\mathbf{\Omega}\mathbf{d}|^2)$ etc., and $\mathbf{d}$ can be arbitrary, the equalities in (52) hold for all real variables of the functions, i.e. we have the differential equations

$$2\alpha' = \alpha\beta^2 \qquad 2\beta' = \beta^3. \tag{56}$$

We can solve the second equation for β, and then taking into account (33), we find that there is a positive constant h such that

$$\alpha(k(x)) = \frac{1}{\sqrt{1-h^2|\mathbf{\Omega}(x-o)|^2}}, \qquad \beta(k(x)) = \frac{h}{\sqrt{1-h^2|\mathbf{\Omega}(x-o)|^2}}. \tag{57}$$

Therefore, we conclue that the Foucault precession in the space points of the rotating observers 2, 3 and 4 listed in Section 4 is not meaningful.

The Foucault precesion is meaningful for the rotating observer 1 (for $h = 1$). Then the angular velocity of the Foucault precession at the observer space point given by o and $\mathbf{d}$ is (see (55))

$$\mathbf{A}(0)^{-1}\dot{\mathbf{A}}(0) = -\beta_0\mathbf{\Omega} + \beta_0^2(\alpha_0\mathbf{u} + \beta_0\mathbf{\Omega}\mathbf{d}) \wedge \mathbf{\Omega}^2\mathbf{d}. \tag{58}$$

Since o is an arbitrary world point of the central world line of the observer and $\mathbf{d}$ is an arbitrary vector in $\mathbf{E_u}$, we get the angular velocity of the Foucault precession at the world point x by replacing α_0 etc. with $\alpha(k(x))$ etc.:

$$-\beta(k(x))\mathbf{\Omega} + \beta(k(x))^2\big(\alpha(k(x))\mathbf{u} + \beta(k(x))\mathbf{\Omega}(x-o)\big) \wedge \mathbf{\Omega}^2(x-o) \tag{59}$$

which is opposite to the angular velocity of the observer at x (see (43)).

5 Counterexamples

5.1 A single world line has no angular velocity

One can be tempted to say that the circular world line (44) has angular velocity $\beta_0\mathbf{\Omega}$. The angular velocity of a single world line, however, cannot be defined: we shall show that the same world line can be a space point of different observers with different angular velocities.

Let r be an arbitrary world line function. Let $\mathbf{s} \mapsto \mathbf{H}(\mathbf{s})$ be a continuously differentiable map such that $\mathbf{H}(\mathbf{s})$ is Lorentz transformation for which $\mathbf{H}(\mathbf{s})\dot{r}(\mathbf{s}) = \dot{r}(0)$ holds. ($\mathbf{H}(\mathbf{s})$ can be for example the Lorentz boost from $\dot{r}(\mathbf{s})$ to $\dot{r}(0)$ or the Fermi-Walker transport along r from $\mathbf{s}$ to 0.) Given such a family $\mathbf{H}(\mathbf{s})$, and an antisymmetric linear map $\Gamma : \mathbf{M} \to \frac{\mathbf{M}}{\mathbf{I}}$ for which $\Gamma \cdot \dot{r}(0) = 0$, the associated family $\mathbf{H}_\Gamma(\mathbf{s}) := e^{\mathbf{s}\Gamma}\mathbf{H}(\mathbf{s})$ is another good choice, so we have some freedom when choosing $\mathbf{H}(\mathbf{s})$.

Taking the nearly standard synchronization instant $\mathbf{s}(x)$ of the world point x (in a neighbourhood of the world line) determined by (3) and putting

$$\mathbf{V}(x) := \dot{r}(\mathbf{s}(x)) - \mathbf{H}(\mathbf{s}(x))^{-1}\dot{\mathbf{H}}(\mathbf{s}(x))\big(x - r(\mathbf{s}(x))\big), \tag{60}$$

we define the observer

$$\mathbf{U}(x) := \frac{\mathbf{V}(x)}{|\mathbf{V}(x)|} \tag{61}$$

where, of course, $|\mathbf{V}| = \sqrt{-\mathbf{V}\cdot\mathbf{V}}$.

Then $\mathrm{D}\mathbf{U} = \frac{\mathrm{D}\mathbf{V}}{|\mathbf{V}|} + \frac{\mathbf{V}\otimes(\mathrm{D}\mathbf{V})\mathbf{V}}{|\mathbf{V}|^3}$, and $\mathbf{V}(r(\mathbf{s})) = \dot{r}(\mathbf{s})$, $|\mathbf{V}(r(\mathbf{s})| = 1$, $\mathrm{D}\mathbf{V}(r(\mathbf{s})) = \ddot{r}(\mathbf{s})\otimes\dot{r}(\mathbf{s}) - \mathbf{H}(\mathbf{s})^{-1}\dot{\mathbf{H}}(\mathbf{s})(1 + \dot{r}(\mathbf{s})\otimes\dot{r}(\mathbf{s}))$ and $(\mathrm{D}\mathbf{V}(r(\mathbf{s}))V(r(\mathbf{s}) = -\ddot{r}(\mathbf{s})$. Since $\mathbf{H}(\mathbf{s})^{-1}\dot{\mathbf{H}}(\mathbf{s})$ is antisymmetric, we find that the angular velocity of the observer $\mathbf{U}$ (see (20)) at $r(\mathbf{s})$ is

$$-\mathbf{P}(\mathbf{s})\mathbf{H}(\mathbf{s})^{-1}\dot{\mathbf{H}}(\mathbf{s})\mathbf{P}(\mathbf{s}). \tag{62}$$

In particular, if $\mathbf{H}(\mathbf{s})$ is the Fermi-Walker transport along r from $\mathbf{s}$ to 0, then $\mathbf{H}(\mathbf{s})^{-1}\dot{\mathbf{H}}(\mathbf{s}) = \dot{r}(\mathbf{s})\wedge\ddot{r}(\mathbf{s})$ and the angular velocity of the observer at $r(\mathbf{s})$ is zero.

We emphasize what this means: *the properties of a world line are in no relation with the angular velocity of an observer having the world line as a space point.*

Therefore, the circular world line (44) can be the space point of several observers having different angular velocities; in partiular, it can be the space point of an observer whose angular velocity at the world points of that circular world line is zero.

5.2 Thomas rotation versus Foucault precession

Now we investigate the relation of the Foucault precession and the angle of Thomas rotation.

First we demonstrate that, for any choice of $\mathbf{H}(\mathbf{s})$, the Foucault precession of the observer (61) in the space point given by r is meaningful.

It is easy to see that the function $\rho(\mathbf{s}) := r(\mathbf{s}) + \mathbf{H}(\mathbf{s})^{-1}\mathbf{h}$ for $\mathbf{h} \in \mathbf{E}_{\dot{r}(0)}$ satisfies $\dot{\rho}(\mathbf{s}) = \mathbf{V}(\rho(\mathbf{s}))$. Thus the range of ρ is a $\mathbf{U}$-line (ρ parameterizes a $\mathbf{U}$-line by the proper time of r). As a consequence, using the notations of Subsection 2.4, for an arbitrary world point x, taking $\mathbf{h} := R(\mathbf{t}(x), x) - r(0)$ and $\mathbf{s} := \mathbf{s}(x)$, we get $x = r(\mathbf{s}(x)) + \mathbf{H}(\mathbf{s}(x))^{-1}(R(\mathbf{t}(x), x) - r(0))$, i.e.

$$R(\mathbf{t}(x), x) = r(0) + \mathbf{H}(\mathbf{s}(x))(x - r(\mathbf{s}(x))) \tag{63}$$

from which we obtain by (4) that

$$\mathbf{A}(\mathbf{s}) = \frac{dR(\mathbf{t}(x), x)}{dx}\Big|_{x=r(\mathbf{s})} = \mathbf{H}(\mathbf{s})\mathbf{P}(\mathbf{s}). \tag{64}$$

Then it follows from $\mathbf{H}(\mathbf{s})\dot{r}(\mathbf{s}) = \dot{r}(0)$ that

$$\mathbf{A}(\mathbf{s}) = \mathbf{H}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(0)\mathbf{H}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(0)\mathbf{H}(\mathbf{s}) \tag{65}$$

and

$$\mathbf{A}(\mathbf{s})^{-1} = \mathbf{H}(\mathbf{s})^{-1}\mathbf{P}(0) = \mathbf{P}(\mathbf{s})\mathbf{H}(\mathbf{s})^{-1}\mathbf{P}(0) = \mathbf{P}(\mathbf{s})\mathbf{H}(\mathbf{s})^{-1}. \tag{66}$$

Consequently,

$$\mathbf{A}(\mathbf{s})^{-1}\dot{\mathbf{A}}(\mathbf{s})\mathbf{P}(\mathbf{s}) = \mathbf{P}(\mathbf{s})\mathbf{H}(\mathbf{s})^{-1}\dot{\mathbf{H}}(\mathbf{s})\mathbf{P}(\mathbf{s}), \tag{67}$$

which is evidently Lorentz antisymmetric.

We obtained that the Foucault precession of the observer (61) in the space point given by r is meaningful and its angular velocity at the proper time value $\mathbf{s}$ equals the negative of the angular velocity of the observer (62). The authors conjecture that the following general statement is true: whenever the Foucault precession in a space point of an observer is meaningful, it equals the negative of the angular velocity of the observer. We also conjecture that the Foucault conjecture is meaningful if and only if the observer is (locally) rigid. (We do not formally define rigidity here, but heuristically it simply means that the distance between any two space-points of the observer does not change in time, i.e. its spatial metric is time-independent, at least locally.) These two general statements would justify the first part of the principle of [14] for rigid observers. We note, however, that in the calculations above only some very special cases were considered.

Note also that the gyroscope whose centre is the world line given by r shows different Foucault precessions with respect to different observers (the same gyroscope, whose centre rests in different observers' spaces, shows different Foucalt precessions with respect to the observers; this is not surprising, having seen that different observers, having r as a space point, may have different angular velocities).

Now, if the Foucault precession is meaningful, then the time-integral of the Foucault precession results in a finite angle and we can investigate the connection between such a Foucault angle and the angle of Thomas rotation. By some examples we will show that, in general, they differ from each other.

First, let us consider a trivial example. Take the world line function given in (44) with $\mathbf{d} = 0$ (a space point of the axis of rotation). Then a gyroscopic vector $\mathbf{z}$ on the inertial world line function $\mathbf{s} \mapsto o + \mathbf{u}\mathbf{s}$ is constant, so the Thomas rotation is meaningful for every proper time value $\mathbf{s}$ of r and equals the identity map (no rotation occurs). On the other hand, the Foucault precession in that space point of the conventional rotating observer has angular velocity $-\mathbf{\Omega}$, so $e^{-\mathbf{s}\mathbf{\Omega}}$ is the rotation arising from the Foucault precession.

More generally, we have shown that the world line function given by (44) with $\mathbf{d} \neq 0$ can be a space point of different observers (61) with different meaningful Foucault precessions.

According to (66), the Euclidean form (19) is the restriction of the Lorentz form for all $\mathbf{s}$. Thus it is meaningful that, after the time period $\mathbf{s}$, the Foucualt precession results in an angle whose cosine is $\frac{\mathbf{z}_0(0)\cdot\mathbf{z}_0(\mathbf{s})}{|\mathbf{z}_0(0)|^2}$, $\mathbf{z}_0$ being the solution of the differential equation (31).

For the conventional rotating observer

$$\dot{\mathbf{A}}(\mathbf{s})\mathbf{A}(\mathbf{s})^{-1} = -\beta_0\mathbf{\Omega} + \beta_0^2\big(\alpha_0\mathbf{u} + \beta_0\mathbf{\Omega}\mathbf{d}\big)\wedge\mathbf{\Omega}^2\mathbf{d} = \\ = -\beta_0\mathbf{\Omega} + \dot{r}(0)\wedge\ddot{r}(0) =: \mathbf{\Omega}_r \quad (68)$$

is independent of $\mathbf{s}$, thus $\mathbf{z}_0(\mathbf{s}) = e^{\mathbf{s}\mathbf{\Omega}_r}\mathbf{z}_0$, therefore the angle in question is $\mathbf{s}|\mathbf{\Omega}_r|$ where $|\mathbf{\Omega}_r| = -\frac{1}{2}\mathrm{Tr}(\mathbf{\Omega}_r^2)$. We find (recall that now $\alpha = \beta$) that

$$|\mathbf{\Omega}_r| = \frac{\beta_0\omega}{\sqrt{1-\omega^2|\mathbf{d}|^2}}; \quad (69)$$

as a consequence, the Foucault angle for $\mathbf{s} = \frac{2\pi}{\beta_0\omega}$ – after a whole revolution – equals the angle of the Thomas rotation (see [14], [10]).

However, for other rotating observers other Foucault angles are obtained after a whole revolution. Namely, let Γ be a Lorentz antisymmetric map for which $\Gamma\dot{r}(0) = 0$ holds and $\mathbf{H}(\mathbf{s}) := e^{\mathbf{s}\Gamma}e^{-\mathbf{s}\beta_0\mathbf{\Omega}}$. Then $\mathbf{z}_0(\mathbf{s}) = e^{\mathbf{s}\Gamma}e^{\mathbf{s}\mathbf{\Omega}_r}\mathbf{z}_0$ and the Foucault angles are different for different Γ-s.

The reason behind these examples seem to be that although the world line r returns to its initial local rest frame, i.e. $\dot{r}(0) = \dot{r}(\mathbf{s}_1)$, the velocity field given by the observer does not (except for the one spacepoint given by r). On the other hand, it is obvious that if the mapping $\mathbf{A}(\mathbf{s}_1)$ is the identity (which happens to be the case for the conventional rotating observer), then the measured angle will indeed give the Thomas rotation angle.

6 Conclusion

In conclusion we can say that there are some limitations in applying the principle of relating the Thomas rotation angle to the angular velocity of a co-moving observer.

First, the angular velocity of the observer does not always equal the negative of the angular velocity of the Foucault precession, because the latter might not even be meaningful. (On the other hand, in our examples whenever the Foucault precession made sense, it was also equal to the negative of the angular velocity of the observer. We conjecture that such is the case for all locally rigid observers.)

Secondly, even if the Foucault precession is meaningful, the Foucault angle after a whole revolution will not necessarily give the angle of Thomas rotation.

Thus we can see why the correct Thomas rotation angle emerged for the conventional rotation observer, and what went wrong in the cases of Trocheris-Takeno and modified-Trocheris-Takeno observers in [4].

References

[1] Costella J.P., McKellar B.H.J., Rawlinson A.A. and Stephenson G.J., *Am. J. of Phys.* **69**, 837, (2001)

[2] Fisher G.P. *Am.J.Phys.* **40**, 1772, (1972)

[3] L. Herrera, *Nuovo Cimento B*, **115**, 37 (2000)

[4] Herrera L. and di Prisco A. *Found.Phys.Lett.* **15**, 373, (2002)

[5] W. L. Kennedy *Eur. J. Phys.* **23** 235, (2002)

[6] T. Matolcsi *A Concept of Mathematical Physics, Models for Spacetime*, Akadémiai Kiadó Budapest, (1984)

[7] T. Matolcsi *Spacetime without Reference Frames*, Akadémiai Kiadó Budapest, (1993)

[8] T. Matolcsi *Found. Phys.* **27**, 1685, (1998)

[9] T. Matolcsi *Stud.Hist.Phil.Mod.Phys.* **32**, 83, (2001)

[10] T. Matolcsi and M. Matolcsi, *Int. J. Theo Phys.* **44, no.1**, 63, (2005)

[11] M.C. Møller, *The Theory of Relativity* 2.ed. Oxford, Clarendon Press (1972)

[12] Philpott R.J. *Am. J. Phys.* **64**, 552, (1996)

[13] Rebilas K. *Am. J. Phys.* **70**, 1163, (2002)

[14] Rindler W. and Perlick V. *Gen.Rel.Grav.* **22**, 1067, (1990)

[15] H. Takeno, *Prog.Theor.Phys.*, **7**, 367, (1952)

[16] Thomas L. H., *Phil. Mag.* **3**, 1, (1927)

[17] M. Trocheris, *Phil. Mag.* **40**, 1143, (1949)

[18] Ungar A. A. *Found. Phys.* **19**, 1385, (1989)

[19] D.C. Wilkins, *Ann. Phys. (N.Y.)* **61**, 277 (1970).