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Spacetime without Reference Frames: An Application to Synchronizations on a Rotating Disk

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Nonstandard synchronizations of inertial observers in special relativity and synchronizations with respect to a uniformly rotating observer are investigated in a setting which avoids coordinates and transformation rules and so removes some misunderstandings.

1. INTRODUCTION

It has long been known that the standard synchronization corresponding to inertial observers in special relativity can be replaced by another in which the one-way speed of light is not supposed to equal its two-way speed,(1) i.e., if a reflected light signal starts from a space point at t_s and arrives back at t_a, the space point of reflection at t_r, then

$$t_r = t_s = \alpha(t_s - t_a) \tag{1}$$

where $\alpha \in [0, 1]$, the synchronization parameter is arbitrary. If c is the two-way light speed, then the starting light speed is $c_s = c/(2\alpha)$ and the arriving light speed is $c_a = c/[2(1-\alpha)]$.

Recently several papers(2–5) have dealt with nonstandard synchronization, extending the investigations to synchronization on the rotating disk. As a result of these examinations it was claimed that the nonstandard synchronization, instead of being a possibility, is a necessity, and absolute simultaneity must be introduced in special relativity.

The arguments are expounded in the usual framework using coordinates and transformation rules. However, coordinates and transformation rules are not inherent objects of spacetime and their use can mislead us. An

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excellent example for such an error is that the usual formulation with coordinates *suggests tacitly that it has an a priori meaning* that two vectors in the spaces of different observers be equal, and under this tacit assumption it has been taken for granted that being equal in different spaces is a transitive relation; this has resulted in the paradox of velocity addition.(6)

Using a formalism of special relativity which is built up without reference frames, coordinates, or transformation rules,(7) we can solve the velocity addition paradox.(8) The same formalism will be applied to investigate nonstandard synchronizations and the synchronization on the rotating disk, and it will be shown that some of the conclusions of the cited papers are erroneous.

As an introductory remark to the absolute (without reference frames) formulation of spacetime, we note that the frequently stated assertion that special relativity is the theory of inertial reference frames and general relativity is the theory of arbitrary reference frames(9) is to be substituted with the one that general relativity describes gravitation and special relativity concerns the lack of gravitation.(10, 11, 7) It is evident nowadays that the *mathematical structure of spacetime can* (*and must*) *be formulated without reference frames*.(12–14) A *general relativistic spacetime model* is a triplet $(M, \mathbf{I}, g)$, where M is a four-dimensional smooth manifold, $\mathbf{I}$ is the measure line of spacetime distances, and g is an $\mathbf{I}\otimes\mathbf{I}$-valued Lorentz form on M(12) (usually one takes real-valued Lorentz forms, but it is obvious that spacetime distances are not real numbers). A *special relativistic spacetime model* is a particular general relativistic one in which M is an affine space and g is constant.(7)

2. SPACETIME WITHOUT REFERENCE FRAMES

Now we recapitulate briefly the fundamental notions and results of the formalism given in Ref. 7.

2.1. Spacetime Model

A special relativistic spacetime model is a triplet $(M, \mathbf{I}, g)$ where:

- M is an oriented four-dimensional affine space, i.e., there is a four-dimensional vector space $\mathbf{M}$ such that to each pair (x, y) of points of M a vector in $\mathbf{M}$, denoted by $x - y$ is assigned.
- $\mathbf{I}$ is an oriented one-dimensional vector space.
- $g\colon \mathbf{M}\times\mathbf{M}\to\mathbf{I}\otimes\mathbf{I}$ is a Lorentz form endowed with an arrow orientation.

Instead of g we shall write a dot product, i.e., $\boldsymbol{x} \cdot \boldsymbol{y} := g(\boldsymbol{x}, \boldsymbol{y})$.

A vector $0 \neq \boldsymbol{x} \in \mathbf{M}$ is called spacelike, timelike, and lightlike if $\boldsymbol{x} \cdot \boldsymbol{x} > 0$, $\boldsymbol{x} \cdot \boldsymbol{x} < 0$, and $\boldsymbol{x} \cdot \boldsymbol{x} = 0$, respectively. The timelike and lightlike vectors are either future directed or past directed, according to the arrow orientation of g.

2.2. Absolute Velocities

The history of a classical mass point is described in $(M, \mathbf{I}, g)$ by a world line, which is a curve whose tangent vectors are timelike.

If x and y are points on a world line C, then the *time passed along C* between x and y is

$$t_C(x, y) := \int_{p^{-1}(x)}^{p^{-1}(y)} \sqrt{-\dot{p}(t) \cdot \dot{p}(t)}\, dt \tag{2}$$

where p is an arbitrary parametrization of C.

Using the time passed along the world line (the proper time of the world line) as a parameter, we get a *world line function* $r: \mathbf{I} \to M$ which is a parametrization of the world line and $\dot{r}(\boldsymbol{s}) \in V(1)$ for all $\boldsymbol{s} \in \mathbf{I}$, where

$$V(1) := \left\{ \boldsymbol{u} \in \frac{\mathbf{M}}{\mathbf{I}} \,\middle|\, \boldsymbol{u} \cdot \boldsymbol{u} = -1,\ \boldsymbol{u} \text{ is future directed} \right\} \tag{3}$$

is the set of *absolute velocities* (corresponding to four-velocities in the usual nomenclature).

A *light signal* is a curve whose tangent vectors are lightlike.

2.3. Observers

First we have to make some comments on the nomenclature because the term observer and reference frame is used in several senses in the literature. Frequently observer means a single world line[10] and reference frame refers to a collection of world lines,[10] but observer, too, can refer to a collection of world lines[11, 13] and reference frame can involve implicitly or explicitly coordinates or a basis ("tetrad").[10, 11, 14] Since world line is a customary notion, there is no need to rename it observer, so a collection of world lines is accepted to be an observer and the name reference frame will be retained for observers with chosen coordinates (Ref. 7, II.9).

A physical observer is a material object consisting of mass points; accordingly, it would be described by a collection of world lines. However, it is more convenient to give it by the tangent vectors of the corresponding world lines, so we accept that

- An *observer* is a smooth vector field $\boldsymbol{U}: M \to V(1)$.

The integral curves of such a vector field are world lines, representing the histories of the material points that the observer is constituted of; thus it is quite evident that an integral curve of $\boldsymbol{U}$ is a space point of the observer; the set $E_{\boldsymbol{U}}$ of the (maximal) integral curves of $\boldsymbol{U}$ is the *space* of the observer, briefly the ***U**-space*.

This is the most important—but trivial—fact concerning observers: *a space point of an observer is a line in spacetime*.

For every spacetime point x there is a unique $\boldsymbol{U}$-space point $q_{\boldsymbol{U}}(x)$ containing x.

Observers and their spaces in spacetime are well defined simple and straightforward notions. The spaces of different observers are evidently different.

The time of an observer, however, is not meaningful, in general.

2.4. Synchronizations

Time in spacetime is given by a synchronization whose physical meaning is that one determines by some procedure which spacetime points are to be considered simultaneous. Accordingly,

- A *synchronization* or *simultaneity* is a smooth equivalence relation on M such that the equivalence classes are connected three-dimensional smooth submanifolds (hypersurfaces) whose tangent spaces are spacelike (Ref. 7, II.6.).

Given a synchronization S, an equivalence class is called *S-instant*; the set I_S of S instance is called *S-time*.

This is the most important—but trivial—fact concerning synchronizations: *a time point* (*instant*) *in a synchronization is a hypersurface in spacetime*.

For every spacetime point x there is a unique S-instant $\tau_S(x)$ containing x.

Smoothness of a synchronization S means the following: the tangent space of $\tau_S(x)$ at x is three dimensional and spacelike, so there is a unique $\boldsymbol{U}_S(x) \in V(1)$, g-orthogonal to that tangent space; we require that the assignment $x \mapsto \boldsymbol{U}_S(x)$ be smooth. Thus a synchronization S determines a unique observer $\boldsymbol{U}_S$.

On the contrary, not every observer determines a synchronization. We call an observer $\boldsymbol{U}$ *regular* if there is a (necessarily unique) synchronization S, called the $\boldsymbol{U}$-synchronization, such that $\boldsymbol{U} = \boldsymbol{U}_S$.

It is worth mentioning that it follows from a well-known theorem in differential geometry[15] that an observer $\boldsymbol{U}$ is regular if and only if for all smooth vector fields $\boldsymbol{X}, \boldsymbol{Y}: M \to \mathbf{M}$ such that $\boldsymbol{U} \cdot \boldsymbol{X} = \boldsymbol{U} \cdot \boldsymbol{Y} = 0$ we have $\boldsymbol{U} \cdot [\boldsymbol{X}, \boldsymbol{Y}] = 0$.

An observer $\boldsymbol{U}$ and a synchronization S define the *splitting* of spacetime into S-time and $\boldsymbol{U}$-space, which means that the corresponding S-instants and $\boldsymbol{U}$-space points are assigned to spacetime points:

$$M \to I_S \times E_{\boldsymbol{U}}, \qquad x \mapsto (\tau_S(x), q_{\boldsymbol{U}}(x)) \tag{4}$$

2.5. Motions and Relative Velocities

The *history* of a material point—a world line C—is perceived by an observer $\boldsymbol{U}$ as a *motion* which is described with the aid of a synchronization S as a function which assigns $\boldsymbol{U}$-space points to S-instants.

The world line C meets every hypersurface $t \in I_S$ at most in one point; let $C * t$ denote this point of intersection. The unique $\boldsymbol{U}$-space point passing through this point is assigned to t, i.e., the motion in question is described by the function

$$I_S \to E_{\boldsymbol{U}}, \qquad t \mapsto q_{\boldsymbol{U}}(C * t) \tag{5}$$

Since $\boldsymbol{U}$ and S are smooth, I_S and $E_{\boldsymbol{U}}$ can be made a one-dimensional and a three-dimensional smooth manifold, respectively. Then the derivative of the above function is the *relative velocity* of the material point with respect to the observer and the synchronization.

The same can be said if C is a light signal.

It is emphasized that *the relative velocity of a material point or a light signal with respect to an observer has a meaning only if a synchronization is given and it depends on the synchronization.*

To avoid misunderstandings, we call attention that in this paper relative velocity means a vector; its magnitude will be called speed.

2.6. Inertial Observers and Standard Synchronizations

An observer having constant value is called *inertial.* An inertial observer will be referred to by its constant value.

The space points—the integral curves—of an inertial observer with value $\boldsymbol{u} \in V(1)$ are straight lines parallel to $\boldsymbol{u}$.

Inertial observers are regular; the corresponding synchronization is the standard one determined by light signals with the assumption that light propagates isotropically with respect to the observer, i.e., the one-way speed of light is the same in all directions. According to the standard synchronization determined by $\boldsymbol{u}$, the spacetime points x and y are $\boldsymbol{u}$-simultaneous if and only if $\boldsymbol{u}\cdot(x-y)=0$. Thus putting

$$\mathbf{E}_{\boldsymbol{u}} := \{\boldsymbol{x}\in\mathbf{M} \mid \boldsymbol{u}\cdot\boldsymbol{x}=0\} \tag{6}$$

we have that the instants of the standard synchronization corresponding to the observer $\boldsymbol{u}$ are hyperplanes parallel to $\mathbf{E}_{\boldsymbol{u}}$; their set, called the $\boldsymbol{u}$-time, will be denoted by $I_{\boldsymbol{u}}$.

Let $t_1, t_2 \in I_{\boldsymbol{u}}$. It is quite trivial that the same time passes on every $\boldsymbol{u}$-space point between t_1 and t_2, namely $-\boldsymbol{u}\cdot(x_2-x_1)$, where x_1 and x_2 are arbitrary elements of t_1 and t_2, respectively. Thus $I_{\boldsymbol{u}}$, endowed with the subtraction

$$t_2-t_1 := -\boldsymbol{u}\cdot(x_2-x_1) \qquad (x_2\in t_2,\ x_1\in t_1) \tag{7}$$

becomes a one dimensional affine space over $\mathbf{I}$.

The $\boldsymbol{u}$-space vector between the $\boldsymbol{u}$-space points $q_1, q_2 \in E_{\boldsymbol{u}}$ is defined as the spacetime vector between $\boldsymbol{u}$-simultaneous points of the straight lines q_1 and q_2, respectively. Thus $E_{\boldsymbol{u}}$, endowed with the subtraction

$$q_2-q_1 := x_2-x_1+\boldsymbol{u}(\boldsymbol{u}\cdot(x_2-x_1)) \qquad (x_2\in q_2,\ x_1\in q_1) \tag{8}$$

becomes a three dimensional affine space over $\mathbf{E}_{\boldsymbol{u}}$.

3. NONSTANDARD SYNCHRONIZATIONS

Nonstandard synchronizations for inertial observers can be simply treated in our model outlined previously. Namely, let us take an inertial observer with value $\boldsymbol{u}\in V(1)$ and an affine synchronization S, i.e., a synchronization in which the instants are parallel hyperplanes. Then there is a unique $\boldsymbol{u}_S\in V(1)$ such that the S-instants are hyperplanes parallel to $\mathbf{E}_{\boldsymbol{u}_S}$; thus, in the previous notations S-time is $I_{\boldsymbol{u}_S}$. Of course, if $\boldsymbol{u}_S=\boldsymbol{u}$, then we get back the standard synchronization corresponding to $\boldsymbol{u}$.

As in the case of standard synchronization corresponding to $\boldsymbol{u}$, the same time passes on every $\boldsymbol{u}$-space point between the S-instants t_1 and t_2, namely $(-\boldsymbol{u}_S\cdot(x_2-x_1))/(-\boldsymbol{u}_S\cdot\boldsymbol{u})$, where x_1 and x_2 are arbitrary elements of t_1 and t_2, respectively. Thus $I_{\boldsymbol{u}_S}$, endowed with the subtraction

$$t_2-t_1 := \frac{-\boldsymbol{u}_S\cdot(x_2-x_1)}{-\boldsymbol{u}_S\cdot\boldsymbol{u}} \qquad (x_2\in t_2,\ x_1\in t_1) \tag{9}$$

becomes a one dimensional affine space over $\mathbf{I}$.

According to Sec. 2.5, if C is a world line, then the motion, corresponding to C, relative to the inertial observer $\boldsymbol{u}$ and the affine synchronization S, is described by the function

$$I_{\boldsymbol{u}_S} \to E_{\boldsymbol{u}}, \qquad t \mapsto (\text{straight line parallel to } \boldsymbol{u}\text{, passing through } C * t) =: q(t) \tag{10}$$

Then for $t_2, t_1 \in I_{\boldsymbol{u}_S}$, using formulae (8) and (9), we get

$$\frac{q(t_2) - q(t_1)}{t_2 - t_1} = \frac{C * s - C * t + \boldsymbol{u}(\boldsymbol{u} \cdot (C * t_2 - C * t_1))}{-\boldsymbol{u}_S \cdot (C * t_2 - C * t_1)} (-\boldsymbol{u}_S \cdot \boldsymbol{u}) \tag{11}$$

Let us suppose now that the world line C is a straight line parallel to $\boldsymbol{u}' \in V(1)$. Then there is a unique $\boldsymbol{s} \in \mathbf{I}$ such that $C * t_2 - C * t_1 = \boldsymbol{u}'\boldsymbol{s}$. Thus we obtain from (11) that the relative velocity corresponding to the above motion, more precisely, *the relative velocity of* $\boldsymbol{u}'$ *with respect to* $\boldsymbol{u}$ *according to the synchronization* S is

$$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u},\, \boldsymbol{u}_S} := \boldsymbol{u}' + \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{u}') \frac{\boldsymbol{u}_S \cdot \boldsymbol{u}}{-\boldsymbol{u}_S \cdot \boldsymbol{u}'} = \boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} \frac{(-\boldsymbol{u}' \cdot \boldsymbol{u})(-\boldsymbol{u}_S \cdot \boldsymbol{u})}{-\boldsymbol{u}_S \cdot \boldsymbol{u}'} \tag{12}$$

where

$$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}} := \frac{\boldsymbol{u}'}{-\boldsymbol{u}' \cdot \boldsymbol{u}} - \boldsymbol{u} \tag{13}$$

is the relative velocity of $\boldsymbol{u}'$ with respect to $\boldsymbol{u}$ according to the standard synchronization of $\boldsymbol{u}$. This shows clearly that *the relative velocity of the same motion depends on the synchronization.*

Formulae (11) and (12) remain valid for a light signal C parallel to a (future directed) lightlike vector $\boldsymbol{w}$ if we replace $\boldsymbol{u}'$ with $\boldsymbol{w}$. In the standard synchronization corresponding to $\boldsymbol{u}$ light speed is unity for all light signals: $|\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u}}| = 1$, thus

$$|\boldsymbol{v}_{\boldsymbol{w}\boldsymbol{u},\, \boldsymbol{u}_S}| = \frac{(-\boldsymbol{w} \cdot \boldsymbol{u})(-\boldsymbol{u}_S \cdot \boldsymbol{u})}{-\boldsymbol{u}_S \cdot \boldsymbol{w}} \tag{14}$$

To analyze this expression, let

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{n}, \qquad \boldsymbol{u}_S = \frac{\boldsymbol{u} + \beta \boldsymbol{n}_S}{\sqrt{1 - \beta^2}} \tag{15}$$

where $\boldsymbol{n}$ and $\boldsymbol{n}_S$ are unit vectors in the $\boldsymbol{u}$-space and $\beta \in [0, 1[$. Then an easy calculation yields that

$$\boldsymbol{v}_{\boldsymbol{wu},\, \boldsymbol{u}_S} = \frac{1}{1 - \beta \boldsymbol{n}_S \cdot \boldsymbol{n}} \boldsymbol{n} \tag{16}$$

Since in our model the standard (or the two-way) light speed is unity, we have that the synchronization parameter in the direction of $\boldsymbol{n}$ is

$$\alpha(\boldsymbol{n}) = \frac{1 - \beta \boldsymbol{n}_S \cdot \boldsymbol{n}}{2} \tag{17}$$

Its value varies between $\alpha_0 := (1 - \beta)/2$ and $1 - \alpha_0 = (1 + \beta)/2$.

Returning to relative velocities of material points, and putting

$$\boldsymbol{u}' = \frac{\boldsymbol{u} + v' \boldsymbol{n}}{\sqrt{1 - (v')^2}} \tag{18}$$

where, of course, $v' = |\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u}}| < 1$, we find that

$$\boldsymbol{v}_{\boldsymbol{u}'\boldsymbol{u},\, \boldsymbol{u}_S} = \frac{v'}{1 - v' \beta \boldsymbol{n}_S \cdot \boldsymbol{n}} \boldsymbol{n} \tag{19}$$

It is evident that $v'/(1 - v'\beta \boldsymbol{n}_S \cdot \boldsymbol{n}) < 1/(1 - \beta \boldsymbol{n}_S \cdot \boldsymbol{n})$; thus in nonstandard synchronization we have, too, that in all directions the magnitude of the relative velocity of a material point is less then the one-way light speed.

Putting $\cos \Theta := -\boldsymbol{n}_S \cdot \boldsymbol{n}$, our formulae, obtained in a transparent and simple way, turn into the known ones.[2]

4. SYNCHRONIZATION AND LIGHT SPEED IN GENERAL

We have seen that one-way light speed with respect to inertial observers depends on the synchronization, and to every inertial observer there is a single synchronization—the standard one—which results in the same one-way light speed in all directions. It is an important and general fact that the *one-way speed of light or material points relative to a* (*non-necessarily inertial*) *observer makes sense only if a synchronization is chosen and it depends on synchronization.*

The question arises: given an arbitrary (noninertial) observer, is there a synchronization which ensures the observer the same one-way light speed in all space points and in all directions? We cannot hope the answer is yes.

Let us take an observer $\boldsymbol{U}$. Suppose S is a synchronization such that all one-way light speeds with respect to $\boldsymbol{U}$ are the same (the unity). We imagine then that the observer and the synchronization infinitesimally are similar to an inertial observer with the standard synchronization, i.e., if q' and q are $\boldsymbol{U}$-space points (integral curves) near to each other and t is an S-instant, then $q' * t - q * t$ is nearly g-orthogonal to $\boldsymbol{U}(q * t)$. Precisely this means that the instants (world surfaces) of the synchronization are g-orthogonal to $\boldsymbol{U}$: $\boldsymbol{U}$ is regular (see 2.4.).

The heuristic consideration above suggests the following conjecture: *for an observer* **U** *there is a synchronization S such that all the one-way light speeds in the space of the observer corresponding to the synchronization S are the same if and only if* **U** *is regular and S is the* **U**-*synchronization*.

5. SYNCHRONIZATION AND LIGHT SPEED ON THE ROTATING DISK

5.1. Uniformly Rotating Rigid Observers

Uniformly rotating rigid observers are described thoroughly in Ref. 7, II.6.8; let us recapitulate the most important formulae. Such an observer is given by an $o \in M$, $\boldsymbol{u}_o \in V(1)$, and a nonzero antisymmetric linear map Ω: $\mathbf{E}_{\boldsymbol{u}_o} \to \mathbf{E}_{\boldsymbol{u}_o}/\mathbf{I}$ as follows:

$$\boldsymbol{U}_{\mathrm{rot}}(o+\boldsymbol{x}) = \frac{\boldsymbol{u}_o + \Omega(\boldsymbol{x}+\boldsymbol{u}_o(\boldsymbol{u}_o\cdot\boldsymbol{x}))}{\sqrt{1-|\Omega(\boldsymbol{x}+\boldsymbol{u}_o(\boldsymbol{u}_o\cdot\boldsymbol{x}))|^2}} \tag{20}$$

for $\boldsymbol{x} \in \mathbf{M}$ such that the denominator makes sense.

The axis of rotation is at rest in the space of the inertial observer with value $\boldsymbol{u}_o$. Every vector in $\mathbf{E}_{\boldsymbol{u}_o}$ can be decomposed into the sum of a vector $\boldsymbol{e}$ in the (one-dimensional) kernel of Ω and a vector $\boldsymbol{q}$ orthogonal to $\boldsymbol{e}$. The integral curve (space point) of the rotating observer passing through $o+\boldsymbol{e}+\boldsymbol{q}$ for $\omega|\boldsymbol{q}| := |\Omega\boldsymbol{q}| < 1$ can be given by the parametrization

$$\boldsymbol{t} \mapsto o+\boldsymbol{e}+\boldsymbol{u}_o\boldsymbol{t}+\exp(\boldsymbol{t}\Omega)\,\boldsymbol{q} \tag{21}$$

where the parameter $\boldsymbol{t} \in \mathbf{I}$ is the $\boldsymbol{u}_o$-time originated in o, i.e., if t_o is the $\boldsymbol{u}_o$-instant assigned to o (t_o is the hyperplane parallel to $\mathbf{E}_{\boldsymbol{u}_o}$ and passing through o), then $\boldsymbol{t} = t - t_o$ for $t \in I_{\boldsymbol{u}_o}$. The proper time of this world line is

$$\boldsymbol{s} = \boldsymbol{t}\sqrt{1-\beta^2} \tag{22}$$

where

$$\beta := \omega r, \qquad r := |\boldsymbol{q}| \tag{23}$$

It can be shown that all the integral curves of $\boldsymbol{U}_{\text{rot}}$ are of the form (20).

5.2. The Sagnac Effect

The Sagnac effect consists in the following. Let two light signals be emitted from a space point of a rotating disk; one of the signals goes round on a circle in the direction of the rotation, the other in the opposite direction. According to the inertial observer in which the axis is at rest, the travel time of the first signal until its return to the emitting point is denoted by $\boldsymbol{t}_+$, the other travel time is denoted by $\boldsymbol{t}_-$. If the angular speed of the disk is ω and the radius of the circle (the distance of the emitting point from the axis) is r, then it has been demonstrated long ago that

$$\boldsymbol{t}_- - \boldsymbol{t}_+ = \frac{2(2\pi r)\,\beta}{1-\beta^2} \tag{24}$$

which is in accordance with experiments.

The Sagnac experiment is discussed from the point of view of the rotating observer in Ref. 5 and on the base of arguments using transformation rules, in particular, Lorentz transformations, it is stated that "special relativity predicts a null-effect (i.e. no time delay) on the rotating platform for the Sagnac experiment" which is taken to deny the validity of special relativity.

Now we shall show by our formalism without coordinates and transformation rules that the above statement is false.

Let the $\boldsymbol{U}_{\text{rot}}$-space point q be the one given by (21) and let $r := |\boldsymbol{q}|$. Then the light signal started from q and circulating in the direction of the rotation is described in spacetime by the function

$$\boldsymbol{t} \mapsto o + \boldsymbol{e} + \boldsymbol{u}_o \boldsymbol{t} + \exp\left(\frac{\boldsymbol{t}\Omega}{\beta}\right)\boldsymbol{q} \tag{25}$$

Indeed, its tangents are lightlike and runs at a distance r from the axis.

The intersections of the lines (21 and (25) are the meeting points of the $\boldsymbol{U}_{\text{rot}}$-space point and the light signal; the intersections occur at $\boldsymbol{t}$ for which

$$\boldsymbol{q} = \exp\left(\boldsymbol{t}\Omega\frac{1-\beta}{\beta}\right)\boldsymbol{q} \tag{26}$$

Thus the first meeting (after starting at $\boldsymbol{t} = 0$) occurs at

$$\boldsymbol{t}_{+} = \frac{2\pi r}{1-\beta} \tag{27}$$

The light signal started from q and circulating in the opposite direction is described in spacetime by the function (25), $-\Omega$ substituted for Ω. Then we get that this light signal arrives back at the starting point at

$$\boldsymbol{t}_{-} = \frac{2\pi r}{1+\beta} \tag{28}$$

Note that we used only world lines and light signals, their meeting points were determined without any reference frame. World lines and light signals were parametrized by $\boldsymbol{u}_o$-time; consequently, the inertial observer with velocity value $\boldsymbol{u}_o$ perceives time periods $\boldsymbol{t}_{+}$ and $\boldsymbol{t}_{-}$ between the meetings of the light signals and the rotating point. Thus this inertial observer measures the time delay (24). If the time passed between the meeting of the light signals and the rotating point (world line) q is measured on q (i.e., the proper time of the space point of the rotating disk is measured), then we get

$$\boldsymbol{s}_{\pm} = \frac{2\pi r\sqrt{1-\beta^2}}{1\mp\beta} \tag{29}$$

so the time delay from the point of the rotating observer is

$$\boldsymbol{s}_{-} - \boldsymbol{s}_{+} = \frac{2(2\pi r)\beta}{\sqrt{1-\beta^2}} \neq 0 \tag{30}$$

5.2. Synchronizations on the Rotating Disk

The uniformly rotating observers are not regular,[7] thus it is not a "mystery" that there is no synchronization in which the light speed is the same in all space points and in all directions.

From (29) we find that

$$\boldsymbol{s}_{\pm} = \frac{2\pi r}{\sqrt{1-\beta^2}}(1\pm\beta) \tag{31}$$

The first factor on the right-hand side is the circumference of the circle measured by the rotating observer; thus we infer that the *circle-way light speed* with respect to the rotating observer is different for the two directions,

$$\hat{c}_{\pm} = \frac{1}{1 \pm \beta} \tag{32}$$

Selleri[5] showed under very general conditions that

$$\frac{\hat{c}_{+}}{\hat{c}_{-}} = \frac{1-\beta}{1+\beta} \tag{33}$$

must hold. Then he states that "$(1-\beta)/(1+\beta)$ does not give the ratio of global light velocities for a full trip around the platform in the two opposite directions, but the local ratio as well: isotropy of space ensures that the velocities of light are the same in all points of the rim and therefore the average value coincides with the local ones."

This statement, however, is not right. Namely, the ratio (33) as well as formulae (32) concern the *circle-way speed* of light, which does not tell anything about the "local", i.e., one-way speed, as two-way or many-way speed does not imply anything for one-way speed. One-way or local speed makes sense only if a synchronization is given (and depends on synchronization); so *any assertion regarding one-way (local) light speeds—e.g., a formula for their ratio—would be meaningful if a synchronization had been specified.*

Then in Ref. 5 it is argued as follows: consider uniformly rotating observers whose angular speed ω is smaller and smaller, and take their small pieces whose distance r from the axis is larger and larger such that $\omega r = \beta =$ constant. Then the ratio of light speeds in the opposite directions is the same $(1-\beta)/(1+\beta) \neq 1$ for all such pieces which become more and more similar to pieces of a limit inertial observer moving with speed β with respect to the axis. Thus the ratio will differ from unity for that limit inertial observer, in contradistinction to special relativity, which asserts that light speed is the same in all directions with respect to inertial observers. Thus, accepting special relativity, we have a discontinuity which is not confirmed by experiments.

This conclusion is erroneous as well. Namely, according to special relativity, *light speed is the same in all directions with respect to an inertial observer if and only if the standard synchronization of the observer is used.* We shall show that the ratio above refers to a synchronization which is not the standard one of the limit inertial observer, so no contradiction and no discontinuity occur.

Let us investigate the one-way light speed on the rotating disk in the standard synchronization corresponding to the inertial observer in which the axis is at rest; then the instants are the hyperplanes parallel to $\mathbf{E}_{\boldsymbol{u}_o}$. Let the time passed between the $\boldsymbol{u}_o$-instants be measured by the proper time of q, i.e., the time passed from 0 to $\boldsymbol{t}$ will be $\boldsymbol{s} := \boldsymbol{t}\sqrt{1-\beta^2}$.

At the $\boldsymbol{u}_o$-instant $\boldsymbol{t}$ the light signal (25) meets an $\boldsymbol{U}_{\mathrm{rot}}$-space point $q(\boldsymbol{t})$; their intersection is a spacetime point of the form

$$o + \boldsymbol{e}_t + \boldsymbol{u}_o \boldsymbol{t} + \exp(\boldsymbol{t}\Omega)\, \boldsymbol{q}_t \tag{34}$$

for some $\boldsymbol{e}_t$ and $\boldsymbol{q}_t$. It is an easy task to find that that $\boldsymbol{e}_t = \boldsymbol{e}$ and

$$\boldsymbol{q}_t = \exp\left(\boldsymbol{t}\Omega \frac{1-\beta}{\beta}\right) \boldsymbol{q} \tag{35}$$

$\boldsymbol{q}$ and $\boldsymbol{q}_t$ are on the same circle of radius r in the (vectorized) space of the inertial observer $\boldsymbol{u}_o$; the angle between them is $\boldsymbol{t}(1-\omega r)/r$, so their distance on the circle is $\boldsymbol{t}(1-\beta)$. It is well known that the tangential distance in the $\boldsymbol{U}_{\mathrm{rot}}$-space (the rotating disk) is contracted by the factor $\sqrt{1-\beta^2}$, thus

$$d_{\boldsymbol{U}_{\mathrm{rot}}}(q, q(\boldsymbol{t})) = \boldsymbol{t}\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \tag{36}$$

As a consequence, the one-way (local) light speed on the rotating disk, corresponding to the synchronization in question, in the direction of the rotation is

$$c_+ := \lim_{\boldsymbol{s}\to 0} \frac{d_{\boldsymbol{U}_{\mathrm{rot}}}(q, q(\boldsymbol{s}/\sqrt{1-\beta^2}))}{\boldsymbol{s}} = \frac{1}{1+\beta} \tag{37}$$

Similarly, the light speed in the opposite direction is

$$c_- = \frac{1}{1-\beta} \tag{38}$$

Consequently, the assertion that the ratio of one-way (local) tangential light speeds on the rotating disk equals the ratio of the circle-way (global) light speeds is true *in the synchronization which is the standard one of the inertial observer with respect to which the axis is at rest.*

In the limit $\omega \to 0$, $r \to \infty$, $\omega r = \beta =$ constant we have that the one-way light speeds relative to the limit inertial observer $\boldsymbol{u}$ moving with relative speed β with respect to the inertial observer $\boldsymbol{u}_o$ are given by (37) and (38) in *a synchronization which is not the standard one of* $\boldsymbol{u}$, more closely *in the affine synchronization corresponding to* $\boldsymbol{u}_o$ as it is described in Sec. 3 (for $\boldsymbol{u}_S = \boldsymbol{u}_o$), so the result is in full agreement with special relativity, according to formula (16). There is no contradiction, no discontinuity.

Now we are interested in whether a synchronization can be given such that both one-way (local) tangential light speeds on the rotating disk are the same (unity). Since $\boldsymbol{U}_{\text{rot}}$ is not regular, we cannot hope so. We shall show, however, that to each $\boldsymbol{U}_{\text{rot}}$-space point (integral curve) q a *local, nearly standard synchronization* S_q exists in which both one-way tangential light speeds in q are the same. It is emphasized that such synchronizations are defined for all $\boldsymbol{U}_{\text{rot}}$-space points, but different synchronizations for different points, and the synchronization S_q is similar to a standard one exclusively for the $\boldsymbol{U}_{\text{rot}}$-space point q; the two tangential one-way light speeds relative to $\boldsymbol{U}_{\text{rot}}$ and corresponding to S_q are equal only in q.

The local nearly standard synchronization S_q is defined in a neighborhood of q as follows: let its instants be subsets of hyperplanes such that for all $x \in q$ the hyperplane passing through x is g-orthogonal to $\boldsymbol{U}_{\text{rot}}(x)$. It can be shown by the inverse function theorem that this synchronization is well defined in a neighborhood of the curve q. The time passed between the hyperplanes t_1 and t_2 is measured by the proper time of q between $q * t_1$ and $q * t_2$.

Let q be the $\boldsymbol{U}_{\text{rot}}$-space point described by (21); use the proper time (22) instead of $\boldsymbol{u}_o$-time, i.e., substitute $\boldsymbol{s}/\sqrt{1-\beta^2}$ for $\boldsymbol{t}$. The absolute velocity (the value of $\boldsymbol{U}_{\text{rot}}$) of this world line at $\boldsymbol{s}$ is

$$\frac{\boldsymbol{u}_o + \Omega \exp(\boldsymbol{s}\Omega/\sqrt{1-\beta^2})\,\boldsymbol{q}}{\sqrt{1-\beta^2}} \tag{39}$$

The spacetime point of q at proper time $\boldsymbol{s}$,

$$o + \boldsymbol{e} + \boldsymbol{u}_o \frac{\boldsymbol{s}}{\sqrt{1-\beta^2}} + \exp\left(\frac{\boldsymbol{s}\Omega}{\sqrt{1-\beta^2}}\right)\boldsymbol{q} \tag{40}$$

is simultaneous, according to S_q, with the spacetime point of the light signal described by (25)

$$o + \boldsymbol{e} + \boldsymbol{u}_o \boldsymbol{t}(\boldsymbol{s}) + \exp\left(\frac{\boldsymbol{t}(\boldsymbol{s})\,\Omega}{\beta}\right)\boldsymbol{q} \tag{41}$$

where $\boldsymbol{t}(\boldsymbol{s})$ is determined by the condition that the difference of (40) and (41) is is g-orthogonal to the velocity value (39). We find that

$$\boldsymbol{t}(\boldsymbol{s}) - \frac{\boldsymbol{s}}{\sqrt{1-\beta^2}} - \Omega \exp\left(\frac{\boldsymbol{s}\Omega}{\sqrt{1-\beta^2}}\right) \boldsymbol{q} \cdot \left(\exp\left(\frac{\boldsymbol{t}(\boldsymbol{s})\,\Omega}{\beta}\right) \boldsymbol{q} \right.$$
$$\left. - \exp\left(\frac{\boldsymbol{s}\Omega}{\sqrt{1-\beta^2}}\right) \boldsymbol{q} \right) = 0 \tag{42}$$

Since $\boldsymbol{t}(0) = 0$, differentiating with respect to $\boldsymbol{s}$ and then taking $\boldsymbol{s} = 0$, we get

$$\dot{\boldsymbol{t}}(0) = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \tag{43}$$

Thus, according to (36), the one-way light speed in the direction of the rotation, measured in q with respect to the synchronization S_q, is

$$c_+ = \lim_{s \to 0} \frac{d_{U_{\text{rot}}}(q, q(\boldsymbol{t}(\boldsymbol{s})))}{\boldsymbol{s}} = 1 \tag{44}$$

We get similarly that the light speed in the opposite direction is $c_- = 1$. In particular, in this synchronization the ratio of the one way (local) light speeds is not given by (33).

Note that in the limit $\omega \to 0$, $r \to \infty$, $\omega r = \beta =$ constant, the synchronization S_q becomes the standard one of a limit inertial observer having relative speed β with respect to the the axis. Again we get a result in full agreement with special relativity: there is no contradiction, no discontinuity.

6. DISCUSSION

A formulation of special relativity which avoids reference frames allows us to avoid some misunderstandings arising from the use of coordinates which are not inherent objects of spacetime.

Recently several papers have been published on the light speed relative to a rotating disk.[2–5] The basic idea is the "clock hypothesis": the rate of an ideal accelerated clock is identical to the rate of a similar clock in the instantaneously comoving inertial frame, which was confirmed by muon storage experiments. From this it is concluded that "the speed of light found locally in the accelerated system should be the same as that observed in the 'tangent' inertial frame".[4]

It is well known, however, from the arguments regarding nonstandard synchronizations that the speed of light—more precisely, the one-way light speed—relative to an observer has a meaning only if a synchronization is specified. Different synchronizations yield different one-way light speeds. Thus the above conclusion would be right with the following completion: *if both the accelerated observer and the instantaneously comoving inertial one apply the same synchronization*.

Disregarding this completion, a contradiction was claimed to be deduced in special relativity (a discontinuity which contradicts experience).[5] Namely, different one-way light speeds were found for the accelerated (in particular, the rotating) observer and the instantaneously comoving inertial observer; tacitly, however, different synchronizations were attached to the two observers.

We have shown that specifying clearly the synchronizations with respect to which the light speed is determined, the contradiction or discontinuity disappears. Furthermore, we have disproved the statement[5] that special relativity predicts a null effect from the point of view of the rotating platform for the Sagnac experiment.

The cited papers, on the ground of their results—which are untenable, as we have seen—argue for the existence of an absolute simultaneity. We make a simple remark regarding the following reasoning:[2] let us consider two twins who are at rest with respect to an inertial observer, at a given distance from each other, and their clocks are synchronized according to the inertial observer. They start accelerating simultaneously according to that inertial observer and then later they stop accelerating simultaneously according to that inertial observer. Then they will be at rest with respect to another inertial observer, distinct from the original one. Their clocks remain synchronized according to the original inertial observer, but will not be synchronized according to the new inertial observer. Then "of course in principle nothing can stop them from resynchronizing their clocks once they have finished accelerating. If they do so, however, they find in general that they have different biological ages at the resynchronized time." This is false, however. The resynchronization means, e.g., that one of them puts the clock back; this affects his nominal age, but not his biological age, as the one who flies from Europe to America will not be younger when putting the clock back.

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