Spacetime without Reference Frames and its Application to the Thomas Rotation

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Abstract. Spacetime structures defined in the language of manifolds admit an absolute formulation of physical theories i.e. a formulation which does not refer to observers (reference frames). Now we consider an affine structure of special relativistic spacetime admitting an absolute form of the Thomas rotation which throws new light on the velocity addition paradox.

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1. Introduction

General relativity is a mathematically developped nice physical theory whose modern setting is based on the global objects of manifolds: vector fields, differential forms, covariant derivations etc. These global objects can be called **absolute** from a physical point of view because they are not related to observers (reference frames, coordinate systems). In the last years several attempts appeared to formalize non-relativistic (Galilean, Euclidean) spacetime in a similar mathematical way ([1],[2],[6],[7],[8],[9],[12]) which shows well the demand for an absolute formulation of physical theories.

It is worth emphasizing: the frequently stated assertion that special relativity is the theory of inertial observers and general relativity is the theory of arbitrary observers ([11]) is to be substituted with the one that general relativity describes gravitation and special relativity concerns the lack of gravitation ([13],[14]). It is evident nowadays that the mathematical structure of spacetime can (and must) be formulated without observers. A **general relativistic spacetime model** is a triplet $(M, \mathbf{I}, g)$ where M is a four dimensional manifold, $\mathbf{I}$ is the measure line of spacetime distances and g is an $\mathbf{I} \otimes \mathbf{I}$ valued Lorentz form on M ([16],[9]). A **special relativistic spacetime model** is a particular general relativistic one in which M is an affine space and g is constant as it was stated even seventy years ago ([17]).

The treatment of special relativity based on the affine structure of spacetime has the great advantage that we can get rid of coordinates which are not inherent objects of spacetime and can cause confusions in theoretical considerations as it occurs e.g. in the velocity addition paradox.

The velocity addition paradox in special relativity has been discussed in the last years ([10],[15],[5]) and it became clear that it is related somehow to the Thomas rotation. However, the mathematical formulae based on coordinates do not give a clear physical explanation of the paradox; even, the Thomas rotation, taken as a real (dynamical) rotation is claimed to be a starting point to refute the theory of special relativity ([3],[4]).

Now the Thomas rotation and the velocity addition paradox will be treated without the use of reference frames (coordinate axes) which shows lucidly that the paradox arose from incorrect tacit assumptions deriving from the use of coordinates.

2. Spacetime vectors and observers

The spacetime vectors in special relativity form an oriented four dimensional vector space $\mathbf{N}$ on which a real valued Lorentz product $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ is given. $0 \neq \mathbf{x} \in \mathbf{N}$ is called timelike, spacelike and lightlike if $\mathbf{x} \cdot \mathbf{x}$ is negative, positive and zero, respectively. The Lorentz product is endowed with an arrow orientation: the set of timelike vectors consists of two disjoint open convex cones and one of them is selected to contain the future directed vectors which determines the set of future directed lightlike vectors, too. The set of **absolute velocities** is

$$V(1) := \{\mathbf{u} \in \mathbf{N} \mid \mathbf{u} \cdot \mathbf{u} = -1, \ \mathbf{u} \text{ is future directed}\}. \tag{1}$$

Every element $\mathbf{u}$ of $V(1)$ represents an **inertial observer**. The well known synchronization procedure by light signals ([13],[11],[9]) (yielding simultaneity)

with respect to an observer gives that

$$\mathbf{H_u} := \{\mathbf{x} \in \mathbf{N} \mid \mathbf{u} \cdot \mathbf{x} = 0\} \qquad (2)$$

is the set of **space vectors of the observer u**. $\mathbf{H_u}$ is a three dimensional vector space which can be oriented in a natural way by the orientation of $\mathbf{N}$ and the arrow orientation of the Lorentz product ([9],II.1.3.4.).The restriction of the Lorentz product onto $\mathbf{H_u}$ is a Euclidean product. The corresponding norm (length of vectors) is denoted by $|\ |$.

It can be shown by usual arguments ([9],II.4.2.) that

$$\mathbf{v_{u'u}} := \frac{\mathbf{u}'}{-\mathbf{u}' \cdot \mathbf{u}} - \mathbf{u} \qquad (3)$$

is the **relative velocity** of $\mathbf{u}'$ with respect to $\mathbf{u}$. We easily find that

(i) $|\mathbf{v_{u'u}}|^2 = |\mathbf{v_{uu'}}|^2 = 1 - \frac{1}{(\mathbf{u}' \cdot \mathbf{u})^2}$ implying

$$-\mathbf{u}' \cdot \mathbf{u} = \frac{1}{\sqrt{1 - |\mathbf{v_{u'u}}|^2}}; \qquad (4)$$

(ii) $\mathbf{v_{u'u}} \in \mathbf{H_u}$;

(iii) $\mathbf{v_{u'u}} = -\mathbf{v_{uu'}}$ if and only if $\mathbf{u} = \mathbf{u}'$ which is equivalent to $\mathbf{v_{u'u}} = \mathbf{v_{uu'}} = 0$. This last one is an important and far reaching fact:

if $\mathbf{u} \neq \mathbf{u}'$ then *the relative velocity of* $\mathbf{u}$ *with respect to* $\mathbf{u}'$ *is not the opposite of the relative velocity of* $\mathbf{u}'$ *with respect to* $\mathbf{u}$.

Furthermore, it is not hard to see that $\mathbf{H_u} \cap \mathbf{H_{u'}}$ is a two dimensional linear subspace of $\mathbf{N}$ if and only if $\mathbf{u} \neq \mathbf{u}'$ and then $\mathbf{v_{u'u}}$ in $\mathbf{H_u}$ and $\mathbf{v_{uu'}}$ in $\mathbf{H_{u'}}$ are orthogonal to $\mathbf{H_u} \cap \mathbf{H_{u'}}$. This is another important and far reaching fact:

the spaces of different observers are different three dimensional vector spaces.

The set of vectors simultaneous with respect to two different observers form a two dimensional vector space which is orthogonal in both spaces to the corresponding relative velocity.

3. PHYSICAL EQUALITY (PARALLELISM) OF VECTORS IN DIFFERENT OBSERVER SPACES

The space vectors of different observers $\mathbf{u}$ and $\mathbf{u}'$ constitute different three dimensional vector spaces $\mathbf{H_u}$ and $\mathbf{H_{u'}}$. Thus it has no "a priori" meaning, in general, that a vector (straight line) in the space of an observer is parallel to a vector (straight line) in the space of another observer. We have seen that the relative velocity of $\mathbf{u}$ with respect to $\mathbf{u}'$ is not the opposite of the relative velocity of $\mathbf{u}'$ with respect to $\mathbf{u}$. However, we can show that a light signal travelling in $\mathbf{H_u}$ in the direction of $\mathbf{v_{u'u}}$ travels in $\mathbf{H_{u'}}$ in the direction of $-\mathbf{v_{uu'}}$.

A light signal is characterized by a future-directed lightlike vector $\mathbf{k}$. The relative velocity of the light signal with respect to the observer $\mathbf{u}$ is found to be

$$\mathbf{v_{k,u}} := \frac{\mathbf{k}}{-\mathbf{k} \cdot \mathbf{u}} - \mathbf{u} \qquad (5)$$

and similar expression gives the relative velocity $\mathbf{v}_{\mathbf{k},\mathbf{u}'}$. We easily obtain that $|\mathbf{v}_{\mathbf{k},\mathbf{u}}| = |\mathbf{v}_{\mathbf{k},\mathbf{u}'}| = 1$ (light speed is the unity) ([9],4.7.)

Let us introduce the unit vectors in the directions of the relative velocities,

$$\mathbf{n}_{\mathbf{u}'\mathbf{u}} := \frac{\mathbf{v}_{\mathbf{u}'\mathbf{u}}}{|\mathbf{v}_{\mathbf{u}'\mathbf{u}}|}, \qquad\qquad \mathbf{n}_{\mathbf{u}\mathbf{u}'} := \frac{\mathbf{v}_{\mathbf{u}\mathbf{u}'}}{|\mathbf{v}_{\mathbf{u}\mathbf{u}'}|}. \qquad (6)$$

Proposition 1. *If*

$$\mathbf{v}_{\mathbf{k},\mathbf{u}} = \mathbf{n}_{\mathbf{u}',\mathbf{u}} \qquad (7)$$

then

$$\mathbf{v}_{\mathbf{k},\mathbf{u}'} = -\mathbf{n}_{\mathbf{u},\mathbf{u}'}. \qquad (8)$$

Proof. Multiplying equality (7) by $\mathbf{k}$, we easily deduce that

$$\frac{-\mathbf{k} \cdot \mathbf{u}}{-\mathbf{k} \cdot \mathbf{u}'} = \sqrt{\frac{1 - v}{1 + v}} \qquad (9)$$

where $v := |\mathbf{v}_{\mathbf{u}'\mathbf{u}}| = |\mathbf{v}_{\mathbf{u}\mathbf{u}'}|$, which yields equality (8).

The previous result suggests a relation between the spaces of different observers which formalizes the usual tacit assumption that the relative velocities of observers are opposite to each other.

Definiton 1. A vector $\mathbf{x}$ in $\mathbf{H}_{\mathbf{u}}$ is considered to be **physically equal** to a vector $\mathbf{x}'$ in $\mathbf{H}_{\mathbf{u}'}$ if

– the orthogonal projection of $\mathbf{x}$ onto $\mathbf{H}_{\mathbf{u}} \cap \mathbf{H}_{\mathbf{u}'}$ (which is the plane in $\mathbf{H}_{\mathbf{u}}$ orthogonal to $\mathbf{v}_{\mathbf{u}'\mathbf{u}}$) equals the orthogonal projection of $\mathbf{x}'$ onto $\mathbf{H}_{\mathbf{u}} \cap \mathbf{H}_{\mathbf{u}'}$ (which is the plane in $\mathbf{H}_{\mathbf{u}'}$ orthogonal to $\mathbf{v}_{\mathbf{u}\mathbf{u}'}$) i.e.

$$\mathbf{x} - (\mathbf{n}_{\mathbf{u}'\mathbf{u}} \cdot \mathbf{x})\mathbf{n}_{\mathbf{u}'\mathbf{u}} = \mathbf{x}' - (\mathbf{n}_{\mathbf{u}\mathbf{u}'} \cdot \mathbf{x}')\mathbf{n}_{\mathbf{u}\mathbf{u}'} \qquad (10)$$

– the orthogonal projection of $\mathbf{x}$ onto the direction of $\mathbf{v}_{\mathbf{u}'\mathbf{u}}$ is opposite to the orthogoanl projection of $\mathbf{x}'$ onto the direction of $\mathbf{v}_{\mathbf{u}\mathbf{u}'}$ i.e.

$$\mathbf{n}_{\mathbf{u}'\mathbf{u}} \cdot \mathbf{x} = -\mathbf{n}_{\mathbf{u}\mathbf{u}'} \cdot \mathbf{x}'. \qquad (11)$$

Then a vector $\mathbf{x}$ in $\mathbf{H}_{\mathbf{u}}$ is **physically parallel** to a vector $\mathbf{x}'$ in $\mathbf{H}_{\mathbf{u}'}$ if there is a real number λ such that $\lambda\mathbf{x}$ is physically equal to $\mathbf{x}'$ or $\mathbf{x}$ is physically equal to $\lambda\mathbf{x}'$.

It is quite evident that physical equality (parallelism) is a symmetric relation. Describing physical equality by a transparent mathematical formula, we shall see that it is not transitive.

The agreement about physical equality establishes a linear bijection $\mathbf{H}_{\mathbf{u}} \to \mathbf{H}_{\mathbf{u}'}$, $\mathbf{x} \mapsto \mathbf{x}'$, $\mathbf{x}'$ is physical equal to $\mathbf{x}$, which can be extended to a linear bijection $\mathbf{N} \to \mathbf{N}$ by the requirement $\mathbf{u} \mapsto \mathbf{u}'$. This linear bijection is uniquely determined by the prescribed properties because they fix its values on vectors spanning $\mathbf{N}$. The explicit form of this linear bijection is given as follows.

Definition 2. Let $\mathbf{u}' \otimes \mathbf{u}$ denote the linear map $\mathbf{N} \to \mathbf{N}$, $\quad \mathbf{x} \mapsto \mathbf{u}'(\mathbf{u} \cdot \mathbf{x})$ and let $\mathbf{1}$ be the identity map of $\mathbf{N}$. Then

$$\mathbf{L}(\mathbf{u}', \mathbf{u}) := \mathbf{1} + \frac{(\mathbf{u}' + \mathbf{u}) \otimes (\mathbf{u}' + \mathbf{u})}{1 - \mathbf{u}' \cdot \mathbf{u}} - 2\mathbf{u}' \otimes \mathbf{u} \tag{12}$$

is called the **Lorentz boost** from $\mathbf{u}$ to $\mathbf{u}'$.

Proposition 2. $\mathbf{L}(\mathbf{u}', \mathbf{u})$ *preserves the orientation of* $\mathbf{N}$, *the Lorentz product, the arrow orientation, and*

    (i) $\mathbf{L}(\mathbf{u}, \mathbf{u}') = \mathbf{L}(\mathbf{u}', \mathbf{u})^{-1}$.

Furthermore,

    (ii) $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{u} = \mathbf{u}'$,

    (iii) $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{x} = \mathbf{x}$ *if* $\mathbf{x} \in \mathbf{H_u} \cap \mathbf{H_{u'}}$,

    (iv) $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{v_{u'u}} = -\mathbf{v_{uu'}}$.

Properties (ii)-(iv) show us that the Lorentz boost establishes the physical equality of vectors in different observer spaces: $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{x}$ is the vector in $\mathbf{H_{u'}}$ which is physically equal to $\mathbf{x}$ in $\mathbf{H_u}$. Later the phrase "$\mathbf{x}'$ boosted (from $\mathbf{H_{u'}}$) into $\mathbf{H_u}$" will mean the vector $\mathbf{x}$ in $\mathbf{H_u}$ physically equal to $\mathbf{x}' \in \mathbf{H_{u'}}$.

Remarks. (i) In usual treatments of special relativity, spacetime is considered to be $\mathbb{R} \times \mathbb{R}^3$ in which $\mathbb{R}$ is "time" and "$\mathbb{R}^3$ is "space". All space vectors – space vectors of different observers – are taken to be elements of the same vector space $\mathbb{R}^3$. This corresponds to the fact that an observer $\mathbf{u}$ and an orthonormal basis in the $\mathbf{u}$-space are chosen to coordinatize spacetime, i.e. an observer is "hidden" in the coordinates and all space vectors are tacitly boosted into the space of the hidden observer.

(ii) The Lorentz boost is the absolute counterpart of the usual "Lorentz transformation without rotation": if $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ is an orthonormal basis (representing coordinate axes) in the $\mathbf{u}$-space then $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{n}_1$, $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{n}_2$, $\mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{n}_3$ determine the coordinate axes in the $\mathbf{u}'$-space that are parallel to those in the $\mathbf{u}$-sapce. Moreover, the matrix of $\mathbf{L}(\mathbf{u}', \mathbf{u})$ in the basis $\mathbf{n}_0 := \mathbf{u}$, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ of $\mathbf{N}$ becomes the well known usual Lorentz matrix (with $\kappa := 1/\sqrt{1 - |\mathbf{v_{u'u}}|^2} := -\mathbf{u}' \cdot \mathbf{u}$, $\quad \kappa v_i := \mathbf{n}_i \cdot \mathbf{u}'$, $\quad i = 1, 2, 3$).

(iii) Lorentz boosts refer to two observers i.e. to *two absolute velocities*. The usal matrix of a Lorentz transformation refers to a *single relative velocity*. Nevertheless, that matrix form, too, refers to two observers, but one of them is "hidden" in the coordinate axes and the relative velocity of another observer is taken with respect to the hidden observer.

Our treatment rules out hidden observers and coordinate axes.

4. Thomas rotation

The explicit form of the Lorentz boost allows us to prove without difficulty that the Lorentz boost from $\mathbf{u}$ to $\mathbf{u}'$ followed by the Lorentz boost from $\mathbf{u}'$ to $\mathbf{u}''$, in general, is not the Lorentz boost from $\mathbf{u}$ to $\mathbf{u}''$. More precisely $\mathbf{L}(\mathbf{u}'', \mathbf{u}')\mathbf{L}(\mathbf{u}', \mathbf{u}) = \mathbf{L}(\mathbf{u}'', \mathbf{u})$ if and only if $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$ are coplanar ([9],II.1.3.9.). This results shows that the **physical equality** (and physical parallelism) of vectors in different observer spaces **is not a transitive relation**: it may be that $\mathbf{x}' \in \mathbf{H_{u'}}$ is physically

equal to $\mathbf{x} \in \mathbf{H_u}$ and $\mathbf{x}'' \in \mathbf{H_{u''}}$ is physically equal to $\mathbf{x}' \in \mathbf{H_{u'}}$ but $\mathbf{x}''$ is not physically equal to $\mathbf{x}$.

The same can be expressed in another way as follows. Suppose $\mathbf{x}' \in \mathbf{H_{u'}}$ is physically equal to $\mathbf{x}'' \in \mathbf{H_{u''}}$. Furthermore, let $\mathbf{x}$ and $\hat{\mathbf{x}}$ be in $\mathbf{H_u}$ physically equal to $\mathbf{x}'$ and $\mathbf{x}''$, respectively. Then $\mathbf{x}$ need not be equal to $\hat{\mathbf{x}}$.

We can reformulate the result about the product of the Lorentz boosts as follows:

Proposition 3.

$$\mathbf{R_u(u', u'')} := \mathbf{L(u, u'')L(u'', u')L(u', u)} \tag{13}$$

is the identity transformation if and only if $\mathbf{u}$, $\mathbf{u}'$ *and* $\mathbf{u}''$ *are coplanar.*

Because of the properties of the Lorentz boosts, it is trivial that $\mathbf{R_u(u', u'')u = u}$ and the restriction of $\mathbf{R_u(u', u'')}$ to $\mathbf{H_u}$ is an orientation and Euclidean product preserving linear bijection from $\mathbf{H_u}$ onto $\mathbf{H_u}$ i.e. it is a rotation. We easily find that − excluding the trivial case when the three absolute velocities are coplanar − the axis of rotation (the set of invariant vectors) is the one dimensional linear subspace $\mathbf{H_u} \cap \mathbf{H_{u'}} \cap \mathbf{H_{u''}}$. We continue to consider the linear bijection defined on $\mathbf{N}$ rather than its restriction to $\mathbf{H_u}$ that is why accept the following definition.

Definition 3. $\mathbf{R_u(u', u'')}$ is called the **Thomas rotation** of $\mathbf{u}$ corresponding to $\mathbf{u}'$ and $\mathbf{u}''$.

Remarks. (i) We emphasize that *three absolute velocities* are involved in Proposition 3. The corresponding statement in the usual matrix formulation refers to *two relative velocities* and their colinearity. This is so because the usual formulation "hides" an observer.

(ii) Note that the Thomas rotation is defined without coordinate axes, so its fundamental meaning is not connected with the rotation of axes and it corresponds to no real rotation. *The Thomas rotation measures somehow the deviation of the physical equality from being transitive.*

The Thomas rotation, too, is given in terms of absolute velocities; three absolute velocities are involved in its definition. In the usual matrix formulation of the Thomas rotation two relative velocities appear and an observer is hidden in the coordinate axes.

Of course, we should like to deduce from the previous definition the expression of the Thomas rotation in terms of relative velocities. Since $\mathbf{R_u(u', u'')}$ is a rotation in the $\mathbf{u}$-space and space in the usual matrix formalism always means the space of the hidden observer, now $\mathbf{u}$ corresponds to the hidden observer and the two relative velocities in question would be $\mathbf{v_{u'u}}$ and $\mathbf{v_{u''u'}}$; however, the latter one is to be boosted in the $\mathbf{u}$-space. Thus

$$\mathbf{v} := \mathbf{v_{u'u}}, \qquad \mathbf{w} := \mathbf{L(u, u')v_{u''u'}} \tag{14}$$

correspond to the usual relative velocites considered in connection with the Thomas rotation. It is easy to check that $\mathbf{v}$ and $\mathbf{w}$ are in the rotation plane of the Thomas rotation i.e. they are orthogonal to the rotation axis $\mathbf{H_u} \cap \mathbf{H_{u'}} \cap \mathbf{H_{u''}}$.

Introducing

$$\alpha := -\mathbf{u}' \cdot \mathbf{u} = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}, \qquad \beta := -\mathbf{u}'' \cdot \mathbf{u}' = \frac{1}{\sqrt{1 - |\mathbf{w}|^2}}, \qquad (15)$$

$$\gamma := -\mathbf{u}'' \cdot \mathbf{u} = \alpha\beta(1 + \mathbf{v} \cdot \mathbf{w}), \qquad (16)$$

we can recover the absolute velocities $\mathbf{u}'$ and $\mathbf{u}''$ from $\mathbf{u}$ and the relative velocites $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{u}' = \alpha(\mathbf{u} + \mathbf{v}), \qquad \mathbf{u}'' = \beta(\mathbf{u}' + \mathbf{L}(\mathbf{u}', \mathbf{u})\mathbf{w}) = \gamma\mathbf{u} + \beta\mathbf{w} + \frac{\alpha(\beta + \gamma)}{1 + \alpha}\mathbf{v}. \quad (17)$$

The Thomas rotation in terms of relative velocities is

$$\mathbf{T_u}(\mathbf{v}, \mathbf{w}) := \mathbf{R_u}\left(\alpha(\mathbf{u} + \mathbf{v}), \gamma\mathbf{u} + \beta\mathbf{w} + \frac{\alpha(\beta + \gamma)}{1 + \alpha}\mathbf{v}\right). \qquad (18)$$

A lengthy but straightforward calculation yields the following result.

Proposition 4.

$$\mathbf{T_u}(\mathbf{v}, \mathbf{w}) = \mathbf{1} + \alpha^2 \frac{1 - \beta}{(1 + \alpha)(1 + \gamma)}\mathbf{v} \otimes \mathbf{v} + \alpha\beta\frac{(1 + \alpha)(1 + \gamma) + (\beta + \gamma)(1 - \alpha)}{(1 + \alpha)(1 + \beta)(1 + \gamma)}\mathbf{v} \otimes \mathbf{w}$$
$$- \alpha\beta\frac{1}{1 + \gamma}\mathbf{w} \otimes \mathbf{v} + \beta^2 \frac{1 - \alpha}{(1 + \beta)(1 + g)}\mathbf{w} \otimes \mathbf{w}. \qquad (19)$$

This very nice form allows us to deduce easily all the results regarding the Thomas rotation which are difficult to obtain in the matrix formalism.

Proposition 5. *Let ϵ denote the angle of rotation of $\mathbf{T_u}(\mathbf{v}, \mathbf{w})$ and let θ denote the angle between $\mathbf{v}$ and $\mathbf{w}$. Then*

$$\cos\epsilon = 1 - \frac{(\alpha - 1)(\beta - 1)}{1 + \gamma}\sin^2\theta. \qquad (20)$$

Proof. It suffices to find the cosine of the angle between $\mathbf{x}$ and $\mathbf{T_u}(\mathbf{v}, \mathbf{w})\mathbf{x}$ for some special $\mathbf{x}$ in the rotation plane. Let us choose $\mathbf{x}$ such that $|\mathbf{x}| = 1$, $\mathbf{w} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{x} > 0$. Then $\mathbf{v} \cdot \mathbf{x} = |\mathbf{v}|\sin\theta$ and $\cos\epsilon = \mathbf{x} \cdot \mathbf{T_u}(\mathbf{v}, \mathbf{w})\mathbf{x}$, so we get the desired result immediately.

The above formula is simple but it contains four quantities which are not independent:

$$\gamma = \alpha\beta + \sqrt{\alpha^2 - 1}\sqrt{\beta^2 - 1}\cos\theta \qquad (21)$$

Eliminating θ, we get

$$\cos\epsilon = 1 - \frac{1 + 2\alpha\beta\gamma - (\alpha^2 + \beta^2 + \gamma^2)}{(1 + \alpha)(1 + \beta)(1 + \gamma)}. \qquad (22)$$

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Eliminating γ and introducing

$$k := \sqrt{\frac{(\alpha + 1)(\beta + 1)}{(\alpha - 1)(\beta - 1)}}, \qquad (23)$$

we get

$$\cos \epsilon = 1 - \frac{2 \sin^2 \theta}{1 + k^2 + 2k \cos \theta} = \frac{(k + \cos \theta)^2 - \sin^2 \theta}{(k + \cos \theta)^2 + \sin^2 \theta}. \qquad (24)$$

The orientation (the positive direction of the rotation axis) of the Thomas rotation is the direction of $\mathbf{x} \times \mathbf{T_u}(\mathbf{v}, \mathbf{w})\mathbf{x}$ where $\mathbf{x}$ is an arbitrary non-zero vector in the rotation plane and $\times$ denotes the vectorial product.

Proposition 6. *The orientation of the Thomas rotation* $\mathbf{T_u}(\mathbf{v}, \mathbf{w})$ *is given by* $\mathbf{w} \times \mathbf{v}$.

Proof. The map $\lambda \mapsto \mathbf{T_u}(\mathbf{v}, \mathbf{w} + \lambda \mathbf{v})$ ($\lambda \in \mathbb{R}$) is continuous. Since there are two disjoint orientations, a continuous map cannot change the orientation; consequently, the orientation is the same for all λ as for $\lambda = 0$. Let λ_0 be such that $(\mathbf{w} + \lambda_0 \mathbf{v}) \cdot \mathbf{v} = 0$. Then we easily find that

$$\mathbf{v} \times \mathbf{T_u}(\mathbf{v}, \mathbf{w} + \lambda_0 \mathbf{v})\mathbf{v} = -\frac{\alpha\beta |\mathbf{v}|^2}{1 + \gamma}(\mathbf{v} \times \mathbf{w}) \qquad (25)$$

which proves our assertion because all the coefficients on the right side are positive.

5. THE VELOCITY ADDITION PARADOX

The paradox can be described as follows. Let me, you and him sit in different spaceships. Your velocity $\mathbf{v}$ relative to me and his velocity $\mathbf{w}$ relative to you determine his velocity $\mathbf{v} \oplus \mathbf{w}$ to me by the formula ([10])

$$\mathbf{v} \oplus \mathbf{w} = \frac{\alpha\beta}{\gamma}\left(\mathbf{v} + \mathbf{w} + \frac{\alpha}{1 + \alpha}\mathbf{v} \times (\mathbf{v} \times \mathbf{w})\right) = \frac{\alpha(\beta + \gamma)}{\gamma(1 + \alpha)}\mathbf{v} + \frac{\beta}{\gamma}\mathbf{w}. \qquad (26)$$

Similarly, your velocity $\hat{\mathbf{w}}$ relative to him and my velocity $\hat{\mathbf{v}}$ relative to you determine my velocity $\hat{\mathbf{w}} \oplus \hat{\mathbf{v}}$ relative to him by the same formula. We "evidently" have $\hat{\mathbf{w}} = -\mathbf{w}$, $\hat{\mathbf{v}} = -\mathbf{v}$ and $\hat{\mathbf{w}} \oplus \hat{\mathbf{v}} = -\mathbf{v} \oplus \mathbf{w}$; however, the actual formula for the addition $\oplus$ shows that, in general,

$$(-\mathbf{w}) \oplus (-\mathbf{v}) \neq -(\mathbf{v} \oplus \mathbf{w}), \qquad \text{or, equivalently,} \qquad \mathbf{w} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{w}.$$

We shall soon see that the paradox arose in the usual matrix formalism from the fact that instead of vectors in the spaces of different observers one considers tacitly the corresponding physically equal vectors in the space of the hidden observer (every space vector is tacitly boosted into the space of the hidden observer), which implies the incorrect tacit assumption that physical equality is a transitive relation. In fact, relative velocites $\mathbf{v}$ and $\mathbf{w}$ as well as $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$ are considered to be elements of $\mathbb{R}^3$, their sum and vectorial product appear in the formulae $\mathbf{v} \oplus \mathbf{w}$ and $\hat{\mathbf{w}} \oplus \hat{\mathbf{v}}$, yielding elements of $\mathbb{R}^3$.

Now let us return from me, you and him to the notations $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{u}''$. Then $\mathbf{v_{u'u}}$ would be $\mathbf{v}$ and $\mathbf{v_{u''u'}}$ would be $\mathbf{w}$. However, $\mathbf{v_{u'u}}$ and $\mathbf{v_{u''u'}}$ are in the different three dimensional vector spaces $\mathbf{H_u}$ and $\mathbf{H_{u'}}$, their sum is meaningful as an element of $\mathbf{N}$ but is not in either $\mathbf{H_u}$ or $\mathbf{H_{u'}}$ and their vectorial product is not meaningful, in general.

The velocity addition formula (26) *is meaningful and holds true only if the second relative velocity is boosted into the space of the observer to which the first velocity and the resulting one are related.*

Thus we have to take $\mathbf{v} = \mathbf{v_{u'u}}$ and $\mathbf{w} = \mathbf{L(u, u')v_{u''u'}}$ in accordance with (14) and then

$$\mathbf{v} \oplus \mathbf{w} = \mathbf{v_{u''u}}. \tag{27}$$

Regarding the other addition in the paradox involving $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$, we must be careful: since the velocity of $\mathbf{u}$ relative to $\mathbf{u}''$ is calculated by the addition formula from the velocity of $\mathbf{u}'$ relative to $\mathbf{u}''$ and from the velocity of $\mathbf{u}$ relative to $\mathbf{u}'$, this last relative velocity must be boosted into the $\mathbf{u}''$-space, so

$$\hat{\mathbf{w}} := \mathbf{v_{u'u''}}, \qquad \hat{\mathbf{v}} := \mathbf{L(u'', u')v_{uu'}} \tag{28}$$

and then

$$\hat{\mathbf{w}} \oplus \hat{\mathbf{v}} = \mathbf{v_{uu''}}. \tag{29}$$

The vector in the $\mathbf{u}$-space, physically equal to $\hat{\mathbf{w}}$ is

$$\begin{aligned}
\mathbf{L(u, u'')}\hat{\mathbf{w}} = \mathbf{L(u, u'')v_{u'u''}} &= -\mathbf{L(u, u'')L(u'', u')v_{u''u'}} \\
&= -\mathbf{L(u, u'')L(u'', u')L(u', u)w} = -\mathbf{R_u(u', u'')w},
\end{aligned} \tag{30}$$

and the vector in the $\mathbf{u}$-space, physically equal to $\hat{\mathbf{v}}$ is

$$\begin{aligned}
\mathbf{L(u, u'')}\hat{\mathbf{v}} = \mathbf{L(u, u'')L(u'', u')v_{uu'}} &= \\
&= -\mathbf{L(u, u'')L(u'', u')L(u', u)v_{u'u}} = -\mathbf{R_u(u', u'')v}.
\end{aligned} \tag{31}$$

Now we see that $\hat{\mathbf{w}}$ *is not physically equal* to $-\mathbf{w}$ and $\hat{\mathbf{v}}$ *is not physically equal* to $-\mathbf{v}$, contrary to the usual "evidence" which leads to the paradox. Then it is not surprising that $\hat{\mathbf{w}} \oplus \hat{\mathbf{v}}$ is not physically equal to $\mathbf{v} \oplus \mathbf{w}$ either. All this is the consequence of the non-transitivity of physical equality.

Formulae (30) and (31) indicate that the velocity addition formula will be "commutative" if we replace $\hat{\mathbf{w}}$ and $\hat{\mathbf{v}}$ with $-\mathbf{R_u(u', u'')w} = \mathbf{T_u(v, w)w}$ and $-\mathbf{R_u(u', u'')v} = \mathbf{T_u(v, w)v}$, respectively. This is so.

Proposition 7.

$$\mathbf{T_u(v, w)w} \oplus \mathbf{T_u(v, w)v} = \mathbf{T_u(v, w)(w \oplus v)} = \mathbf{v} \oplus \mathbf{w}. \tag{32}$$

Proof. In the next lemma we demonstrate that a Lorentz transformation (in particular, the Thomas rotation) is "linear" with respect to the addition $\oplus$ which implies the first equality.

To prove the second equality, we apply the inverse of the Thomas rotation and we take into account the properties of Lorentz boosts as well as equalities (27)-(29):

$$
\begin{aligned}
\mathbf{T_u(v,w)}^{-1}(\mathbf{v}\oplus\mathbf{w}) =&\mathbf{L(u,u')L(u',u'')L(u'',u)v_{u''u}} = -\mathbf{L(u,u')L(u',u'')v_{uu''}} \\
=& -\mathbf{L(u,u')L(u',u'')}\big(\mathbf{v_{u'u''}}\oplus\mathbf{L(u'',u')v_{uu'}}\big) \\
=&\mathbf{L(u,u')L(u',u'')}\big(\mathbf{L(u'',u')v_{u''u'}}\oplus\mathbf{L(u'',u')L(u',u)v_{u'u}}\big) \\
=&\big(\mathbf{L(u,u')v_{u''u'}}\big)\oplus\mathbf{v_{u'u}} = \mathbf{w}\oplus\mathbf{v}. \qquad (33)
\end{aligned}
$$

Lemma. *If* $\mathbf{L}$ *is a Lorentz transformation – i.e. a Lorentz product preserving linear bijection of* $\mathbf{N}$ *– then*

$$
\mathbf{L(v\oplus w) = (Lv)\oplus(Lw).} \qquad (34)
$$

Proof. According to the formula (26), $\mathbf{v}\oplus\mathbf{w}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$ with coefficients composed from $|\mathbf{v}|^2$, $|\mathbf{w}|^2$ and $\mathbf{v}\cdot\mathbf{w}$. Then our assertion is quite trivial, since $\mathbf{L}$ is linear and $|\mathbf{Lv}|^2=|\mathbf{v}|^2$, $|\mathbf{Lw}|^2=|\mathbf{w}|^2$, $(\mathbf{Lv}\cdot\mathbf{Lw})=\mathbf{v}\cdot\mathbf{w}$.

6. DISCUSSION

A spacetime structure which is free of observsrs, reference frames and coordinates admits a treatise of special relativity based on **absolute** objects i.e. objects not involving reference frames and coordinates. Such a treatise makes it evident that the spaces of different observers are different. A clear and rigorous definition of "Lorentz transformations whitout rotation", called Lorentz boosts are used to relate the spaces of different observers to each other, establishing a notion of physical equality and physical parallelism (called quasi-parallelism in [5]) of vectors in different observer spaces. The explicit form of the Lorentz boost in terms of absolute velocities makes very simple the proof of such statements as "the succession of two Lorentz transformations whitout rotation, in general, is not a Lorentz transformation without rotation" which yields that the physical equality is not a transitive relation. The Thomas rotation, simply defined by the succession of three Lorentz boosts, measures how much the relation of physical equality of vectors deviates from being transitive. An explicit form of the Thomas rotation is deduced which is much more convenient in applications than the usual matrix forms.

The absolute formulation of spacetime illuminates that the velocity addition paradox is a consequence of the facts that in the treatements applying coordinates

1. the space of every observer is tacitly considered through the corresponding physically equal vectors in the space of the observer hidden in the coordinates,

2. physical equality is tacitly taken to be a transitive relation.

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