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SPECTRAL PROPERTIES OF VECTOR OPERATORS

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1. Introduction

Usual quantum mechanical observables are self-adjoint operators, or better said, families of self-adjoint operators. For instance, position, a so-called vectorial observable, is considered as a family of three self-adjoint operators that are interpreted as the components of position relative to a basis of the physical space. If we want to get rid of bases and to look for a coordinate-free description, we face the problem, what mathematical objects represent quantum mechanical observables. The notion of vector operator is introduced to answer this question. Here we investigate only mathematical properties of vector operators and we do not enter into physical applications.

2. Preliminaries

In the sequel H and Z denote a complex Hilbert space and a finite dimensional complex vector space, respectively.

Inner products are denoted by the symbol $\langle , \rangle$ and are taken to be linear in the second variable.

$H\otimes Z$ is the algebraic tensor product of H and Z. It is well-known (see [1], Ch. II. 4) that if we equip Z with an inner product then $H\otimes Z$ turns into a Hilbert space with the inner product defined by

$$\langle h\otimes z,\ g\otimes y\rangle := \langle h, g\rangle\langle z, y\rangle \qquad (h, g\in H,\ z, y\in Z).$$

The corresponding topology on $H\otimes Z$ is independent of the particular inner product chosen on Z. That is why we consider $H\otimes Z$ as a topological vector space without specifying an inner product on Z.

If $z_1, \dots, z_N$ is a basis of Z then every element of $H\otimes Z$ can be written in the form $\sum_{k=1}^{N} h_k\otimes z_k$.

Z^* stands for the dual of Z and the bilinear map of duality is denoted by $(|)$. We are given a continuous bilinear map

$$((|)) \colon Z^*\times(H\otimes Z) \to H,$$

defined by

$$((p|h\otimes z)) := (p|z)h \qquad (p\in Z^*,\ h\otimes z\in H\otimes Z),$$

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and a continuous sesquilinear map

$$\langle\langle\,,\,\rangle\rangle\colon H\times(H\otimes Z)\to Z$$

defined by

$$\langle\langle g, h\otimes z\rangle\rangle := \langle g, h\rangle z \qquad (g\in H,\ h\otimes z\in H\otimes Z).$$

We have the following relation:

$$\langle g, ((p|a))\rangle = (p|\langle\langle g, a\rangle\rangle) \qquad (p\in Z^*,\ g\in H,\ a\in H\otimes Z).$$

If $p_1, \ldots, p_N$ is a basis of Z^* then the elements a and b of $H\otimes Z$ are equal if and only if $((p_k|a))=((p_k|b))$ $(k=1, \ldots, N)$.

3. Basic facts about vector operators

DEFINITION 1. A linear map defined in H and having values in $H\otimes Z$ is called a *Z valued vector operator* in H.

If A is a vector operator and $p\in Z^*$ then we define the linear map

$$((p|A))\colon H\supset \operatorname{Dom} A\to H, \quad h\mapsto ((p|Ah)).$$

REMARKS. (i) A complex valued vector operator is a usual operator.

(ii) Since $H\otimes Z$ has a topology, we can speak about continuous and closed vector operators.

(iii) Let $z_1, \ldots, z_N$ be a basis of Z and let $p_1, \ldots, p_N$ be the corresponding dual basis of Z^*. Then we can consider $((p_k|A))$ $(k=1, \ldots, N)$ as the *components* of the vector operator A relative to the given basis of Z. We have the equality

$$Ah = \sum_{k=1}^{N}[((p_k|Ah))]\otimes z_k \qquad (h\in\operatorname{Dom} A).$$

Consequently, if we are given a family $A_1, \ldots, A_N$ of operators with common domain D in H, then we can construct the vector operator

$$h\mapsto \sum_{k=1}^{N}(A_k h)\otimes z_k \qquad (h\in D)$$

whose components are precisely the given operators.

As a consequence, two Z valued vector operators are equal if and only if their components relative to any basis of Z coincide.

EXAMPLES. (i) If $u\in Z$ then $\otimes u\colon H\to H\otimes Z$, $h\mapsto h\otimes u$ is a continuous vector operator and $((p|\otimes u))=(p|u)\,\mathrm{id}_H$.

(ii) Let V be a finite dimensional real vector space. Then $L^2(V)\otimes Z$ is identified, through the prescription $f\otimes z=(v\mapsto f(v)z)$, with the vector space of Z valued square integrable function classes. The *identity multiplication operator M* defined on

$$\operatorname{Dom} M := \{f\in L^2(V)\colon f\,\mathrm{id}_V\in L^2(V)\otimes V_{\mathbf{C}}\}$$

by

$$f\mapsto f\,\mathrm{id}_V := (v\mapsto f(v)v)$$

is a $V_{\mathbf{C}}$ valued vector operator in $L^2(V)$ where $V_{\mathbf{C}}$ stands for the complexification of V. If $r_1, \ldots, r_N$ is a basis in V then $((r_k|M))$ is contained in the operator of multiplication by the k-th coordinate.

If $f\colon V\to\mathbf{C}$ is differentiable, then $\mathrm{D}f(v)$, its derivative at $v\in V$, is a linear map $V\to\mathbf{C}$ which can be extended uniquely to a complex linear map $V_{\mathbf{C}}\to\mathbf{C}$; in other words, we can consider $\mathrm{D}f$ as a map $V\to(V_{\mathbf{C}})^*=(V^*)_{\mathbf{C}}=:V^*_{\mathbf{C}}$. Then the differentiation operator D defined on

$$\mathrm{Dom}\,\mathrm{D}:=\{f\in L^2(V)\colon f \text{ is differentiable, } \mathrm{D}f\in L^2(V)\otimes V^*_{\mathbf{C}}\}$$

is a $V^*_{\mathbf{C}}$ valued vector operator in $L^2(V)$. If $v_1, \ldots, v_N$ is a basis in $V=(V^*)^*$ then $((v_k|D))$ is contained in the k-th partial differentiation operator.

Definition 2. A bounded operator L is said to *commute* with the vector operator A if $AL\supset(L\otimes\mathrm{id}_Z)A$.

Proposition 1. *L commutes with A if and only if L commutes with $((p|A))$ for all $p\in Z^*$ which holds if and only if L commutes with $((p_k|A))$ $(k=1, \ldots, N)$ for an arbitrary basis $p_1, \ldots, p_N$ of Z^*.*

4. The spectrum of a vector operator

In the sequel A denotes a fixed densely defined vector operator.

Definition 3. A linear subspace D of $\mathrm{Dom}\,A$ is called *invariant* under A if $A(D)\subset D\otimes Z$.

Proposition 2. *D is invariant under A if and only if D is invariant under $((p|A))$ for all $p\in Z^*$ which holds if and only if D is invariant under $((p_k|A))$ $(k=1, \ldots, N)$ for an arbitrary basis $p_1, \ldots, p_N$ of Z^*.*

Definition 4. An element λ of Z is called an *eigenvalue* of A *if there is a nonzero* $h\in\mathrm{Dom}\,A$ such that $Ah=h\otimes\lambda$. The linear subspace $\{h\in\mathrm{Dom}\,A\colon Ah=h\otimes\lambda\}$ is the *eigenspace* of A corresponding to λ. The set of eigenvalues of A is denoted by $\mathrm{Eig}\,A$.

Definition 5. A linear subspace T of $H\otimes Z$ is called *bulky* if there is no proper closed linear subspace D of H such that $T\subset D\otimes Z$.

Proposition 3. *A linear subspace T of $H\otimes Z$ is bulky if and only if H is spanned by* $\bigcup_{p\in Z^*}\{((p|a))\colon a\in T\}$.

Definition 6. An element λ of Z is a *regular value* of A if
(i) $A-\otimes\lambda$ is injective,
(ii) $\mathrm{Ran}\,(A-\otimes\lambda)$ is bulky,
(iii) $(A-\otimes\lambda)^{-1}$ is continuous.

The set
$\mathrm{Sp}\,A:=\{\lambda\in Z\colon \lambda \text{ is not a regular value of } A\}$ is the *spectrum* of A.

PROPOSITION 4. (i) $\operatorname{Eig} A \subset \operatorname{Sp} A$, *and for all* $p \in Z^*$
(ii) $(p|\operatorname{Eig} A) \subset \operatorname{Eig}((p|A))$,
(iii) $(p|\operatorname{Sp} A) \subset \operatorname{Sp}((p|A))$.

PROOF. (i) and (ii) are evident. To prove (iii) suppose that $\lambda \in \operatorname{Sp} A$, $S(\lambda):=$ $:=A-\otimes\lambda$ is injective, and distinguish the following two cases.

Firstly, assume that $\bigcup_{p\in Z^*}((p|\operatorname{Ran} S(\lambda))$ does not span H. Then $\operatorname{Ran}((p|S(\lambda))=$ $=((p|\operatorname{Ran} S(\lambda)))$ cannot be dense in H, thus $(p|\lambda) \in \operatorname{Sp}((p|A))$ for all $p \in Z^*$.

Secondly, suppose that the inverse of $S(\lambda)$ is not continuous. Then there is an unbounded sequence h_n $(n \in \mathbf{N})$ in H such that $S(\lambda)h_n$ is bounded. Consequently, the sequence $((p|S(\lambda)h_n))$ is bounded, thus $((p|S(\lambda)))$ cannot have a continuous inverse (it may have no inverse at all), and $(p|\lambda) \in \operatorname{Sp}((p|A))$ $(p \in Z^*)$.

PROPOSITION 5. *The spectrum of a vector operator is closed.*

PROOF. To demonstrate this assertion let us equip Z with an inner product. Then for all $u \in Z$ the norm of the vector operator $\otimes u$ equals the norm of the vector u: $\|h\otimes u\|=\|u\|\|h\|$ for all $h \in H$. As a consequence, one can show as in usual operator theory that if B is a vector operator having a continuous inverse then $B-\otimes u$ has a continuous inverse for u in a convenient neighbourhood of the zero of Z. Furthermore, suppose that $\operatorname{Ran} B$ is bulky, i.e. for any $g \in H$ there are $p \in Z^*$ and $h \in \operatorname{Dom} B$ such that $\langle g, ((p|Bh))\rangle \neq 0$; then $\langle g, ((p|B-\otimes u))h\rangle = \langle g, ((p|Bh))\rangle -$ $-\langle g, h\rangle (p|u) \neq 0$ if u is small enough, hence $\operatorname{Ran}(B-\otimes u)$ is bulky. Substitute $A-\otimes\lambda$ for B with a regular value λ of A to have the desired result.

PROPOSITION 6. *Let A be continuous. Equip Z with an inner product. Then the set* $\{z \in Z\colon \|z\| > \|A\|\}$ *is disjoint from* $\operatorname{Sp} A$.

PROOF. If $\|z\|>\|A\|$ then $\|(A-\otimes z)h\| \geqq \big|\|Ah\|-\|z\|\,\|h\|\big| \geqq (\|z\|-\|A\|)\|h\|$ for all $h \in \operatorname{Dom} A$, hence $A-\otimes z$ has a continuous inverse. We have to show now that $\operatorname{Ran}(A-\otimes z)$ is bulky. Let $\dot{z}$ denote that element of Z^* for which $(\dot{z}|y)=\langle z, y\rangle$ $(y \in Z)$. Then $((\dot{z}|A))=(\otimes z)^*A$, so $\|((\dot{z}|A))\| \leqq \|z\|\|A\| < \|z\|^2$, and thus $\|z\|^2=$ $=(\dot{z}|z)$ is not in the spectrum of $((\dot{z}|A))$ as it is well-known from usual operator theory. Consequently, $\operatorname{Ran}[((\dot{z}|A))-(\dot{z}|z)\,\mathrm{id}_H]=((\dot{z}|\operatorname{Ran}(A-\otimes z)))$ is dense in H; apply Proposition 3 to end the proof.

PROPOSITION 7. *Let Y be a finite dimensional vector space containing Z as a linear subspace. Then $H\otimes Z \subset H\otimes Y$ and a Z valued vector operator is also a Y valued vector operator. The spectrum of A is independent of whether A is considered as Z valued or Y valued.*

PROOF. We have to show that if $y \in Y$ and $y \notin Z$ then y is a regular value of A. Choose an inner product on Y and write $y=u+v$ such that u is in Z and $v \neq 0$ is orthogonal to Z. Then for all $h \in \operatorname{Dom} A$, $\|(A-\otimes y)h\|^2=\|(A-\otimes u)h\|^2+\|v\|^2\|h\|^2 \geqq$ $\geqq \|v\|^2\|h\|^2$, hence $A-\otimes y$ has a continuous inverse. Furthermore, using the notation introduced in Proposition 6, we have $((\dot{v}|(A-\otimes y)h))=-\|v\|^2 h$ $(h \in \operatorname{Dom} A)$ which yields that $\operatorname{Ran}(A-\otimes y)$ is bulky.

REMARKS. (i) If $Z=\mathbf{C}$, Definition 6 gives back the usual definition of the spectrum. If Z is one-dimensional, the spectrum of a Z valued vector operator has the usual properties.

(ii) To construct examples that the spectrum of a vector operator does not exhibit in geneıal all the properties of the usual spectrum, we take two dimensional spaces. Let h_1, h_2 and z_1, z_2 be an orthonormal basis of H and a basis of Z, respectively, and let us consider vector operators of the form $H \to H \otimes Z$, $h \mapsto (A_1 h) \otimes z_1 + (A_2 h) \otimes z_2$.

— The vector operator given by $A_1 h_1 := h_1$, $A_2 h_2 := 0$, $A_2 h_1 := A_2 h_2 := h_1 + h_2$ has a void spectrum.

— The spectrum of the vector operator given by $A_1 h_1 := A_1 h_2 := h_1 + h_2$, $A_1 h_2 := := A_2 h_1 := 0$ contains zero, but not as an eigenvalue.

(iii) Observe that the norm of vector operators depends on the inner product on Z. It is interesting that even the set $\{z \in Z\colon \|z\| > \|A\|\}$ depends on it. To see this let H and Z be as in (ii) and let A_1 and A_2 be the projections onto the subspaces spanned by h_1 and h_2, respectively. Then the corresponding vector operator has one and the same norm whatever be the inner product on Z such that $\|z_1\| = \|z_2\| = 1$.

(iv) If $A_1, \ldots, A_N$ are opeıators defined on a common dense lineaı subspace in H, the spectrum of the $\mathbf{C}^N$ valued vector operator whose components relative to the standard basis are the given operators is some sort of *joint spectrum* for $A_1, \ldots, A_N$.

5. Spectral theorem for vector operators

If T is a Hausdorff topological space, $B(T)$ denotes the algebra of Borel subsets of T. If P is a projection valued measure defined on $B(T)$ and having values in the set of projections of H then for all $h, g \in H$, $E \mapsto P_{h,g}(E) := \langle h, P(E)\, g \rangle$ is a complex measure on $B(T)$.

An element t of T is called a *sharp value* of P if $P(\{t\}) \neq 0$. The set of sharp values of P is denoted by Sharp P.

The *support* of P is the set

$$\operatorname{Supp} P := \{t \in T\colon\ P(G) \neq 0 \ \text{ for all open } G \text{ with } \ t \in G\}.$$

Definition 7. A Z valued vector operator A in H is called

(i) *partially normal* if

$((p|A))$ is closable and its closure is normal for all $p \in Z^*$,

$$\operatorname{Dom} A = \bigcup_{p \in Z^*} \operatorname{Dom} \overline{((p|A))};$$

(ii) *totally normal* if it is partially normal and $\overline{((p|A))}$ and $\overline{((q|A))}$ strongly commute for all $p, q \in Z^*$.

Proposition 8. (i) *A partially normal vector operator is densely defined and closed.*

(ii) *A continuous partially normal vector operator is totally normal.*

Proof. (i) is quite easy. To show (ii) observe the continuity of A implies that $\overline{((p|A))} = ((p|A))$. Take the bounded normal operators $((p+q|A)) = ((p|A)) + ((q|A))$ and $((p+iq|A))$ to obtain that $((p|A))$ commutes with $((q|A))^*$ which implies the commutativity of $((p|A))$ and $((q|A))$ $(p, q \in Z^*)$.

PROPOSITION 9. *Let A be a totally normal vector operator. Then there exists a unique projection valued measure R on $B(Z)$ such that*

$$\langle\langle h, Ag\rangle\rangle = \int_Z \mathrm{id}_Z dR_{h,g} \qquad (h\in H,\ g\in \mathrm{Dom}\, A).$$

PROOF. Let $p_1, \ldots, p_N$ be a basis in Z^* and let R_k be the spectral resolution of the normal operator $\overline{((p_k|A))}$ $(k=1, \ldots, N)$. Then $R_1, \ldots, R_N$ are commuting projection valued measures, hence their product $\bigotimes_{k=1}^{N} R_k$ exists and is the unique projection valued measure on $B(\mathbf{C}^N)$ determined by $\left(\bigotimes_{k=1}^{N} R_k\right)\left(\bigtimes_{k=1}^{N} E_k\right) = \prod_{k=1}^{N} R_k(E_k)$. Let b denote the inverse of the linear bijection $Z\to\mathbf{C}^N$, $z\mapsto\{(p_k|z)\colon k=1, \ldots, N\}$, and put $R:=\left(\bigotimes_{k=1}^{N} R_k\right)\circ b^{-1}$. Then for all $k=1, \ldots, N$, $h\in H$ and $g\in\mathrm{Dom}\, A$

$$(p_k|\langle\langle h, Ag\rangle\rangle) = \langle h, ((p_k|A))g\rangle = \int_{\mathbf{C}} \mathrm{id}_{\mathbf{C}} d(R_k)_{h,g} =$$

$$= \int_{\mathbf{C}^N} \mathrm{pr}_k d\left(\bigotimes_{i=1}^{N} R_i\right)_{h,g} = \int_Z p_k\, dR_{h,g} = \left(p_k\Big| \int_Z \mathrm{id}_Z dR_{h,g}\right)$$

where $\mathrm{pr}_k\colon \mathbf{C}^N\to\mathbf{C}$ is the k-th canonical projection; we also used the relation $p_k\circ b= =\mathrm{pr}_k$ and the well-known integral transformation formula. The uniqueness of R follows from the uniqueness of the R_k's and from the equalities

$$R = \left[\bigotimes_{k=1}^{N} (R\circ p_k^{-1})\right]\circ b^{-1}, \qquad R_k = R\circ p_k^{-1}.$$

REMARK. We can define the integral of measurable functions $T\to Z$ with respect to projection valued measures on $B(T)$ as Z valued vector operators. It can be shown that all such vector operators are totally normal. In other words, only the totally normal vector operators have spectral resolutions, i.e. are integrals of id_z with respect to projection valued measures.

PROPOSITION 10. *A bounded operator L commutes with a totally normal vector operator A if and only if L commutes with the spectral resolution of A.*

The proof of the following assertion requires a number of notions and particular results from the theory of integration with respect to projection valued measures. Who is familiar with them, can argue similarly as in the case of usual normal operators (see [2]), needing only one new step, a consideration on bulky subspaces. We omit these details.

PROPOSITION 11. *Let A be a totally normal vector operator having R as its spectral resolution. Then*

$$\mathrm{Eig}\, A = \mathrm{Sharp}\, R, \qquad \mathrm{Sp}\, A = \mathrm{Supp}\, R.$$

DEFINITION 8. Let V be a finite dimensional real vector space. A $V_{\mathbf{C}}$ valued vector operator A in H is called

(i) *partially self-adjoint* if

$\overline{((r|A))}$ is closable and its closure is self-adjoint for all $r\in V^*$,

$$\mathrm{Dom}\, A = \bigcap_{r\in V^*} \mathrm{Dom}\, \overline{((r|V))};$$

(ii) *totally self-adjoint* if it is partially self-adjoint and $\overline{((r|A))}$ and $\overline{((s|A))}$ strongly commute for all $r, s\in V^*$.

REMARKS. (i) A partially self-adjoint vector operator is densely defined and closed.

(ii) A partially self-adjoint vector operator need not be partially normal. For instance, the first operator given in Remaık (ii) at the end of Section 3, if $Z=V_C$, $z_1, z_2\in V$, is partially self-adjoint without being partially normal.

(iii) Taking a basis $r_1, ..., r_N$ in V^* (it is a basis in V_C^*, too, with respect to the complex structure) and repeating the argument of the proof of Proposition 9, this time considering $((r_k|A))$ instead of $((p_k|A))$, we find that a totally self-adjoint vector operator is the integral of id_{V_C} with respect to a projection valued measure whose support is in V. As a consequence, by the Remark to Proposition 9, a totally self-adjoint vector operator is totally normal, and its spectrum is contained in V.

EXAMPLES. (i) For $u\in z$, the vector operator $\otimes u$ is totally noımal, its spectral resolution is the projection valued measure concentrated at u.

(ii) The identity multiplication operator in $L^2(V)$ is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to $E\in B(V)$ the operator of multiplication by the characteristic function of E (which is the projection onto $L^2(E)\subset L^2(V)$).

(iii) The differentiation operator in $L^2(V)$ is closable, its closure multiplied by the imaginary unit is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to $S\in B(V^*)$ the projection $F^{-1}K(S)F$ where $K(S)$ is the projection onto $L^2(S)\subset L^2(V^*)$ and F: $L^2(V)\to L^2(V^*)$ is the Fourier transformation defined by

$$(Ff)(r) := \int_V e^{i(r|v)} f(v)\, dv \qquad (f\in L^2(V)\cap L^1(V),\ r\in V^*)$$

with the translation invariant measure on $B(V^*)$ chosen in such a way that F be unitary.

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