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Strict Asymptotic Stability*

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Abstract. A new notion of aymptotic stability for sets consisting of equilibria of an ordinary differential equation is introduced and sufficient conditions are established that imply strict asymptotic stability.

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1. Inroduction

In thermodynamical applications a new notion of stability appears [1] which is related to stability of sets [2]. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a function and consider the differential equation

$$\dot{x} = f \circ x \,. \tag{1}$$

Recall that a set S in the domain of f is called *stable* if for all $\epsilon > 0$ there is a $\delta > 0$ such that if r is a solution of (1) and $\operatorname{dist}(r(0), S) < \delta$ then $\operatorname{dist}(r(t), S) < \epsilon$ for all $t > 0$; S is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that if r is a solution of (1) and $\operatorname{dist}(r(0), S) < \eta$, then $\lim_{t\to\infty} \operatorname{dist}(r(t), S) = 0$.

If S consists of a single equilibrium x_0, then the (asymptotic) stability of $S = \{x_0\}$ coincides with the usual stability of x_0.

It is of special importance when S is a subset of

$$E := f^{-1}(0) = \{x \in \operatorname{Dom} f \mid f(x) = 0\}, \tag{2}$$

the set of equilibria of the differential equation (1).

Then the following two strange situations may occur.

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(i) S is stable though none of the equilibria in S is stable; for this consider the differential equations

$$
\begin{aligned}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -(x_1)^2 x_2
\end{aligned}
\tag{3}
$$

and $S := \{(0, x_2, 0) \mid x_2 \in \mathbf{R}\}$.

(ii) S is not stable though each equilibrium in S is stable: for this consider the previous differential equation and $S := \{(x_1, 0, 0) \mid x_1 > 0\}$.

We shall be interested in some kind of asymptotic stability of such sets.

DEFINITION 1. A subset S of E is called *strictly asymptotically stable* if

(i) each equilibribrium in S is stable,

(ii) for all $x \in S$ there is an $\eta > 0$ such that if r is a solution of (1) and $\operatorname{dist}(r(0), x) < \eta$ then $\lim_{t\to\infty} r(t)$ is in the closure of S.

Remark 1. Condition (ii) in the definition is equivalent to the following: there is an open set N containing S such that if r is a solution and $r(0) \in N$ then $\lim_{t\to\infty} r(t)$ is in the closure of S.

2. If S consists of isolated equilibria then S is strictly asymptotically stable if and only if all the equilibria in S are asymptotically stable in the usual sense.

3. Consider the linear equation $\dot{x} = Ax$. It is easy to check that $E = \operatorname{Ker} A$ is strictly asymptotically stable if and only if the geometric and algebraic multiplicity of the zero eigenvalue of A are equal and all the other eigenvalues have negative real parts.

4. The following example is of basic importance because some general cases can be reduced to it. Take the decomposition $\mathbf{R}^n = \mathbf{R}^m \times \mathbf{R}^k$, $x = (\alpha, \xi)$ and the differential equation

$$
\begin{aligned}
\dot{\alpha} &= 0 \\
\dot{\xi} &= \phi(\alpha, \xi)
\end{aligned}
\tag{4}
$$

where $\phi : \Omega \times \Delta \to \mathbf{R}^k$ is a continuously differentiable function, Ω is an open subset in $\mathbf{R}^m$, Δ is a neighbourhood of zero in $\mathbf{R}^k$, $\phi(\alpha, 0) = 0$ and $\phi(\alpha, \xi) \neq 0$ for all $\alpha \in \Omega$ and $0 \neq \xi \in \Delta$. Then

$$
E = \{(\alpha, 0) \mid \alpha \in \Omega\}
\tag{5}
$$

in the set of equilibria.

Evidently, for each fixed α the differential equation

$$\dot{\xi} = \phi(\alpha, \xi) \tag{6}$$

has the unique equilibrium $\xi = 0$ which is asymptotically stable if E is strictly asymptotically stable.

Conversely, if for each fixed α the unique equilibrium of the differential equation (6) is asymptotically stable in some "uniform way" with respect to α then E is strictly asymptotically stable. This "uniformity" can be formulated as follows:

Uniformity condition: For all compact subsets Γ of Ω there is a $\sigma > 0$ such that if $\alpha \in \Gamma$ and ρ_α is a solution of (6), $|\rho_\alpha(0)| < \sigma$ then $\lim_{t\to\infty} \rho_\alpha(t) = 0$.

2. Strict asymptotic stability and invariant submanifolds

Strict asymptotic stability of a set S of equilibria cannot be examined by known methods; we may encounter difficulties even in showing that the elements of S are stable. Namely, if the elements of S are not isolated equilibria, in general, we can hardly find Lyapunov functions to assure the stability of all the equilibria in S. To see the problem, the reader is advised to try to construct Lyapunov functions for the equilibria of the linear differential equation whose matrix is negative semidefinite.

Now a special important case of strict asymptotic stability - arising in thermodynamical applications - will be studied.

First of all some notions will be listed.

A subset Z of $\mathbf{R}^n$ is an *m-dimensional s times differentiable submanifold* if for every $x \in Z$ there are

(i) a neghbourhood U of x,

(ii) an s times continuously differentiable function $F : U \to \mathbf{R}^{n-m}$,

(iii) a $c \in \operatorname{Ran} F$, such that

- the derivative of F at every $x \in U$ (which is a linear map $DF(x) : \mathbf{R}^n \to \mathbf{R}^{n-m}$) is a surjection,
- $Z \cap U = F^{-1}(c) = \{x \in \operatorname{Dom} F \mid F(x) = c\}$.

Such an F is called a *level function* for Z.

The *tangent space* $T_x(Z)$ of Z at $x \in Z$ is $\operatorname{Ker} DF(x)$ for an arbitrary level function F having x in its domain.

Two submanifolds Z and A in $\mathbf{R}^n$ are *transversal* to each other if for every $x \in Z \cap A$ the tangent spaces $T_x(Z)$ and $T_x(A)$ are complementary subspaces (consequently, $\dim Z + \dim A = \mathrm{n}$).

An r times differentiable *transversal foliation* of the m-dimensional submanifold Z is a set of $(n-m)$-dimensional submanifolds, $\{A(x) \mid x \in Z\}$ such that for every $x \in Z$ there are

(i) a neighbourhood U of x,

(ii) an r times continuously differentiable function $G : U \to \mathbf{R}^m$,

and $A(x) \cap U = G^{-1}(G(x))$ for all $x \in Z \cap U$.

A subset H of $\operatorname{Dom} f$ in $\mathbf{R}^n$ is *invariant* for the differential equation (1) if every solution starting in H remains in H. A submanifold Z is invariant for the differential equation if and only if $f(x)$ is in $T_x(Z)$ for all $x \in Z$, i.e. if and only if $\mathrm{D}F(x)f(x) = 0$ for all level functions F for Z and for all $x \in \operatorname{Dom} F$.

PROPOSITION 1. *Suppose that*

(i) the function f in differential equation (1) is twice continuously differentiable,

(ii) the set E of equilibria is an $m-$dimensional twice differentiable submanifold in $\mathbf{R}^n$,

(iii) there is a three times differentiable transversal foliation $\{A(x) \mid x \in E\}$ such that every $A(x)$ is invariant for the differential equation,

(iv) there is a three times continuously differentiable function $L : \mathbf{R}^n \to \mathbf{R}$ defined on a neighbourhod of E such that

- *L is zero on E,*
- *L restricted to $A(x)$ has strict maximum at x in such a way that its second derivative restricted to $T_x(A(x))$ is negative definite for all $x \in E$,*
- *$DL \cdot f$ (the derivative of L along f) restricted to $A(x)$ has a strict minimum at x in such a way that its second derivative restricted to $T_x(A(x))$ is positive definite for all $x \in E$.*

Then E is strictly asymptotically stable.

Proof. Every element of E has a neighbourhood U in which there are twice continuously differentiable functions $G : U \to \mathbf{R}^m$ and $F : U \to \mathbf{R}^{n-m}$ such that $E \cap U = F^{-1}(0)$, $A(x) \cap U = G^{-1}(G(x))$ for all $x \in E \cap U$ and $(G, F) : \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^{n-m}$ is a coordinatization i.e. it is injective and its inverse is twice continuously differentiable as well. Then with the notations $\Omega := G[U]$, $\Delta := F[U]$, $\alpha := G(x)$, $\xi := F(x)$ for $x \in U$, $\Psi := (G, F)^{-1}$ and $\phi(\alpha, \xi) := DF(\Psi(\alpha, \xi)) f(\Psi(\alpha, \xi))$, this coordinatization transforms the differential equation (1) into the differential equation (4).

Moreover, we have that $E \cap U = \{\Psi(\alpha, 0)) \cap \mid \alpha \in \Omega\}$ and $A(\Psi(\alpha, 0)) \cap U = \{\Psi(\alpha, \xi) \mid \xi \in \Delta\}$ for all $\alpha \in \Omega$.

Then for all fixed α, $\xi \mapsto L(\Psi(\alpha, \xi)) =: \Lambda_\alpha(\xi)$ is Lyapunov function for the differential equation (6) assuring the asymptotic stability of the unique equilibrium $\xi = 0$. Indeed, Λ_α is given by the restriction of L onto $A(\Psi(\alpha, 0))$, thus it has a strict maximum (it is "negative definite") at $\xi = 0$ and its derivative along the differential equation (6) is given by the restriction of $DL \cdot f$ onto $A(\Psi(\alpha, 0))$:

$$\begin{aligned} D\Lambda_\alpha(\xi)\phi(\alpha, \xi) &= DL(\Psi(\alpha, \xi)) \cdot \frac{\partial \phi(\alpha, \xi)}{\partial \xi} \cdot \phi(\alpha, \xi) \\ &= DL(\Psi(\alpha, \xi)) \cdot D\Psi(\alpha, \xi) \cdot D\Psi^{-1}(\Psi((\alpha, \xi)) \cdot f(\Psi(\alpha, \xi)) \\ &= (DL \cdot f)(\Psi(\alpha, \xi)); \end{aligned}$$

thus it has a strict minimum (it is "positive definite") at $\xi = 0$.

Since $\alpha \mapsto (L \circ \Psi)(\alpha, 0) = 0$ and $\xi \mapsto (L \circ \Psi)(\alpha, \xi)$ has a maximum at $\xi = 0$ for all α, we have $D(L \circ \Psi)(\alpha, 0) = 0$ implying $DL(\Psi(\alpha, 0)) = 0$ for all α. Consequently,

$$D^2\Lambda_\alpha(0) = D^2L(\Psi(\alpha, \xi)) \left(\frac{\partial \Psi(\alpha, \xi)}{\partial \xi}, \frac{\partial \Psi(\alpha, \xi)}{\partial \xi} \right) \Big|_{\xi=0}$$

i.e. the second derivative of Λ_α at zero is given by the restriction of the second derivative of L to the tangent space of $A(\Psi(\alpha, 0))$ at $\Psi(\alpha, 0)$; thus the second derivative of Λ_α at zero is negative definite. Similarly, we can state that the second derivative of $\xi \mapsto D\Lambda_\alpha(\xi)\phi(\alpha, \xi) = (DL \cdot f)(\Psi(\alpha, \xi))$ at zero is positive definite.

Now we have that for every $\alpha \in \Omega$ there is a $\sigma_\alpha > 0$ such that Λ_α and its derivative along the differential equation (6) have strict extrema in the neighbourhood of radius σ_α of zero and every solution starting in this neighbourhood tends to zero at infinity. Because of the continuity and the definitness of the second derivatives, $\alpha \mapsto \sigma_\alpha$ is continuous function. Consequently, if Γ is a compact subset of Ω then $\sigma := \inf_{\alpha\in\Gamma} \sigma_\alpha > 0$ satisfies the requirement of the uniformity condition.

3. A particular result

Let us consider the following differential equation in $\mathbf{R}^2$ (arising in the thermodynamical description of a chemical reaction):

$$\begin{aligned} \dot{x}_1 &= F(x_1, x_2) \\ \dot{x}_2 &= F(x_1, x_2)h(x_1, x_2) \end{aligned} \tag{7}$$

where F and h are continuously differentiable real valued functions defined on a connected open subset of $\mathbf{R}^2$.

Then $E = F^{-1}(0)$.

PROPOSITION 2. *Suppose that*

(i) $\partial_2 F \mid_E < 0$,

(ii) $(\partial_1 F + h\partial_2 F) \mid_E < 0$.

Then E is strictly asymptotically stable.

Proof. Because of (i), E is a one dimensional submanifold (a curve) in $\mathbf{R}^2$. Moreover, there is (at least locally) a continuously differentiable function $a : \mathbf{R} \to \mathbf{R}$ such that (a part of) E is the graph of a.

Condition (ii) implies that the first order linear partial differential equation

$$\partial_1 G + h\partial_2 G = 0 \tag{8}$$

with boundary condition

$$G(x_1, x_2) = x_1 \qquad \text{if} \qquad (x_1, x_2) \in E \tag{9}$$

has a unique continuously differentiable solution defined in a neighbourhood of E ([3]) and the derivative of G on E is nowhere zero; indeed, the boundary condition can be written in the form $G(x_1, a(x_1)) = x_1$ from which we infer $\partial_1 G(x_1, a(x_1)) + a'(x_1)\partial_2 G(x_1, a(x_1)) = 1$.

As a consequence, $G^{-1}(c)$ is a one dimensional submanifold, invariant for the differential equation (7), if c is in the range of G, and G defines a transversal foliation of E.

Then with the notiations $\Psi := (G,F)^{-1}$, $\alpha := G(x_1,x_2)$, $\xi := F(x_1,x_2)$ and $\lambda(\alpha,\xi) := (\partial_1 F + h\partial_2 F)(\Psi(\alpha,\xi))$ we transform the differential equation (7) into the differential equation (4), where $m = k = 1$ and $\phi(\alpha,\xi) = \lambda(\alpha,\xi)\xi$.

Condition (ii) and the the continuity of λ imply that for each α, the function $\xi \mapsto \lambda(\alpha,\xi)$ is negative in a neighbourhood of zero; thus for each fixed α, $\xi = 0$ is an asymptotically stable equilibrium of the differential equation (4). Moreover, the uniformity condition, too, follows from the continuity of λ.

References

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