Tensor product of Hilbert lattices and free orthodistributive product of orthomodular lattices

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I. Introduction

In probability theory one defines the product probability space of a set of probability spaces as the product measure space known from measure theory ([1], [2]). However, the meaning of the product probability space is not generally clarified in probability theory except for a particular case. The distribution space of a random variable is a probability space built on the real line in a natural way. Given a set of independent random variables, the product of their distribution spaces can be interpreted as the distribution space of the random variables together.

Physics uses probability theory in the description of physical phenomena. However, it does from a point of view which is somewhat different from that of the classical probability theory. In the simplest physical theory, in mechanics, one assigns to each physical system a so-called logic which is the analogue of the algebraic structure of events and one is concerned with a set of probability measures on the logic, called the states of the physical system ([3]). Independence of events has generally no sense in this case, because independence is formulated with respect to a *given* probability measure. Events or random variables independent for one probability measure can be not independent for another. States of a physical system change one into another and it would be too restrictive to define independence with respect to all sates.

Therefore we see that the definition of product probability spaces as stated in classical probability theory may not work in physics. Moreover, there is another difficulty: one cannot assume in general that the logic is a σ-field of subsets of a set ([3]).

Nevertheless, there is a need for something like product probability space. Namely, if we are given two physical systems, how can we get a new physical system consisting of these two together?

In this paper mathematical aspects will be studied and physics appears again only in Discussion.

In the following sections there are given a definition and a solution of a problem without any relevance to general probability theory. A probabilistic point of view is given only in Discussion, together with a general formulation of the problem.

II. Hilbert lattices

In the sequel a Hilbert space means a non-zero finite or countably infinite dimensional complex or real Hilbert space.

The set $P(H)$ of closed linear subspaces of the Hilbert space H form a σ-lattice under the set theoretical ordering. That is, every denumerable subset of $P(H)$ has a least upper bound (called union, denoted by $\vee$) and a greatest lower bound (called meet, denoted by $\wedge$). (In fact $P(H)$ is a complete lattice.) This lattice has a minimal element — the zero subspace — and a maximal element — the whole space. Moreover, there is a unique orthocomplement of each $M \in P(H)$, denoted in the sequel by $M^{\perp}$. We write $M \perp N$ if M is contained in $N^{\perp}$, in other words, if M is orthogonal to N.

Let H and H' be Hilbert spaces. A map $u: P(H) \to P(H')$ will be called a *σ-orthohomomorphism* if it preserves σ-meets and orthocomplements (consequently, it preserves σ-unions, maximal and minimal elements as well). A *σ-orthoisomorphism* is a σ-orthohomomorphism which is one-to-one and onto. The following facts are known and easily verifiable.

Proposition 1. *$u: P(H) \to P(H')$ is a σ-orthohomomorphism if and only if*

(i) $u\left(\bigwedge_{n=1}^{\infty} M_n\right) = \bigwedge_{n=1}^{\infty} u(M_n)$ for $M_n \in P(H)$,

(ii) $u(H) = H'$,

(iii) $u(M) \perp u(N)$ *and* $u(M \vee N) = u(M) \vee u(N)$ *for* $M, N \in P(H)$, $M \perp N$.

Proposition 2. *A σ-orthohomomorphism u is injective if and only if $u(M)=0$ implies $M=0$.*

We shall use the notation $[x]$ for the subspace generated by the element x of a Hilbert space.

Proposition 3. *Let $u: P(H) \to P(H')$ be a σ-orthohomomorphism. Then* $\dim u([x]) = \dim u([y])$ *for all non-zero $x, y \in H$.*

Proof. One knows that for all $M, N \in P(H')$ satisfying $M \wedge N = 0$,

$$\dim\left((M \vee N) \wedge N^{\perp}\right) = \dim M.$$

Now, let $x, y \in H$ be non-zero vectors, $x \neq y$. Then

$$[x+y] \wedge [x] = [x+y] \wedge [y] = [y] \wedge [x] = 0,$$

$$[x+y] \vee [x] = [x+y] \vee [y] = [y] \vee [x]$$

and the same relations hold for the images by u. Hence, according to the previous remark, we have

$$\dim u([x]) = \dim u([x+y]) = \dim u([y]).$$

Proposition 4. *A σ-orthohomomorphism between Hilbert lattices is necessarily injective.*

Proof. Suppose that a σ-orthohomomorphism is not injective. Then there is non-zero subspace and even a one-dimensional subspace whose image is zero. Consequently, by Proposition 3, the image of any one-dimensional subspace is zero, hence all images are zero, which is a contradiction: the image of the whole space must be the whole space.

Corollary. *If there is a σ-orthohomomorphism from $P(H)$ into $P(H')$, then there is a finite or countably infinite number r such that* $\dim H' = r \cdot \dim H$.

Proof. If e_n $(n=1, 2, \ldots)$ is an orthogonal basis in H, then $u([e_n])$ are pairwise orthogonal subspaces spanning H'. r is the dimension of $u([x])$ for an arbitrary non-zero $x \in H$.

III. Tensor products of Hilbert lattices

Definition 1. Let H_1, H_2 and H be Hilbert spaces, all complex or all real. $(P(H);\ u_1, u_2)$ is called a *tensor product* of $P(H_1)$ and $P(H_2)$ if

(i) $u_i\colon P(H_i) \to P(H)$ is a σ-orthohomomorphism $(i = 1, 2)$,

(ii) $$\bigvee_{n=1}^{\infty} \bigvee_{m=1}^{\infty} \left(u_1(M_1^n) \wedge u_2(M_2^m)\right) = \left(\bigvee_{n=1}^{\infty} u_1(M_1^n)\right) \wedge \left(\bigvee_{m=1}^{\infty} u_2(M_2^m)\right)$$

for any pairwise orthogonal elements M_1^n of $P(H_1)$ and any pairwise orthogonal elements M_2^m of $P(H_2)$,

(iii) $u_1(P(H_1))$ and $u_2(P(H_2))$ generate $P(H)$, that is the smallest orthocomplemented subspace lattice containing both $u_1(P(H_1))$ and $u_2(P(H_2))$ is $P(H)$.

Definition 2. Let $(P(H); u_1, u_2)$ and $(P(H'); u_1', u_2')$ be tensor products of $P(H_1)$ and $P(H_2)$. We say that $(P(H'); u_1', u_2')$ is *subordinated* to $(P(H); u_1, u_2)$ if there is a σ-orthohomomorphism $u: P(H) \to P(H')$ such that $u'_i = u \circ u_i$ $(i=1, 2)$. If $(P(H); u_1, u_2)$ is also subordinated to $(P(H'); u'_1, u'_2)$ then the two tensor products are said to be *equivalent*.

Notice the trivial facts that u in Definition 2 is necessarily surjective and it is unique. Indeed, the image of u is an orthocomplemented sublattice of $P(H')$ and it contains a subset — the image of u_1' and of u_2' — generating $P(H')$. Furthermore, if there were two σ-orthohomomorphisms defining the same subordination, they would coincide on a subset — on the image of u_1 and of u_2 — generating $P(H)$, hence they would be equal. By the same reasons, equivalent tensor products are related by σ-orthoisomorphisms.

Subordination is a quasi-ordering on the tensor products of two given Hilbert lattices. After identification of equivalent tensor products the subordination will be an ordering. Our main task is to examine this ordered set. The notations will be as in Definition 1.

Proposition 5. *The only possible subordination between tensor products of Hilbert lattices is equivalence.*

Proof. A σ-orthohomomorphism establishing a subordination is necessarily surjective and also injective by Proposition 4, hence it is a σ-orthoisomorphism.

Proposition 6. *Let $M_2 \in P(H_2)$, $M_2 \neq 0$ be fixed. Then the map f_{1,M_2} from $P(H_1)$ into $P(u_2(M_2))$ defined by*

$$f_{1,M_2}(M_1) = u_1(M_1) \wedge u_2(M_2) \quad (M_1 \in P(H_1))$$

is a σ-orthohomomorphism. The same is true for the map f_{2,M_1} defined similarly for a fixed non-zero element M_1 of $P(H_1)$.

Proof. We show that f_{1,M_2} satisfies conditions (i)—(iii) of Proposition 1. Conditions (i), (ii) are trivially fullfilled. Let now $M_1, N_1 \in P(H_1)$, $M_1 \perp N_1$, and write $f = f_{1,M_2}$. Then $u_1(M_1) \perp u_1(N_1)$ and so $f(M_1) \perp f(N_1)$ as well. Furthermore,

$$\begin{aligned} f(M_1 \vee N_1) &= u_1(M_1 \vee N_1) \wedge u_2(M_2) = \big(u_1(M_1) \vee u_1(N_1)\big) \wedge u_2(M_2) = \\ &= \big(u_1(M_1) \wedge u_2(M_2)\big) \vee \big(u_1(N_1) \vee u_2(M_2)\big) = \\ &= f(M_1) \vee f(N_1), \end{aligned}$$

where we used condition (ii) of Definition 1.

Proposition 7. $u_1(M_1) \wedge u_2(M_2) = 0$ *if and only if either* $M_1 = 0$ *or* $M_2 = 0$.

Proof. f_{1,M_2} of Proposition 6 is injective by Proposition 4. Thus, by Proposition 2, for fixed $M_2 \neq 0$

$$u_1(M_1) \wedge u_2(M_2) = 0 \text{ if and only if } M_1 = 0,$$

and a similar relation holds for a fixed $M_1 \neq 0$.

Proposition 8. $\dim\big(u_1([x_1]) \wedge u_2([x_2])\big)$ *is the same for all* $0 \neq x_1 \in H_1$, $0 \neq x_2 \in H_2$.

Proof. Let us fix $x_2 \in H_2$, $x_2 \neq 0$. Then

$$\dim\big(f_{1,[x_2]}([x_1])\big) = \dim\big(u_1([x_1]) \wedge u_2([x_2])\big)$$

is independent of x_1 (Proposition 3). Similarly, it is independent of x_2.

Proposition 9. *Let* e_1^n $(n=1, 2, \ldots)$ *and* e_2^m $(m=1, 2, \ldots)$ *be maximal orthogonal systems in* H_1 *and in* H_2, *respectively. Then*

$$u([e_1^n]) \wedge u_2([e_2^m]) \qquad (n, m = 1, 2, \ldots)$$

are pairwise orthogonal subspaces which span H.

Proof. They are orthogonal because $u_1([e_1^n]) \perp u_1([e_1^{n'}])$ and $u_2([e_2^m]) \perp u_2([e_2^{m'}])$ if $n \neq n'$ and $m \neq m'$. Their span is

$$\bigvee_{n=1}^{\infty} \bigvee_{m=1}^{\infty} \big(u_1([e_1^n]) \wedge u_2([e_2^m])\big)$$

which equals H by condition (ii) in Definition 1 and the equalities $u_1(H_1) = u_2(H_2) = H$.

Corollary. *There is a finite or countably infinite number r such that* $\dim H = r \cdot \dim H_1 \cdot \dim H_2$.

Indeed, $r = \dim \big(u_1([x_1]) \wedge u_2([x_2])\big)$ for non-zero $x_1 \in H_1$, $x_2 \in H_2$.

Now we impose a further condition on tensor products. We expect, roughly speaking, that the image of u_1 and of u_2 fill $P(H)$ the possible fullest. It follows from Proposition 9 that the image, by the map f_{1, M_2}, of a one-dimensional subspace is one-dimensional if and only if $r=1$ and M_2 is one-dimensional. Hence f_{1, M_2} can be surjective only in that case. Now, our requirement of a maximality reads as follows:

Condition of fullness. The σ-orthohomomorphisms $f_{1, [x_2]}$ and $f_{2, [x_1]}$ are surjective for all non-zero $x_2 \in H_2$, $x_1 \in H_1$.

At this point we introduce a new notation. If K is a complex Hilbert space, $\bar{K}$ denotes its conjugate Hilbert space, that is a Hilbert space whose elements can be canonically identified with the elements of K such that if $\bar{x}$ and $\bar{y}$ in $\bar{K}$ correspond to x and y in K, then $\bar{x} + \bar{y}$ corresponds to $x + y$, $\bar{\lambda} \bar{x}$ corresponds to λx and $\langle \bar{x}, \bar{y} \rangle = \langle y, x \rangle$, where λ is an arbitrary complex number and $\langle , \rangle$ denotes the inner product both in $\bar{K}$ and in K. If K is real, $\bar{K} = K$.

Theorem 1. *Let* H_1 *and* H_2 *be Hilbert spaces*, $\dim H_1 \geqq 3$, $\dim H_2 \geqq 3$. *If the Hilbert spaces are complex, then there exist exactly two (non-equivalent) tensor products of* $P(H_1)$ *and* $P(H_2)$ *satisfying the condition of fullness. They are given by*

(i) $H = H_1 \otimes H_2, \quad u_1(M_1) = M_1 \otimes H_2, \quad u_2(M_2) = H_1 \otimes M_2;$

(ii) $H = \bar{H}_1 \otimes H_2, \quad u_1(M_1) = \bar{M}_1 \otimes H_2, \quad u_2(M_2) = \bar{H}_1 \otimes M_2,$

where $\otimes$ *denotes the usual tensor products of Hilbert spaces.*

If the Hilbert spaces are real, there is only one tensor product of $P(H_1)$ *and* $P(H_2)$ *satisfying the condition of fullness. It can be obtained from the above formulae, taking the case* (i).

Proof. Only the complex case will be considered, it reflects the real case as well.

Let us choose a vector of norm one in each one-dimensional subspace of H_1 and H_2. Let us denote their set by H_1^0 and H_2^0, respectively.

Let $r_2 \in H_2^0$ be fixed. Now the σ-orthohomomorphism $f_{1,[r_2]}$ is surjective by hypothesis and it is injective by nature, so it is a σ-orthoisomorphism. Hence, by a theorem of E. P. WIGNER ([3], pp. 166—169) there exists a unitary or antiunitary map $U_1^{(r_2)}: H_1 \to u_2([r_2])$, determined up to a scalar factor, such that

$$[U_1^{(r_2)} r_1] = u_1([r_1]) \wedge u_2([r_2]) \tag{2}$$

for all $r_1 \in H_1^0$. In the same way, for all $r_1 \in H_1^0$ one finds a unitary or antiunitary map $U_2^{(r_1)}: H_2 \to u_1([r_1])$ such that

$$[U_2^{(r_1)} r_2] = u_1([r_1]) \wedge u_2([r_2])$$

for all $r_2 \in H_2^0$. As a consequence, we are given a map ϑ from $H_1^0 \times H_2^0$ into the complex unit circle such that

$$U_1^{(r_2)} r_1 = \vartheta(r_1, r_2) U_2^{(r_1)} r_2$$

for all $r_1 \in H_1^0$ and $r_2 \in H_2^0$.

Our first aim is to show that ϑ is a product of two maps, one from H_1^0 and the other from H_2^0.

Let $r_i, s_i, t_i \in H_i^0$ $(i=1, 2)$ and $t_i = \lambda(t_i)(r_i + s_i)$, where $\lambda(t_i)$ is an appropriate complex number. We shall write

$$\lambda(t_1)^{(r_2)} = \begin{cases} \lambda(t_1) & \text{if } U_1^{(r_2)} \text{ is unitary,} \\ \overline{\lambda(t_1)} & \text{if } U_1^{(r_2)} \text{ is antiunitary,} \end{cases}$$

and similarly for all other possible choices of indices and representatives. Now we have:

$$U_1^{(t_2)} t_1 = \lambda(t_1)^{(t_2)} \{U_1^{(t_2)} r_1 + U_1^{(t_2)} s_1\} =$$

$$= \lambda(t_1)^{(t_2)} \{\vartheta(r_1, t_2) \lambda(t_2)^{(r_1)} [U_2^{(r_1)} r_2 + U_2^{(r_1)} s_2] + \vartheta(s_1, t_2) \lambda(t_2)^{(s_1)} [U_2^{(s_1)} r_2 + U_2^{(s_1)} s_2]\} =$$

$$= \lambda(t_1)^{(t_2)} \{\vartheta(r_1, t_2) \lambda(t_2)^{(r_1)} [\overline{\vartheta(r_1, r_2)} U_1^{(r_2)} r_1 + \overline{\vartheta(r_1, s_2)} U_1^{(s_2)} r_1] +$$

$$+ \vartheta(s_1, t_2) \lambda(t_2)^{(s_1)} [\overline{\vartheta(s_1, r_2)} U_1^{(r_2)} s_1 + \overline{\vartheta(s_1, s_2)} U_1^{(s_2)} s_1]\}.$$

On the other hand,

$$U_1^{(t_2)} t_1 = \vartheta(t_1, t_2) U_2^{(t_1)} t_2 =$$

$$= \vartheta(t_1, t_2) \lambda(t_2)^{(t_1)} \{\overline{\vartheta(t_1, r_2)} \lambda(t_1)^{(r_2)} [U_1^{(r_2)} r_1 + U_1^{(r_2)} s_1] + \overline{\vartheta(t_1, s_2)} \lambda(t_1)^{(s_2)} [U_1^{(s_2)} r_1 + U_1^{(s_2)} s_1]\}.$$

If $r_1 \neq s_1$ and $r_2 \neq s_2$ then $U_1^{(r_2)} r_1$, $U_1^{(r_2)} s_1$, $U_1^{(s_2)} r_1$ and $U_1^{(s_2)} s_1$ are linearly independent. Indeed, arbitrary two of them are linearly independent and the first two ones generate a subspace whose intersection with the subspace generated by the second two ones consists of zero only.

As a consequence, the two expressions for $U_1^{(t_2)}t_1$ can be equal only if all the corresponding coefficients are equal. In the second equality the coefficient of $U_1^{(r_2)}r_1$ resp. $U_1^{(s_2)}r_1$ equals that of $U_1^{(r_2)}s_1$ resp. $U_1^{(s_2)}s_1$. The same relation must hold in the first equality, whence we obtain

$$\vartheta(r_1, r_2)\vartheta(s_1, s_2) = \vartheta(r_1, s_2)\vartheta(s_1, r_2)$$

for all $r_1, s_1 \in H_1^0$ and $r_2, s_2 \in H_2^0$. It follows that $\vartheta(r_1, r_2) = \varphi_1(r_1)\varphi_2(r_2)$ for some function φ_1 on H_1^0 resp. φ_2 on H_2^0.

Consequently, the unitary or antiunitary maps, in the sense of Wigner's theorem, can be chosen so that

$$U_1^{(r_2)}r_1 = U_2^{(r_1)}r_2 \quad \text{for all} \quad r_1 \in H_1^0, \quad r_2 \in H_2^0.$$

Now we can assert that $U_1^{(r_2)}$ resp. $U_2^{(r_1)}$ are either unitary or antiunitary for all r_2 resp. for all r_1. We have this result from the equalities written for $U_1^{(t_2)}t_1$ taking $\vartheta = 1$.

It is now possible to define a map $U_1^{(x_2)}$ resp. $U_2^{(x_1)}$ for all $x_2 \in H_2$ resp. $x_1 \in H_1$. Let $x_2 = \lambda r_2$ where $r_2 \in H_2^0$ and λ is an appropriate complex number. Then we define

$$U_1^{(x_2)} = \begin{cases} \lambda U_1^{(r_2)} & \text{if} \quad U_2^{(r_1)} \quad \text{is unitary for all} \quad r_1 \in H_1^0, \\ \bar{\lambda} U_1^{(r_2)} & \text{if} \quad U_2^{(r_1)} \quad \text{is antiunitary for all} \quad r_1 \in H_1^0. \end{cases}$$

A similar definition is made for $U_2^{(x_1)}$.

As a consequence of these definitions, we have a map $b: H_1 \times H_2 \to H$ such that

$$(3) \qquad b(x_1, x_2) = U_1^{(x_2)}x_1 = U_2^{(x_1)}x_2 \quad (x_1 \in H_1,\ x_2 \in H_2)$$

and b is bilinear, or sesquilinear with respect to the first or to the second variable, or conjugate bilinear, according to the unitary or antiunitary nature of the $U_1^{(r_2)}$'s and $U_2^{(r_1)}$'s.

Consider the case when b is bilinear. Then there is a unique densely defined linear map F: $H_1 \otimes H_2 \to H$ such that

$$(4) \qquad F(x_1 \otimes x_2) = b(x_1, x_2).$$

If e_1^n $(n = 1, 2, \ldots)$ and e_2^m $(m = 1, 2, \ldots)$ are maximal orthogonal systems in H_1 and in H_2 respectively, then one knows that $e_1^n \otimes e_2^m$ $(n, m = 1, 2, \ldots)$ is a maximal orthogonal system in $H_1 \otimes H_2$. By Proposition 9, by (2) and (3), $b(e_1^n, e_2^m)$ $(n, m = 1, 2, \ldots)$ is a maximal orthogonal system in H. Thus F can be extended to a unitary map. From (4) one deduces that

$$[x_1] \otimes [x_2] = [x_1 \otimes x_2] = F^{-1}(u_1([x_1]) \wedge u_2([x_2]))$$

for all $x_1 \in H_1$, $x_2 \in H_2$, and it follows by condition 3 in Definition 1 that

$$F^{-1}(u_1(M_1)) = M_1 \otimes H_2 \quad \text{for all} \quad M_1 \in P(H_1),$$
$$F^{-1}(u_2(M_2)) = H_1 \otimes M_2 \quad \text{for all} \quad M_2 \in P(H_2)$$

which establishes an equivalence between the investigated lattice tensor product and the tensor product of the form (i) of Theorem 1.

If b is sesquilinear, we obtain $\overline{H}_1 \otimes H_2$ or $H_1 \otimes \overline{H}_2$. If b is conjugate bilinear, we arrive at $\overline{H}_1 \otimes \overline{H}_2$. There is a canonical antiunitary map between $\overline{H}_1 \otimes H_2$ and $H_1 \otimes \overline{H}_2$ as well as between $H_1 \otimes H_2$ and $\overline{H}_1 \otimes \overline{H}_2$, which are easily seen to establish an equivalence between the corresponding lattice tensor products.

On the contrary, the lattice tensor products corresponding to $H_1 \otimes H_2$ and to $\overline{H}_1 \otimes H_2$ are not equivalent. To see this, assume that there is a σ-orthoisomorphism $u\colon P(H_1 \otimes H_2) \to P(\overline{H}_1 \otimes H_2)$ such that

$$u(M_1 \otimes H_2) = \overline{M}_1 \otimes H_2 \quad \text{for all} \quad M_1 \in P(H_1),$$
$$u(H_1 \otimes M_2) = \overline{H}_1 \otimes M_2 \quad \text{for all} \quad M_2 \in P(H_2).$$

One knows that $(M_1 \otimes H_2) \wedge (H_1 \otimes M_2) = M_1 \otimes M_2$, thus

$$u(M_1 \otimes M_2) = \overline{M}_1 \otimes M_2 \quad \text{for all} \quad M_1 \in P(H_1), \quad M_2 \in P(H_2)$$

because u preserves meet. This implies that there is a unitary or antiunitary map $U\colon H_1 \otimes H_2 \to \overline{H}_1 \otimes H_2$ and a map τ from $H_1 \times H_2$ into the complex unit circle such that

$$U(x_1 \otimes x_2) = \tau(x_1, x_2)\bar{x}_1 \otimes x_2$$

for all $x_1 \in H_1$, $x_2 \in H_2$. It is routine to check that this can hold only for $\dim H_1 = \dim H_2 = 1$.

To end this section, let us observe that we can define the tensor product of finitely many Hilbert lattices as well by an easy generalization of Definition 1. It is given explicitely in Discussion in a more general context. Propositions 5, 6, 7, 8 and 9 can be stated and the condition of fullness can be defined in an obviously generalized manner for the case of finitely many Hilbert lattices. Then we have the following result.

Let m be a fixed natural number and let H_i $(i=1, 2, \ldots, m)$ be Hilbert spaces. Take an integer s, $0 \leqq s \leqq m$, and let C_s^m be the set of all combinations of order s of $1, \ldots, m$. We write for $M_i \in P(H_i)$ and for $p_s \in C_s^m$

$$M_i^{p_s} = \begin{cases} \overline{M}_i & \text{if} \quad i \in p_s \\ M_i & \text{if} \quad i \notin p_s; \end{cases}$$

$[m/2]$ will denote the integral part of $m/2$.

Theorem 2. *Let H_i be Hilbert spaces,* $\dim H_i \geqq 3$ $(i=1, 2, \dots, m)$. *If the Hilbert spaces are complex then there exist exactly 2^{m-1} different (non-equivalent) tensor products of $P(H_i)$ satisfying the condition of fullness. They are given by*

$$H = \bigotimes_{i=1}^{m} H_i^{p_s} \qquad (p_s \in C_s^m,\ s = 0, 1, \dots, [m/2],\ \text{if } m \text{ is even},\ m/2 \in p_{m/2}),$$

$$u_i(M_i) = H_1^{p_s} \otimes H_2^{p_s} \otimes \dots \otimes H_{i-1}^{p_s} \otimes M_i^{p_s} \otimes H_{i+1}^{p_s} \otimes \dots \otimes H_m^{p_s} \quad \textit{for all} \quad M_i \in P(H_i)$$

$$(i = 1, 2, \dots, m).$$

If the Hilbert spaces are real, there is only one tensor product satisfying the condition of fullness. It can be obtained from the above formulae putting $s=0$.

IV. Discussion

As it was pointed out in the Introduction we are interested in composed (or product) probability spaces in general. The proper subject of our investigations should be orthomodular σ-lattices; they are general enough to include the basic consepts both of classical and of the most important non-classical probability theory: σ-algebras of subsets as well as Hilbert lattices are orthomodular σ-lattices. The definition and fundamental properties of orthomodular σ-lattices can be found in [3], [4], [5]. We shall consider orthomodular σ-lattices and σ-orthohomomorphisms between them as objects and morphism of a category. For details on categories we refer to [6].

In the sequel $\mathbf{N}$ denotes the set of natural numbers and I is an arbitrarily chosen non-void set.

Definition 3. Let $\mathscr{C}$ be a subcategory of the category of orthomodular σ-lattices. Assume L_i $(i \in I)$ and L are objects of $\mathscr{C}$. Then $(L, (u_i)_{i\in I})$ is a *tensor product* (or *free orthodistributive product*) of the L_i's if

(i) $u_i\colon L_i \to L$ are injections in $\mathscr{C}$ $(i \in I)$;

(ii) $\bigcup_{i\in I} u_i(L_i)$ generates L;

for every finite or countable subset F of I

(iii) $\bigwedge_{i\in F} u_i(a_i) = 0$ for $a_i \in L_i$ if and only if at least one a_i is zero;

(iv) if $(a_i^n)_{n\in\mathbf{N}} \subset L_i$ $(i \in F)$ are subsets consisting of pairwise orthogonal elements, then

$$\bigwedge_{i\in F} \bigvee_{n\in\mathbf{N}} u_i(a_i^n) = \bigvee_{n\in\mathbf{N}^F} \bigwedge_{i\in F} u_i(a_i^{n_i}).$$

The subordination of tensor products can be defined similarly as in Definition 2.

The definition of tensor products is motivated by physical considerations, outlined here briefly. The L_i's ($i \in I$) are logics (see in the Introduction) of given physical systems and we are seeking the logic of the physical system consisting of the given ones. Condition (i) in Definition 3 requires no comment. Condition (ii) expresses that the component physical systems determine somehow the composite system. Condition (iii) reflects that the component systems are independent, that is there are no constraints among them; interactions, however, may occur. Condition (iv) is an expression of the requirement that the events of different components shall be compatible (the notion of compatibility can be found in [3]).

We defined tensor product in the special case of Hilbert lattices only for finitely many objects because we can give a characterization only in that case. Observe that the u_i's are not required to be injections in Definition 1, because they are injections by Proposition 4. Similarly, condition (iii) of Definition 3 is missing from Definition 1 in view of Proposition 6.

Results are available mostly for the full subcategory of Boolean σ-algebras. For instance, if I is finite, then condition (iv) in Definition 3 is void because of the distributivity in L. Then we know that there exists a maximal tensor product in the ordered set of equivalent tensor products ("free Boolean σ-products" [7] p. 177). It is known as well that in the full subcategory of σ-algebras of sets — where condition (iv) of Definition 3 is void for an arbitrary I — there is only one tensor product (up to equivalence) and this is the σ-algebra generated by "cylinders" in the Descartes product of the underlying sets, well-known from measure theory ([7], p. 186).

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