Unification mechanism for gauge and spacetime symmetries

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Abstract. A group theoretical mechanism for unification of local gauge and spacetime symmetries is introduced. No-go theorems prohibiting such unification are circumvented by slightly relaxing the usual requirement on the gauge group: only the so called Levi factor of the gauge group needs to be compact semisimple, not the entire gauge group. This allows a non-conventional supersymmetry-like extension of the gauge group, glueing together the gauge and spacetime symmetries, but not needing any new exotic gauge particles. It is shown that this new relaxed requirement on the gauge group is nothing but the minimal condition for energy positivity. The mechanism is demonstrated to be mathematically possible and physically plausible on a U(1) based gauge theory setting. The unified group, being an extension of the group of spacetime symmetries, is shown to be different than that of the conventional supersymmetry group, thus overcoming the McGlinn and Coleman-Mandula no-go theorems in a non-supersymmetric way.

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1. Introduction

Unification attempts of internal (gauge) and spacetime symmetries is a long pursued subject in particle field theory. If such unification exists, it would relate coupling factors in the Lagrangian to each-other, which is a strong theoretical motivation. The non-trivialness of the problematics of such unification, however, is well-known. The Coleman-Mandula no-go theorem [1] forbids the most simple unification scenarios. Namely, any larger symmetry group, satisfying a set of plausible properties required by a particle field theory context, and containing the group of spacetime symmetries as a subgroup as well as a gauge group, must be of the trivial form: gauge group × group of spacetime symmetries‡. Also, the earlier theorem of McGlinn [2] concluded in the same direction. The classification result of O'Raifeartaigh [3] on Poincaré group extensions is also usually interpreted in a similar manner. After the discovery of these results, the simple unification attempts of gauge symmetries with spacetime symmetries were not pursued further. Instead, a large amount of research was carried out along the question: can the Poincaré Lie algebra be extended at all in at least by means of some mathematically generalized manner? The answer was positive, as stated by the result of Haag, Lopuszanski and Sohnius [4], and hence the era of supersymmetry (SUSY) was born.

By studying the details of the proof of McGlinn and Coleman-Mandula theorems [5] one finds that the assumption of the presence of a positive definite non-degenerate invariant scalar product on the Lie algebra of the gauge group is essential. Equivalently, these no-go theorems assume that the gauge group is of the form $\mathrm{U}(1)\times\ldots\times\mathrm{U}(1)\times$ a semisimple compact Lie group. The motivations behind this requirement are threefold:

(i) Group theoretical convenience: the classification of semisimple Lie groups is well understood.

(ii) Experimental justification: the Standard Model (SM) has a gauge group of the form $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$, which satisfies the requirement.

(iii) Positive energy condition: the energy density expression of a Yang-Mills (gauge) field involves the pertinent invariant scalar product on the Lie algebra of the gauge group, and that is required to be positive definite.

Traditionally, gauge groups not obeying the above rule are believed to violate positive energy condition, and therefore are considered to be unphysical. However, looking more carefully, the positive energy condition merely requires that the invariant scalar product on the Lie algebra of the gauge group must be positive *semidefinite*.

‡ Whenever a particle field theory model is considered on a fixed flat background spacetime, i.e. not considered as coupled to General Relativity (GR), then the group of spacetime symmetries is simply the Poincaré group. On the other hand, whenever a fully general relativistic field theory is studied, the group of spacetime symmetries is the full diffeomorphism group of the spacetime manifold, acting on the field configurations. Eventually, a general relativistic field theory might be also conformally invariant, in which case the group of spacetime symmetries is the diffeomorphism group along with conformal rescalings (Weyl rescalings) of the spacetime metric tensor field.

In this paper we construct an example when this relaxed condition is considered, and show that this case is mathematically possible, physically plausible, and can be a key to unification of gauge and spacetime symmetries. The proposed mechanism can serve as an alternative to (extended) SUSY.

The structure of the paper is as follows. In Section 2 the Levi decomposition of Lie groups and Lie algebras are recalled, along with O'Raifeartaigh theorem and the elements of SUSY. In Section 3 the proposed structure for a unified gauge–Poincaré group is presented, which survives the previously recalled group theoretical constaints. In Section 4 a concrete example group is presented for such unification, with U(1) being the compact gauge group component. In Section 5 our construction is compared to the mechanism of SUSY or extended SUSY. In Section 6 a conclusion is presented. The paper is closed by a set of Appendices, which expose further technical details on the concrete U(1) based example group.

2. Structure of Lie groups and supersymmetry

2.1. Levi decomposition theorem

Recall that the symmetry group of flat spacetime, the Poincaré group $\mathcal{P}$ is composed of the group of spacetime translations $\mathcal{T}$ and of the homogeneous Lorentz group $\mathcal{L}$. Moreover, the group of spacetime translations $\mathcal{T}$ form a *normal subgroup*§ within the Poincaré group $\mathcal{P}$. Also recall that the Poincaré group can be written as $\mathcal{P}=\mathcal{T}\rtimes\mathcal{L}$, where $\rtimes$ denotes *semi-direct product*‖. It is seen that in the above formula $\mathcal{T}$ is an abelian normal subgroup of $\mathcal{P}$, and that the subgroup $\mathcal{L}$ of $\mathcal{P}$ is a simple matrix group. The Levi decomposition theorem [6] states that such decomposition property is generic to all Lie groups. Namely, any Lie group, assumed now to be connected and simply connected for simplicity, has the structure $R\rtimes L$, R being a solvable normal subgroup called the *radical* and L being a semisimple subgroup called the *Levi factor*. The *semisimpleness* of L means that the *Killing form* $(x,y)\mapsto\mathrm{Tr}(\mathrm{ad}_x\,\mathrm{ad}_y)$ is non-degenerate on the Lie algebra of L, using the symbol $\mathrm{ad}_x(\cdot):=[x,\cdot]$ for any Lie algebra element x. The *solvability* of R means that it represents the degenerate directions of the Killing form. It may also be formulated in terms of an equivalent property: for the Lie algebra r of R with the definition $r^0:=r$, $r^1:=[r^0,r^0]$, $r^2:=[r^1,r^1]$, $\ldots$, $r^k:=[r^{k-1},r^{k-1}]$, $\ldots$, one has $r^k=\{0\}$ for finite k. A special case is when the radical R is said to be *nilpotent*: there exists a finite k for which for all $x_1,\ldots,x_k\in r$ one has $\mathrm{ad}_{x_1}\ldots\mathrm{ad}_{x_k}=0$. An even more special case is when the radical R is *abelian*: for all

§ A subgroup N within a larger group is called normal subgroup whenever it is invariant to the adjoint action of the larger group, i.e. whenever one has $g\,N\,g^{-1}\subset N$ for all elements g of the larger group.

‖ Semi-direct product means that any element of the larger group can uniquely be written as a product of elements from the coefficient groups, and that at least the leftmost coefficient group is normal subgroup. The two coefficient groups are not required to commute. When they commute, then also the rightmost coefficient group is normal subgroup, and the semi-direct product becomes a direct product, denoted by $\times$.

$x \in r$, one has $\mathrm{ad}_x = 0$.

The (proper) Poincaré group with its structure $\mathcal{T} \rtimes \mathcal{L}$ is a demonstration of Levi decomposition theorem, where $\mathcal{T}$ is the abelian normal subgroup consisting of spacetime translations, being the radical, and where $\mathcal{L}$ is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. Groups like $\mathrm{SU}(N)$, often turning up as gauge groups in Yang-Mills models, however are semisimple, and therefore their radical vanishes, i.e. such a group consists purely of its Levi factor. Historically, groups with nonvanishing radical are usually not studied in context with physical field theory models, even though the symmetry group of flat spacetime readily provides an archetypical example for such groups.

2.2. Levi structure of supersymmetry group

The Levi decomposition theorem also sheds a light on the group structure of supersymmetry transformations, being an extension of the Poincaré group. That Lie group has a Levi decomposition of the form $\mathcal{S} \rtimes \mathcal{L}$, where $\mathcal{S}$ is the nilpotent normal subgroup consisting of *supertranslations*, being the radical, and where $\mathcal{L}$ is the semisimple subgroup consisting of the (proper) homogeneous Lorentz transformations, being the Levi factor. The supertranslations are defined as transformations on the vector bundle of superfields [7, 8, 9]. With supertranslation parameters $(\epsilon^A,\, d^a)$ they are of the form

$$\begin{pmatrix} \theta^A \\ x^a \end{pmatrix} \mapsto \begin{pmatrix} \theta^A + \epsilon^A \\ x^a + d^a + \sigma^a_{AA'}\mathrm{i}(\theta^A\bar{\epsilon}^{A'} - \epsilon^A\bar{\theta}^{A'}) \end{pmatrix} \tag{1}$$

in terms of "supercoordinates" (Grassmann valued two-spinors) and affine spacetime coordinates.¶ From Eq.(1) it is seen that although the pure spacetime translations $\mathcal{T}$ form an abelian normal subgroup inside $\mathcal{S}$, but $\mathcal{S} \neq \mathcal{T} \rtimes \{\text{some other subgroup}\}$, and thus such splitting is not applicable for the entire supersymmetry group. A geometric consequence of that phenomenon is illustrated in Figure 1: a pure supertranslation with parameter $(\epsilon^A, 0)$ does not act *pointwise* (or *fibrewise*), but it transforms a superfield value at a point of spacetime to an other superfield value over a point shifted by a corresponding spacetime translation. Note that such shift cannot be compensated by a counter-translation, because the introduced spacetime point shift depends on the field value in the fiber, i.e. is not constant as a function of the supercoordinate.

In this paper, however, we shall present a different nontrivial Poincaré group extension, enlarged both on the side of the radical and of the Levi factor, containing

¶ A note about the presentation of supersymmetry transformations: usually, they are presented in the infinitesimal form and in a parametrization which is often referred to as a "graded Lie algebra", or "super Lie algebra". That form, however, may be reparametrized in order to form a conventional Lie algebra, as shown in [7, 8, 9], see also Section 5. This Lie algebra presentation, when exponentiated, shall form a conventional Lie group discussed above. This simple reparametrization, although is known in the literature [7, 8, 9], is mostly not used in the traditional way of presentation of SUSY.

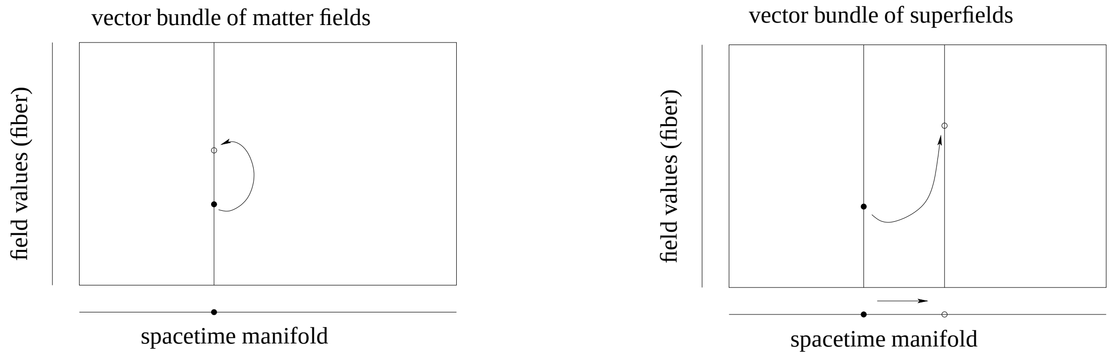

Figure 1. Left panel: Illustration of how in a conventional gauge theory the gauge symmetries, i.e. the transformations complementing the spacetime symmetries act on the vector bundle of matter fields. The action of the such transformations do preserve the spacetime points, i.e. they act "pointwise" on the matter fields. Our construction for a unified gauge – spacetime symmetry group shall also possess such property. Right panel: Illustration of how in a supersymmetric theory the transformations complementing the spacetime symmetries (i.e. the pure supertranslations) act on the vector bundle of superfields. Such a transformation does not act "pointwise", but maps a field value into a field value over a shifted point of spacetime. The amount of shift depends also on the field value, and therefore cannot be compensated by a counter-translation.

both the gauge and the spacetime symmetries, and being of the form

$$\mathcal{T} \rtimes \{\text{some group acting at points of spacetime}\}, \tag{2}$$

and thus rather acting pointwise, similarly as conventional gauge groups do, as illustrated in the left panel of Figure 1.

2.3. O'Raifeartaigh classification of Poincaré group extensions

Let us take a larger symmetry group E with its Levi decomposition $E = R \rtimes L$, containing the Poincaré group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$ as a subgroup. Then the theorem of O'Raifeartaigh [3] states that either one has $\mathcal{T} \subset R$ and $\mathcal{L} \subset L$ (radical embedded into radical, Levi factor embedded into Levi factor), or one has $\mathcal{T} \rtimes \mathcal{L} \subset L$ (the entire Poincaré group is embedded into the Levi factor of a much larger symmetry group). This result leads to the following classification theorem of O'Raifeartaigh [3] on the possible extensions of the Poincaré group:

(i) $R = \mathcal{T}$, and $L = \{\text{some semisimple Lie group}\} \times \mathcal{L}$. This means that whenever the radical R of the larger symmetry group solely consists of the spacetime translations, then one has only the trivial group extension $E = \mathcal{P} \times \{\text{some extra symmetries}\}$. This group theoretical phenomenon drives the no-go theorems of McGlinn and Coleman-Mandula.

(ii) R is an abelian extension of $\mathcal{T}$, and $\mathcal{L} \subset L$. This means that in the radical R of the larger symmetry group one has the spacetime translations and some abelian extension. The Levi factor L of the extended symmetries E may be larger than $\mathcal{L}$.

(iii) R is a non-abelian extension of $\mathcal{T}$, and $\mathcal{L} \subset L$. In this case the radical R contains the spacetime translations and some non-abelian solvable extension. The Levi factor L of the extended symmetries E can be larger than $\mathcal{L}$. SUSY, extended SUSY, as well as the example to be presented in this paper falls into this case.

(iv) $\mathcal{T} \rtimes \mathcal{L} \subset L$ and L is a simple Lie group. This case means that the entire Poincaré group is fully embedded into a much larger simple Lie group. Conformal theories, i.e. theories having the conformal Poincaré transformations as symmetry group+ are typical examples. Also an SO(1, 13) based theory [10], as well as an E_8 based theory [11] provide such examples. All of these models do need a symmetry breaking to explain a Standard Model-like limit of the corresponding theory, since the embedding group is rather large.

Consequently: for nontrivially extending the Poincaré group, its radical must necessarily be extended, as shown by cases (ii)–(iii), or the extended group must be a spontaneously broken large simple Lie group, as shown for the case (iv).

It is seen that the supersymmetry group is of type (iii) in the classification theorem of O'Raifeartaigh: its radical is extended and therefore the no-go theorems of McGlinn and Coleman-Mandula are not applicable. The unification mechanism for gauge and spacetime symmetries proposed in the followings uses the same group theoretical possibility as well, but in a very different way in comparison to SUSY: our extended group shall have the structure Eq.(2), which is not the case for the SUSY group.

3. Unification mechanism for gauge and spacetime symmetries

In terms of global symmetries, our proposed unification mechanism for gauge and spacetime symmetries assumes a structure

$$\underbrace{\Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \overbrace{\underbrace{\mathcal{N}}_{\text{solvable internal}}\Big) \rtimes \Big(\underbrace{\mathcal{G}}_{\text{compact internal}}}^{\text{full gauge (internal) group}} \times \underbrace{\mathcal{L}}_{\text{Lorentz (or Weyl) group}}\Big)}_{\text{global symmetries of matter fields when considered over flat spacetime}} \tag{3}$$

for the unified group. Here, $\mathcal{G}$ symbolizes the usual compact gauge group, being $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ in case of Standard Model, $\mathcal{L}$ denotes the homogeneous part of the spacetime symmetry group, being the homogeneous Lorentz (or possibly, the Weyl*) group, and $\mathcal{N}$ stands for a non-usual extension of the group of internal symmetries, allowed to be a solvable normal subgroup. The arrows indicate which subgroup acts nontrivially on which normal subgroup, i.e. subgroups not connected by arrows do

+ Conformal Poincaré group is isomorphic to SO(2, 4), hence it is a simple Lie group.

* Weyl group: the homogeneous Lorentz group augmented by the group of metric rescalings with a constant conformal factor.

commute, whereas the others do not. Clearly, such group structure as a Poincaré group extension is potentially allowed by the case (iii) of O'Raifeartaigh classification theorem. Using the semi-associativity of $\rtimes$ and $\times$, the global unified group described by Eq.(3) can be rewritten in an equivalent form

$$\underbrace{\underbrace{\mathcal{T}}_{\text{translations}} \rtimes \underbrace{\Big(\underbrace{\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \Big(\underbrace{\mathcal{G}}_{\text{compact internal}}}_{\text{full gauge (internal) group}} \times \underbrace{\mathcal{L}}_{\text{Lorentz (or Weyl) group}} \Big)\Big)}_{\text{symmetries of matter fields at points of spacetime}}}_{\text{global symmetries of matter fields when considered over flat spacetime}} \tag{4}$$

which shows that our unified group, as global symmetries, are of the form of Eq.(2). That naturally motivates to search for a local unified group of gauge and spacetime symmetries in the form

$$\underbrace{\underbrace{\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \Big(\underbrace{\mathcal{G}}_{\text{compact internal}}}_{\text{full gauge (internal) group}} \times \underbrace{\mathcal{L}}_{\text{Lorentz (or Weyl) group}} \Big)}_{\text{local symmetries of matter fields at points of spacetime}} \tag{5}$$

which is just Eq.(4) without the translations, acting as local symmetry on the matter fields independently at each point of a spacetime manifold. Again, using the semi-associativity of $\rtimes$ and $\times$, the local unified group described by Eq.(5) can be rewritten in the equivalent form

$$\underbrace{\underbrace{\Big(\underbrace{\mathcal{N}}_{\text{solvable internal}} \rtimes \underbrace{\mathcal{G}}_{\text{compact internal}} \Big)}_{\text{full gauge (internal) group}} \rtimes \underbrace{\mathcal{L}}_{\text{Lorentz (or Weyl) group}}}_{\text{local symmetries of matter fields at points of spacetime}} \tag{6}$$

which implies that there exists a *homomorphism*♯ from the local unified group Eq.(6) onto the local group of spacetime symmetries $\mathcal{L}$, and the kernel of that homomorphism is the local group of internal (gauge) symmetries $\mathcal{N} \rtimes \mathcal{G}$. This finding implies the following consequences:

(i) The full local unified group Eq.(5) has a four-vector representation through the homomorphism onto $\mathcal{L}$.

♯ Group homomorphism: a product preserving mapping from one group to another.

(ii) The group of local internal (gauge) symmetries $\mathcal{N} \rtimes \mathcal{G}$ act trivially on such four-vector representation — hence the name: they act trivially on the spacetime vectors.

(iii) The full local unified group Eq.(5) acts as the Lorentz (or Weyl) group on such four-vector representation.

(iv) Because of the previous point, there exists a uniquely determined Lorentz metric conformal equivalence class on the four-vector representation, preserved by the local unified group Eq.(5).

(v) Because of the previous point, there exists a uniquely determined Lorentz causal structure preserved by the local unified group Eq.(5).

(vi) Due to the presence of $\mathcal{N}$, the local unified group Eq.(5) is indecomposable, i.e. is not of the form of a direct product.

In conclusion, Eq.(5) shows that the local gauge group and the group of local spacetime symmetries would decompose into a direct product $\mathcal{G} \times \mathcal{L}$ as dictated by the McGlinn and Coleman-Mandula no-go theorems, however the solvable normal subgroup $\mathcal{N}$ of local gauge symmetries glues them together, making the unification. With that, the full local gauge group shall be an extended one, $\mathcal{N} \rtimes \mathcal{G}$, as a price to pay. Since $\mathcal{N}$ represents the degenerate directions of the Killing form over the full gauge group $\mathcal{N} \rtimes \mathcal{G}$, it only adds some zero-energy gauge field modes to a field theoretical model having local unified symmetries as Eq.(5). These zero-energy gauge field modes shall also have vanishing Yang-Mills kinetic Lagrangian term, and therefore such unification mechanism does not cost adding new propagating gauge particle fields to the system. They do contribute, however, to other parts of the Lagrangian involving matter fields and their covariant derivatives, restricting the forms of possible Lagrangians compatible with the extended symmetry requirement. It is remarkable, that the proposed unification mechanism does not necessarily need a breaking of the large symmetry group, as the non-conventional part $\mathcal{N}$ of internal symmetries is inapparent in terms of detectable gauge particles. Also, one should note that the allowed more relaxed structure $\mathcal{N} \rtimes \mathcal{G}$ of the full gauge group means a softer regularity condition than traditionally required in gauge theory: only the Levi factor of the gauge group needs to be compact, not the entire gauge group itself. This is equivalent to the positive semidefiniteness of the Killing form on the gauge group, and hence is the minimal requirement for the non-negativity of the energy density expression of the Yang-Mills fields in a system with such unified symmetries.

In the coming section we shall construct a minimal version of a unified local symmetry group as in Eq.(5), with $\mathcal{G} = \mathrm{U}(1)$. There is strong indication that the same mechanism can also be performed for the full Standard Model gauge group, e.g. using the approach of [12].

4. Concrete example for the U(1) case

Our example for a local unified symmetry group having the structure like Eq.(5) with $\mathcal{G} = \mathrm{U}(1)$ shall be described below. It is a non-supersymmetric extension of the (proper) homogeneous Lorentz (or rather, of the Weyl) group. It is detailed in [13, 14] and in Appendix A.

Let A be a finite dimensional complex unital associative algebra, with its unit denoted by $\mathbb{1}$. Whenever A is also equipped with a conjugate-linear involution $(\cdot)^+ : A \to A$ such that for all $x, y \in A$ one has $(xy)^+ = x^+ y^+$, then it shall be called a $^+$-*algebra*. Note that this notion differs from the well-known mathematical notion of *-algebra as here the $^+$-adjoining does not exchange the order of products. Let now A be a finite dimensional complex associative algebra with unit, being also $^+$-algebra, and possessing a minimal generator system (e_1, e_2, e_3, e_4) obeying the identity

$$\begin{aligned}
e_i e_j + e_j e_i &= 0 \quad (i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}), \\
e_i e_j - e_j e_i &= 0 \quad (i \in \{1,2\} \text{ and } j \in \{3,4\}), \\
e_3 &= e_1^+, \\
e_4 &= e_2^+, \\
e_{i_1} e_{i_2} \ldots e_{i_k} & \quad (1 \le i_1 < i_2 < \ldots < i_k \le 4,\ 0 \le k \le 4) \\
& \quad \text{are linearly independent.}
\end{aligned} \tag{7}$$

Then we call A *spin algebra*, and we call a minimal generator system obeying Eq.(7) a *canonical generator system*, whereas the $^+$-operation is called *charge conjugation*. That is, spin algebra is a freely generated unital complex associative algebra with four generators, and the generators admit two sectors within which the generators anticommute, whereas the two sectors commute with each-other, and are charge conjugate to each-other. It is easy to check that if S^* is a complex two dimensional vector space (called the *cospinor space*), and $\bar{S}^*$ is its complex conjugate vector space, then $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ naturally becomes spin algebra, where $\Lambda(\cdot)$ denotes the exterior algebra of its argument. It is also seen that any spin algebra is isomorphic (not naturally) to this algebra, i.e. they all have the same structure, but there is a freedom in matching the canonical generators. Some properties of the pertinent mathematical structure is listed in [13]. In terms of a formal quantum field theory (QFT) analogy, the spin algebra can be regarded as the creation operator algebra of a fermion particle with two internal degrees of freedom along with its antiparticle, at a fixed point of spacetime, or equivalently, at a fixed point of momentum space. It is important to understand, however, that in this construction the creation operators of antiparticles are not yet identified with the annihilation operators of particles, i.e. it is not a canonical anticommutation relation (CAR) algebra. As such, the spin algebra reflects the following physical picture:

(i) The basic ingredients of the system are particles obeying Pauli's exclusion principle.

(ii) These particles have finite (two) internal degrees of freedom.

(iii) Corresponding charge conjugate particles are present in the system.

Our extension of the homogeneous Weyl group shall be nothing but $\mathrm{Aut}(A)$, the *automorphism group* of the spin algebra A. That consists of those invertible $A \to A$ linear transformations, which preserve the algebraic product as well as the charge conjugation operation.

It can be shown that if the discrete symmetries are omitted, i.e. if the unit connected component of $\mathrm{Aut}(A)$ is considered, then it has a structure of the form

$$\underbrace{\underbrace{\underbrace{N}_{\text{nilpotent internal}} \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{compact internal}}}_{\text{full gauge (internal) symmetries}} \times \underbrace{\mathcal{L}}_{\text{Weyl group}} \Big)}_{\text{symmetries of } A\text{−valued fields at a point of spacetime or momentum space}} \tag{8}$$

which exactly has a structure like Eq.(5). For details we refer to Appendix A and [13, 14]. The nilpotent normal subgroup N of internal symmetries transform a system of canonical generators in such a way, that it adds higher polynomials of the generators to pure generators, and hence they are named "dressing transformations". Note that the pertinent example group $\mathrm{Aut}(A)$ can also be restricted so that it does not contain the conformal (Weyl) dilatations, but merely the Lorentz group instead of the Weyl group. That is, the inclusion or exclusion of the conformal dilatations to the unified group is optional: both constructions are group theoretically possible.

The nature of our example construction shows that the proposed unified symmetry group can be considered as the symmetries of a limiting scenario in QFT, when the position (or momentum) of fields are fixed and only the internal degrees of freedom are allowed to behave according to the algebra rules of fermionic particle and antiparticle creation operators. In that picture, the new nilpotent symmetries N can be understood to mix higher particle contributions to single particle creation operators, and this is how the mechanism bypasses Coleman-Mandula theorem. (Coleman-Mandula theorem implicitly assumes that the symmetries do map single particle creation operators to single particle creation operators, which is apparently violated here.)

5. Comparison to SUSY and extended SUSY

In this section we show in a detailed manner that a unified gauge and spacetime symmetry group of the form Eq.(3) is inequivalent to SUSY or extended SUSY, however, they are along a similar group theoretical philosophy: both the (extended) SUSY and our construction use the case (iii) of O'Raifeartaigh theorem. On the other hand, the detailed group structure of the two constructions are different, and they use slightly different means to bypass Coleman-Mandula theorem.

Traditionally, the SUSY algebra is presented in a *graded Lie algebra* (also called *super Lie algebra*) form, with the following generating operators:

$$\begin{array}{ll} \Sigma_{ab} & \text{(generators of Lorentz Lie algebra),} \\ Q_A \text{ and } \bar{Q}_{A'} & \text{(supercharges),} \\ P_a & \text{(generators of translation Lie algebra)} \end{array} \qquad (9)$$

obeying the usual super Lie algebra relations [4, 7, 8, 9]. Here, conventional Penrose abstract index notation is used [15, 16]. The super Lie algebra presentation of the SUSY algebra might look paradoxical at a first glance for the following reason. Given a set of transformations, in which subsequent application of transformations is within the set, along with the identity transformation as well as inverse transformation, then that collection of transformations automatically obey the group axioms. (This is how the group axioms were distilled, at the first place.) Then, if such a set of transformations are parametrized by some finite tuple of real parameters, and the multiplication and inverting of transformations are continuously differentiable operations with respect to the parameters, then this set of transformations will automatically obey Lie group axioms. As such, their infinitesimal versions, i.e. their derivatives with respect to the parameters around the unity, automatically obey the Lie algebra axioms. Therefore, if one presents a graded or super Lie algebra, which does not obey ordinary Lie algebra relations, one needs to explain that in what sense these can be considered as infinitesimal version of some parametric transformations. This seemingly paradoxical question can be resolved by recognizing that the super Lie algebra of SUSY can be re-parametrized to obey ordinary Lie algebra relations [7, 8, 9]. In order to show that, take a basis $\left(\epsilon^{A}{}_{(1)}, \epsilon^{A}{}_{(2)}\right)$ of the Grassmann valued two-spinor space, and take the definitions of the following operators:

$$\begin{array}{ll} \Sigma_{ab} & \text{(generators of Lorentz Lie algebra),} \\ \delta_{(i)} := \epsilon^{A}{}_{(i)} Q_A \text{ and } \bar{\delta}_{(i)} := \bar{\epsilon}^{A'}{}_{(i)} \bar{Q}_{A'} & \text{(generators of pure supertranslations),} \\ \quad (i = 1, 2) & \\ P_a & \text{(generators of translation Lie algebra).} \end{array} \qquad (10)$$

It is seen that the Lorentz generators span the Lorentz Lie algebra, let us denote that by ℓ, the translation generators span the translation Lie algebra, let us denote that by t, whereas the pure supertranslation generators span a subspace, which shall be denoted by q. It is seen that by considering the δ-s (variation of superfields upon an infinitesimal pure supertranslation) instead of Q-s (supercharges) as operators acting on the superfields, the super Lie algebra of SUSY has an equivalent ordinary Lie algebra view, due to the "sign flipping trick" by the Grassmann valued two-spinor basis. It is evident, by construction, that such intertwining map between the SUSY super Lie algebra and the corresponding ordinary Lie algebra presentation is one-to-one and onto, furthermore that it really intertwines between the super and the ordinary Lie bracket in the two presentations. (Although this ordinary Lie algebra view of SUSY is known in the literature [7, 8, 9], it is not very commonly used.) Taking now the corresponding ordinary Lie algebra, consisting of $t \oplus q \oplus \ell$, it is seen that by exponentiating it one gets a corresponding Lie group, as discussed in Section 2.2 and [7, 8, 9]. Due to the

SUSY relations, one has that the sub-Lie algebra of translations (t) is a normal sub-Lie algebra, i.e. it is invariant to the adjoint action of the entire Lie algebra $t \oplus q \oplus \ell$. Also the sub-Lie algebra of supertranslations ($s := t \oplus q$) is a normal sub-Lie algebra. The subspace q, residing within s is merely a linear subspace, not even a sub-Lie algebra, since it does not close without t under the Lie bracket. The subspace ℓ is a sub-Lie algebra, but it is not normal, since it acts on $s = t \oplus q$ nontrivially by the adjoint action. It is important to note that the normal sub-Lie algebra of translations (t) is abelian, and that the quotient Lie algebra $q \equiv s/t$ (supertranslations without considering the spacetime translation component) is also abelian. Exactly this structure makes it possible to perform the "sign flipping trick", i.e. to have a super Lie algebra view. Let us introduce the notation $s = t_{\bullet} q$ for denoting the fact that the Lie algebra $s = t \oplus q$ is an extension of the normal sub-Lie algebra t, but its complementing subspace q is not a standalone sub-Lie algebra ("semi-semi-direct product"). Then, the Lie algebra view of SUSY can be presented as:

$$\underbrace{\underbrace{\Big(\underbrace{t}_{\text{translations}} \bullet \underbrace{q}_{\text{pure supertranslations}} \Big)}_{\text{all supertranslations } (=s)} \rtimes \underbrace{\ell}_{\text{Lorentz symmetries}}}_{\text{supersymmetries of superfields}} \tag{11}$$

where again the arrow diagram clarifies which sub-Lie algebra acts nontrivially on which part of the Lie algebra, via the adjoint action. It is seen that $s = t_{\bullet} q$ is the radical of the full supersymmetry Lie algebra $(t_{\bullet} q) \rtimes \ell$, and that s is a nilpotent extension of t. This group theoretical structure is allowed by the case (iii) of the O'Raifeartaigh theorem. The so called extended SUSY has very similar Lie algebra structure, with merely the abelian part of the radical being extended by the so called *central charges*, being $z := \mathrm{u}(1) \times \ldots \times \mathrm{u}(1)$, and the Levi factor being extended by the Lie algebra of a compact gauge (internal) group g:

$$\underbrace{\underbrace{\Big(\big(\underbrace{t}_{\text{translations}} \times \underbrace{z}_{\text{central charges}} \big)_{\bullet} \underbrace{q_{\text{ext}}}_{\substack{\text{pure extended}\\ \text{supertranslations}}} \Big)}_{\text{all extended supertranslations}} \rtimes \Big(\underbrace{g}_{\substack{\text{compact internal}\\ \text{symmetries}}} \times \underbrace{\ell}_{\substack{\text{Lorentz}\\ \text{symmetries}}} \Big)}_{\text{extended supersymmetries}} \tag{12}$$

Here, the arrow diagram explicitly shows that the glueing of the compact gauge (internal) symmetries and the Lorentz symmetries are possible due to their common adjoint Lie group action on some subspace (q_{ext}) of the radical. In that sense, the unification mechanism Eq.(3) proposed in the present paper is kind of similar to that of the mechanism of extended SUSY, shown in Eq.(12).

Given the group theoretical similarities of the (extended) SUSY illustrated in Eq.(12), and our proposed mechanism outlined in Eq.(3), the question naturally arises: are these constructions inequivalent? The answer is yes, which shall be demonstrated in multiple ways, in the closing part of this section.

Recall that a normal sub-Lie algebra within a Lie algebra is an invariant subspace, and as such is independent of the choice of generators (i.e. of a Lie algebra basis). Therefore, if the list of normal sub-Lie algebras within two Lie algebras cannot be identified to each-other, then these Lie algebras cannot be isomorphic. Using Eq.(12) and Eq.(3) one can list the normal sub-Lie algebras in the two constructions, and can see that they are different in number and are different in terms of dimensions, i.e. cannot be identified to each-other.

An other way to show the inequivalence of (extended) SUSY and our unification mechanism is to observe that our group, by construction, can be regarded of the form of Eq.(2), whereas (extended) SUSY cannot be transformed into that form. That is seen via referring again to the invariance of the normal sub-Lie algebras, as a consequence of which the definition of the sub-Lie algebra consisting of translations and central charges is independent of the choice of generators in (extended) SUSY. Clearly, its complementing subspace (q) does not form a standalone sub-Lie algebra, which obstructs Eq.(2). It can be shown that by taking a different complementing subspace (q'), being some mixture of q and of the translations and central charges, this cannot be avoided. That is evidently seen by taking new generators $\delta'_{(i)}$ $(i=1,2)$ being linear combinations of $\delta_{(i)}$ and of translations and central charges, and then by using SUSY relations. It becomes evident that the structure relations of $\delta'_{(i)}$ $(i=1,2)$ shall be the same as of $\delta_{(i)}$, due to:

- abelian nature of translations and central charges,
- $\delta_{(i)}$ $(i=1,2)$ commute with translations and central charges.

Therefore, no complementing sub-Lie algebra q' to translations and central charges can be found, merely a complementing sub-linear space can exist, which then indeed obstructs Eq.(2) to hold for the (extended) SUSY.

A further way to see the inequivalence of the proposed unification mechanism and of (extended) SUSY is to observe that our construction Eq.(3) can be regarded as $(\mathcal{N} \rtimes \mathcal{G}) \rtimes \{\text{Poincaré group}\}$. That implies the existence of a homomorphism from that group onto the Poincaré group. The (extended) SUSY does not possess such homomorphism onto the Poincaré group, since as pointed out above, the pure supertranslation generators cannot be collected into a normal sub-Lie algebra (not even to an ordinary sub-Lie algebra) which does not contain the translations. As such, in the (extended) SUSY Lie algebra one cannot find a normal sub-Lie algebra complementing to Poincaré transformations, which obstructs the existence of a homomorphism *onto* the Poincaré Lie algebra from the (extended) SUSY Lie algebra. Only homomorphic injection of the Poincaré Lie algebra *into* the (extended) SUSY Lie algebra exists, which is just the reverse way.

6. Concluding remarks

A unification mechanism for local gauge and spacetime symmetries was presented. The key ingredient is to allow a solvable normal subgroup in the full gauge group, and to only require the Levi factor of the full gauge group to be compact, not the entire gauge group itself. This relaxed regularity property of allowed gauge groups is the minimal requirement for energy non-negativity. The solvable extension of the gauge group is seen not to introduce new propagating gauge boson degrees of freedom, which would contradict present experimental understanding. It is rather seen to be a set of inapparent symmetries, representing "dressing transformations" for pure one-particle states in a formal quantum field theory setting. The unification mechanism also provides an example for a non-supersymmetric extension of the group of spacetime symmetries, circumventing the McGlinn and Coleman-Mandula no-go theorems in a non-SUSY way. Therefore, the construction of invariant Lagrangians to such a local unified symmetry group is worth to study. That involves representation theory of non-semisimple Lie groups, which is a contemporary branch of research in group theory.

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Appendix A. Details of the concrete example for the U(1) case

The spin algebra A has several important linear subspaces. Given a canonical generator system (e_1, e_2, e_1^+, e_2^+) of A, the followings can be defined: $\Lambda_{\bar{p}q}$ are the linear subspaces of p,q-forms, i.e. the polynomials consisting of p powers of $\{e_1, e_2\}$ and q powers of $\{e_1^+, e_2^+\}$ $(p, q \in \{0, 1, 2\})$, and one has $A = \bigoplus_{p,q=0}^{2} \Lambda_{\bar{p}q}$, called to be the $\mathbb{Z} \times \mathbb{Z}$-*grading* of A. Then, there are the linear subspaces of k-forms, Λ_k, i.e. the polynomials consisting of k powers of $\{e_1, e_2, e_1^+, e_2^+\}$ $(k \in \{0, 1, 2, 3, 4\})$, and one has $A = \bigoplus_{k=0}^{4} \Lambda_k$, called to be the

$\mathbb{Z}$*-grading* of A. Finally, there are the subspaces Λ_{ev} and Λ_{od} being the even and odd polynomials of $\{e_1, e_2, e_1^+, e_2^+\}$, and one has $A = \Lambda_{\mathrm{ev}} \oplus \Lambda_{\mathrm{od}}$, called to be the $\mathbb{Z}_2$*-grading* of A. The subspace $B := \Lambda_{\bar{0}\bar{0}} = \mathbb{C}\,\mathbb{1}$ of zero-forms and the subspace $M := \overset{4}{\underset{k=1}{\oplus}} \Lambda_k$ of at-least-1-forms shall play an important role as well, and one has $A = B \oplus M$. B is a one-dimensional unital associative subalgebra of A, spanned by the unity and called the *unit algebra*, whereas M is the so called *maximal ideal* of A. An other important subspace is $Z = \Lambda_{\bar{0}\bar{0}} \oplus \Lambda_{\bar{2}\bar{0}} \oplus \Lambda_{\bar{0}\bar{2}} \oplus \Lambda_{\bar{2}\bar{2}}$, the *center* of A, being the largest unital associative subalgebra in A commuting with all elements of A. All these are illustrated in Figure A1.

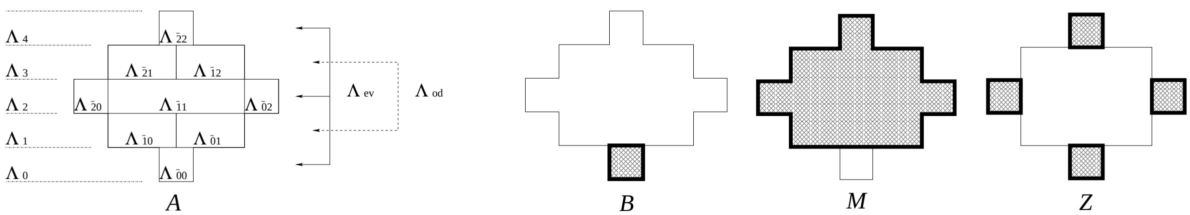

Figure A1. Leftmost panel: illustration of the $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}_2$ grading structure of the spin algebra A. The unit element $\mathbb{1}$ resides in the subspace $\Lambda_{\bar{0}\bar{0}}$, whereas the canonical generators span the subspace $\Lambda_{\bar{1}\bar{0}} \oplus \Lambda_{\bar{0}\bar{1}}$. Other panels: illustration of the important subspaces of the spin algebra, namely the unit subalgebra B, the maximal ideal M, and the center Z. One unit box depicts one complex dimension on all panels, shaded regions depict the subspaces B, M and Z, respectively.

In order to study the structure of $\mathrm{Aut}(A)$, it is important to note that an element of $\mathrm{Aut}(A)$ maps a canonical generator system to a canonical generator system, and that an element of $\mathrm{Aut}(A)$ can be uniquely characterized by its group action on an arbitrary preferred canonical generator system. Let us take such a system (e_1, e_2, e_1^+, e_2^+), with occasional notation $e_3 = e_1^+$, $e_4 = e_2^+$. The group structure of $\mathrm{Aut}(A)$ can then be characterized with the following four subgroups:

(i) Let $\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)$ be the group of $\mathbb{Z} \times \mathbb{Z}$-grading preserving automorphisms: they act on the canonical generators as $e_i \mapsto \sum_{j=1}^{2} \alpha_{ij} e_j$ and $e_i^+ \mapsto \sum_{j=1}^{2} \bar{\alpha}_{ij} e_j^+$ $(i \in \{1, 2\})$, the bar $\bar{(\cdot)}$ meaning complex conjugation and the 2×2 complex matrix $(\alpha_{ij})_{i,j\in\{1,2\}}$ being invertible.

(ii) Let $\mathcal{J} := \{I, J\}$ be the two element subgroup of $\mathbb{Z}$-grading preserving automorphisms, I being the identity and J being the involutive complex-linear operator of *particle-antiparticle label exchanging* acting as $e_1 \mapsto e_3$, $e_2 \mapsto e_4$, $e_3 \mapsto e_1$, $e_4 \mapsto e_2$.

(iii) Let $\tilde{N}_{\mathrm{ev}}$ be a subgroup of the $\mathbb{Z}_2$-grading preserving automorphisms defined by the relations $e_i \mapsto e_i + b_i$ and $e_i^+ \mapsto e_i^+ + b_i^+$ with uniquely determined parameters $b_i \in \Lambda_{\bar{1}\bar{2}}$ $(i \in \{1, 2\})$.

(iv) Let $\mathrm{InAut}(A)$ be the subgroup of inner automorphisms, i.e. the ones of the form $\exp(a)(\cdot)\exp(a)^{-1}$ with some $a \in \mathrm{Re}(A)$. These are of the form $e_i \mapsto$

$e_i + [a, e_i] + \frac{1}{2}[a, [a, e_i]]$ $(i \in \{1, 2, 3, 4\})$ with uniquely determined parameter $a \in \mathrm{Re}(\Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}1} \oplus \Lambda_{\bar{2}1} \oplus \Lambda_{\bar{1}2})$.

With these, the semi-direct product splitting

$$\mathrm{Aut}(A) = \underbrace{\mathrm{InAut}(A) \rtimes \tilde{N}_{\mathrm{ev}}}_{=:N} \rtimes \underbrace{\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \rtimes \mathcal{J}}_{=\mathrm{Aut}_{\mathbb{Z}}(A)} \tag{A.1}$$

holds. It is seen that a $\mathbb{Z}$-grading almost determines the underlying $\mathbb{Z} \times \mathbb{Z}$-grading: only the two-element discrete group of label exchanging transformations $\mathcal{J}$ introduces an ambiguity. The subgroup N shall be called the group of *dressing transformations*, being a nilpotent normal subgroup of $\mathrm{Aut}(A)$. These transformations are mixing higher forms to lower forms, i.e. do not preserve the $\mathbb{Z}$ and $\mathbb{Z}_2$-grading defined by our preferred canonical generator system: they map a system of canonical generators like $e_i \mapsto e_i + \beta_i$, the elements β_i residing in the space of at-least-2-forms M^2 $(i \in \{1, 2, 3, 4\})$, deforming the original $\mathbb{Z}$ and $\mathbb{Z}_2$-grading to an other one. By direct substitution it is seen that the transformations (i)–(iv) indeed define independent subgroups of $\mathrm{Aut}(A)$, however the proof of decomposition theorem Eq.(A.1) needs a bit more complex mathematical apparatus [14]. The principle of the proof is motivated by [17], studying the automorphism group of ordinary finite dimensional complex Grassmann (exterior) algebras.

By scrutinizing the subgroups, it is seen that the group $\mathcal{J}$ of label exchanging transformations has the structure of $\mathbb{Z}_2$. On the other hand, one has

$$\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \equiv \mathrm{GL}(2, \mathbb{C}) \equiv \mathrm{U}(1) \times \mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C}), \tag{A.2}$$

where $\mathrm{D}(1)$ is the dilatation group, i.e. $\mathbb{R}^+$ with the real multiplication. Note that $\mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})$ is nothing but the universal covering group of the (proper) homogeneous Weyl group. As far as a fixed $\mathbb{Z} \times \mathbb{Z}$-grading is taken, A can be always represented via ordinary two-spinor calculus, and the algebra identification $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ can greatly ease the calculations due to well-known identities in that formalism [15, 16]. The group of dressing transformations N, however, does not fit automatically into that framework: it needs the proper apparatus of the introduced spin algebra formalism, or care is needed when represented in terms of two-spinors.

Appendix A.1. Important representations of the example group

Due to the presence of the nilpotent normal subgroup N, $\mathrm{Aut}(A)$ is not semisimple. As a consequence, there can be nontrivial invariant subspaces even in the defining representation, i.e. when $\mathrm{Aut}(A)$ acts on A. However, for the same reason, the existence of an invariant subspace in a representation of $\mathrm{Aut}(A)$ does not imply the existence of an invariant complement. The indecomposable $\mathrm{Aut}(A)$-invariant subspaces of A are listed and illustrated in Figure A2. The invariance of these is seen via the orbits of the subspaces $\Lambda_{\bar{p}q}$ $(p, q \in \{0, 1, 2\})$ by the group action of $\mathcal{J}$ and of N.

The group $\mathrm{Aut}(A)$ naturally acts on A^*, the dual vector space of the spin algebra A with the transpose group action. It may be easily seen that the $\mathrm{Aut}(A)$-invariant

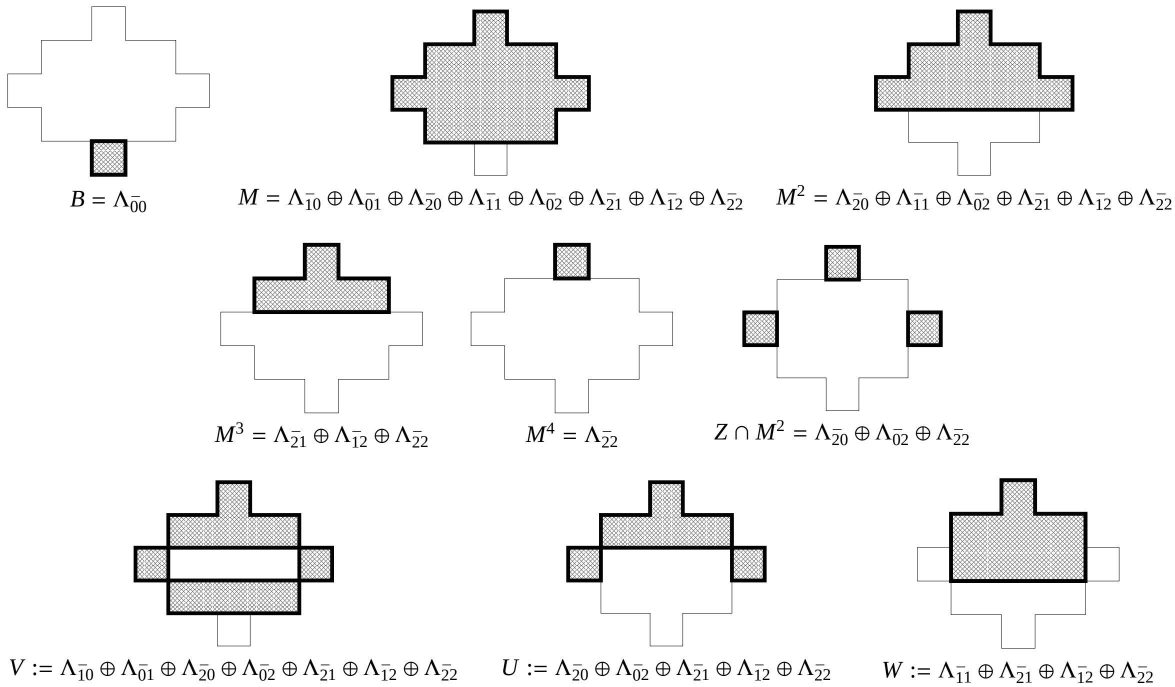

Figure A2. Illustration of the $\mathrm{Aut}(A)$-invariant indecomposable subspaces of the spin algebra A. One unit box depicts one complex dimension, shaded regions denote the invariant subspaces on all panels.

subspaces of A^* can be obtained as annulators of $\mathrm{Aut}(A)$-invariant subspaces of A itself.†† The indecomposable $\mathrm{Aut}(A)$-invariant subspaces of A^* are listed and illustrated in Figure A3.

In Figure A3 it is seen that the $\mathrm{Aut}(A)$-invariant subspace

$$\mathrm{Ann}(B \oplus V) \equiv \Lambda^*_{\bar{1}1} \tag{A.3}$$

is nothing but a four-vector representation of $\mathrm{Aut}(A)$, on which $\mathrm{Aut}(A)$ acts as the homogeneous Weyl group. In the two-spinor representation $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ one has simply $\Lambda^*_{\bar{1}1} \equiv \bar{S} \otimes S$. The kernel of the corresponding homomorphism of $\mathrm{Aut}(A)$ onto the homogeneous Weyl group is said to be the *full gauge group*, having the structure $N \rtimes \mathrm{U}(1)$. Given a four dimensional real vector space T, any injection $T \to \mathrm{Re}(\Lambda^*_{\bar{1}1})$ is called a *Pauli injection*, which is the analogue of the "soldering form" in the traditional two-spinor calculus [15, 16], extending the group action of $\mathrm{Aut}(A)$ onto the real four dimensional vector space T. In the usual Penrose abstract index notation that is nothing but the usual mapping $\sigma_a^{AA'}$ between spacetime vectors T and hermitian mixed spinor-tensors $\mathrm{Re}(\bar{S} \otimes S)$. It is seen that the group of dressing transformations N respects this basic relation of two-spinor calculus and hence realizes the group action of $\mathrm{Aut}(A)$ on the spacetime vectors T as the homogeneous Weyl group.

From Eq.(A.1) it is seen that the connected component $\mathrm{Aut}_0(A)$ of our concrete

††Given a linear subspace $X \subset A$, its annulator subspace $\mathrm{Ann}(X) \subset A^*$ is the set of all A^* elements which maps the subspace X to zero.

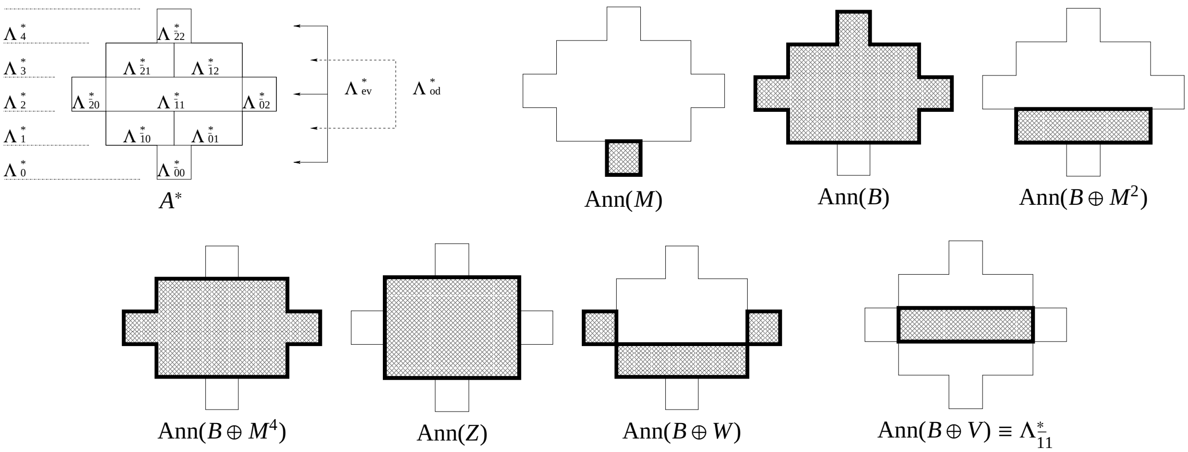

Figure A3. Top left panel: illustration of the $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}_2$ grading structure of the dual vector space A^* of the spin algebra A. Other panels: illustration of the $\mathrm{Aut}(A)$-invariant indecomposable subspaces of the dual vector space A^* of the spin algebra A. One unit box depicts one complex dimension, shaded regions denote the invariant subspaces on all panels. Note that the subspace $\mathrm{Ann}(B\oplus V)\equiv\Lambda^*_{\bar{1}1}$, illustrated on the bottom right panel, is a four-vector representation of $\mathrm{Aut}(A)$ and the pertinent group acts there as the homogeneous Weyl group.

example $\mathrm{Aut}(A)$ has the group structure

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes\Big(\underbrace{\mathrm{U}(1)}_{\text{compact internal}}}_{\text{full gauge (internal) group}} \times \underbrace{\mathrm{D}(1)\times\mathrm{SL}(2,\mathbb{C})}_{\text{Weyl symmetries}}\Big)}_{\text{symmetries of } A-\text{valued fields at a point of spacetime or momentum space}} \tag{A.4}$$

which indeed follows the pattern of Eq.(5), providing a demonstrative example of the proposed unification mechanism. Again, the arrow diagram is meant to indicate that which subgroup acts nontrivially on which normal subgroup. Subgroups not connected by arrows do not act on each-other.

Appendix A.2. Optional exclusion of the Weyl dilatations

It was seen that $\mathrm{Aut}(A)$ provides a nontrivial unification of the Weyl and the $\mathrm{U}(1)$ internal symmetry group. Clearly, $\mathrm{Aut}(A)$ acts on the invariant subspace of the maximal forms M^4 by only a scaling due to the dilatation group $\mathrm{D}(1)$. We call the subgroup of $\mathrm{Aut}(A)$ acting trivially on the maximal forms M^4 as *special automorphism group of* A, and denote it by $\mathrm{SAut}(A)$. By construction, the connected component of $\mathrm{SAut}(A)$ has the group structure

$$\underbrace{\underbrace{\underbrace{N}_{\text{dressing transformations}} \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{compact internal}}}_{\text{full gauge (internal) group}} \times \underbrace{\mathrm{SL}(2,\mathbb{C})}_{\text{Lorentz symmetries}} \Big)}_{M^4-\text{preserving symmetries of } A-\text{valued fields at a point of spacetime or momentum space}} \tag{A.5}$$

and is the same as Eq.(A.4), but without the Weyl dilatations. This shows that the inclusion of the subgroup of dilatations is not crucial for such unification to happen, but is very natural to include those.

Appendix A.3. Adding the translation or diffeomorphism group

Adding translations to the presented homogeneous Weyl (or Lorentz) group extension is trivial. One simply takes a four dimensional real affine space $\mathcal{M}$ as the model of the flat spacetime manifold, with underlying vector space ("tangent space") T. One takes in addition the spin algebra A, and constructs the trivial vector bundle $\mathcal{M} \times A$. The algebraic product on A extends to the sections of this vector bundle (i.e. to the A-valued fields) pointwise, being translationally invariant. Given a Pauli injection (soldering form) between T and $\mathrm{Re}(\Lambda^{*}_{\bar{1}1})$, $\mathrm{Aut}(A)$ acts on T as the homogeneous Weyl group (or $\mathrm{SAut}(A)$ acts on T as the homogeneous Lorentz group). The vector bundle automorphisms of $\mathcal{M} \times A$ preserving the algebraic product of fields as well as preserving the Pauli injection shall have the desired group structure including both the spacetime translations and $\mathrm{Aut}(A)$ in a semi-direct product:

$$\mathcal{T} \rtimes \mathrm{Aut}_0(A) =$$

$$\underbrace{\Big(\underbrace{\mathcal{T}}_{\text{translations}} \times \underbrace{\underbrace{N}_{\text{dressing transformations}} \Big) \rtimes \Big(\underbrace{\mathrm{U}(1)}_{\text{compact internal}}}_{\text{full gauge (internal) group}} \times \underbrace{\mathrm{D}(1) \times \mathrm{SL}(2,\mathbb{C})}_{\text{spacetime related}} \Big)}_{\text{global symmetries of } A-\text{valued fields when considered over flat spacetime}} \tag{A.6}$$

as a global symmetry of fields, following the pattern of Eq.(3). When acting on $\mathcal{M}$, it shall act as the Poincaré group combined with global metric rescalings. This also implies a causal structure on $\mathcal{M}$. Clearly, Eq.(A.6) is a non-supersymmetric extension of the Poincaré group, circumventing McGlinn and Coleman-Mandula no-go theorems. As noted previously, using $\mathrm{SAut}(A)$ the whole construction can be performed also without including the metric dilatations.

The "gauging" of $\mathrm{Aut}(A)$, i.e. making $\mathrm{Aut}(A)$ (or $\mathrm{SAut}(A)$) a local symmetry is also trivial. Let $\mathcal{M}$ be a four dimensional real manifold modeling the spacetime manifold, with tangent bundle $T(\mathcal{M})$. Take in addition a vector bundle $A(\mathcal{M})$ whose fiber in each point is spin algebra. Take also a pointwise Pauli injection between $T(\mathcal{M})$ and $\mathrm{Re}(\Lambda^{*}_{\bar{1}1})(\mathcal{M})$. The gauged version of $\mathrm{Aut}(A)$ shall be nothing but the product preserving vector bundle automorphisms of $A(\mathcal{M})$, and they act on $T(\mathcal{M})$ as the combined group of diffeomorphisms and pointwise spacetime metric conformal rescalings, being the symmetries of (conformal) GR.

Appendix A.4. Meaning of dressing transformations

In the presented example the physical meaning of the nilpotent normal subgroup N can be understood as the "dressing" of pure one-particle states of a formal QFT model at a fixed spacetime point or momentum. Note, that spin algebra differs from a CAR algebra of QFT with the fact that the antiparticle creation operators are not yet identified with particle annihilation operators. It can be shown however [14], that an $\mathrm{Aut}(A)$-covariant family of self-dual CAR algebras can be associated to the spin algebra A, and vice-versa. Here, the self-dual CAR algebra is a mathematical structure, introduced by Araki [18], formally describing the algebraic behavior of quantum field operators. With the use of this relation, the spin algebra is a convenient reparametrization of the quantum field algebra of a QFT at a fixed point of spacetime or momentum space, revealing the hidden internal symmetry subgroup N. The details of the spin algebra $\leftrightarrow$ self-dual CAR algebra family correspondence is, however, out of the scope of the present paper mainly focusing on unification, and shall be rather discussed in [14].

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