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# Vector operators

by

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## 1. Introduction

Usual quantum mechanical observables are self-adjoint operators, or better to say, families of self-adjoint operators. For instance, position, a so called vectorial observable, is considered as a family of three self-adjoint operators that are interpreted as the components of position relative to a basis of the physical space. If we want to get rid of bases of the physical space and to look for a coordinate free description, we are faced the problem, what mathematical objects represent quantum mechanical vectorial observables. The notion of vector operators is introduced to answer this question. Here we investigate only mathematical properties of vector operators and we do not enter into physical applications.

## 2. Preliminaries

In the sequel  $H$  and  $Z$  denote a complex Hilbert space and a finite dimensional complex vector space, respectively.

Inner products are denoted by the symbol  $\langle , \rangle$  and

are taken to be linear in the second variable.

$H \otimes Z$  is the algebraic tensor product of  $H$  and  $Z$ . It is well known that if we equip  $Z$  with an inner product then  $H \otimes Z$  turns into a Hilbert space with the inner product defined by

$$\langle h \otimes z, g \otimes x \rangle_{H \otimes Z} := \langle h, g \rangle_H \langle z, x \rangle_Z$$

$$(h, g \in H, z, x \in Z).$$

The corresponding topology on  $H \otimes Z$  is independent of the particular inner product chosen on  $Z$ . That is why we consider  $H \otimes Z$  as a topological vector space without giving an inner product on  $Z$ .

If  $z_1, z_2, \dots, z_N$  is a basis of  $Z$  then every element of  $H \otimes Z$  can be written in the form  $\sum_{k=1}^N h_k \otimes z_k$ .

$Z^*$  stands for the dual of  $Z$  and the bilinear map of duality is denoted by  $( | )$ . We are given a continuous bilinear map

$$( | ): Z^* \times (H \otimes Z) \rightarrow H, \quad \text{defined by}$$

$$(p | h \otimes z) := (p | z) h \quad (p \in Z^*, h \otimes z \in H \otimes Z),$$

and a continuous sesquilinear map

$$\langle , \rangle : H \times (H \otimes Z) \rightarrow Z, \quad \text{defined by}$$

$$\langle g, h \otimes z \rangle := \langle g, h \rangle_H z \quad (g \in H, h \otimes z \in H \otimes Z).$$

We have the following relation between these two maps:

$$\langle g, (p|a) \rangle_H = (p| \langle g, a \rangle) \quad (p \in Z^*, g \in H, a \in H \otimes Z).$$

If  $p_1, p_2, \dots, p_N$  is a basis of  $Z^*$  then the elements  $a$  and  $b$  of  $H \otimes Z$  are equal if and only if  $(p_k|a) = (p_k|b)$  ( $k = 1, \dots, N$ ).

### 3. Basic facts about vector operators

Definition 1. A linear map defined in  $H$  and having values in  $H \otimes Z$  is called a  $Z$  valued vector operator in  $H$ .

If  $A$  is a vector operator and  $p \in Z^*$  then we define the linear operator

$$(p|A) : H \supset \text{Dom } A \rightarrow H, \quad h \mapsto (p|Ah)$$

Remark 1. (i) A  $\mathbb{C}$  valued vector operator is a usual operator.

(ii) Since  $H \otimes Z$  has a distinguished topology, we can speak about continuous vector operators and closed vector operators. The continuity of a vector operator is equivalent to its boundedness relative to every inner product derived in the previously given way.

(iii) Let  $z_1, z_2, \dots, z_N$  be a basis of  $Z$  and let  $p_1, p_2, \dots, p_N$  be the corresponding dual basis of  $Z^*$  (i.e.  $(p_i|z_k) = \delta_{ik}$ ,  $i, k = 1, \dots, N$ ). Then we can consider

$$(p_k|A) \quad (k = 1, \dots, N).$$

as the components of the vector operator  $A$  relative to the given basis of  $Z$ . We have the equality

$$Ah = \sum_{k=1}^N \left[ (p_k | A) h \right] \otimes z_k \quad (h \in \text{Dom } A).$$

Consequently, if we are given a family  $A_1, A_2, \dots, A_N$  of operators with common domain  $D$  in  $H$ , then we can construct the vector operator

$$h \mapsto \sum_{k=1}^N (A_k h) \otimes z_k \quad (h \in D)$$

whose components are precisely the given operators.

As a consequence, two  $Z$  valued vector operators are equal if and only if their components relative to any basis of  $Z$  coincide.

Examples. (i) If  $z \in Z$  then

$$\otimes z : H \rightarrow H \otimes Z, \quad h \mapsto h \otimes z$$

is a continuous vector operator, and  $(p | \otimes z) = (p | z) \text{id}_H$ .

(ii) Let  $V$  be a real vector space of finite dimensions.  $L^2(V)$  denotes the Hilbert space of the equivalence classes of complex valued functions defined on  $V$  that are square integrable by the translation invariant measure of  $V$  (this latter is unique up to a constant factor). Similarly, if  $Z$  is a finite dimensional complex vector space,  $L^2(V, Z)$  denotes the vector space of the equivalence classes of  $Z$  valued functions defined on  $V$  that are square integrable relative to some (hence for every) norm on  $Z$ . We use the following canonical identification

$$L^2(V) \otimes Z \cong L^2(V, Z) , \quad f \otimes z \equiv (v \mapsto f(v)z) .$$

The symbol  $V_{\mathbb{C}}$  stands for the complexification of  $V$ .

-- The so called identity multiplication operator  $M$  defined on

$$\text{Dom } M := \left\{ f \in L^2(V) : f \text{ id}_V \in L^2(V, V_{\mathbb{C}}) \right\}$$

by

$$f \mapsto f \text{ id}_V := (v \mapsto f(v)v)$$

is a  $V_{\mathbb{C}}$  valued vector operator in  $L^2(V)$ . If  $r_1, \dots, r_N$  is a basis in  $V^*$  then  $(r_k | M)$  is contained in the operator of multiplication by the  $k$ -th coordinate ( $k = 1, \dots, N$ ).

-- Let  $f : V \rightarrow \mathbb{C}$  be a differentiable function. Then  $Df(v)$ , its derivative at  $v \in V$ , is a linear map  $V \rightarrow \mathbb{C}$  that can be extended uniquely to a complex linear map  $V_{\mathbb{C}} \rightarrow \mathbb{C}$ ; in other words, we can consider  $Df$  as a map  $V \rightarrow (V_{\mathbb{C}}^*)_{\mathbb{C}} = (V^*)_{\mathbb{C}}$ .

The so called differentiation operator  $D$  defined on

$$\text{Dom } D := \left\{ f \in L^2(V) : f \text{ is differentiable, } Df \in L^2(V, V_{\mathbb{C}}^*) \right\}$$

by

$$f \mapsto Df$$

is a  $(V_{\mathbb{C}}^*)_{\mathbb{C}}$  valued vector operator in  $L^2(V)$ . If  $v_1, \dots, v_N$  is a basis in  $V = (V^*)^*$  then  $(v_k | D)$  is contained in the  $k$ -th partial differentiation operator ( $k = 1, \dots, N$ ).

Proposition 1. Let  $A$  and  $L$  be a  $Z$  valued vector operator and an operator, respectively. Then for all  $p \in Z^*$

$$(i) (p | AL) = (p | A) L , \quad (ii) (p | (L \otimes \text{id}_Z) A) = L (p | A) .$$

Proof. (i) is trivial. To show (ii) write  $Ah =: \sum_{k=1}^N h_k \otimes z_k$  and use that  $(L \otimes \text{id}_Z)(h_k \otimes z_k) = (Lh_k) \otimes z_k$ .

Definition 2. A bounded operator  $L$  is said to commute with the vector operator  $A$  if  $AL \supset (\text{id}_Z \otimes L)A$ .

Proposition 2.  $L$  commutes with  $A$  if and only if  $L$  commutes with  $(p|A)$  for all  $p \in Z^*$  which holds if and only if  $L$  commutes with  $(p_k|A)$  ( $k = 1, \dots, N$ ) for an arbitrary basis  $p_1, \dots, p_N$  of  $Z^*$ .

#### 4. The Neumann series for vector operators

Let us equip  $Z$  with an inner product. For  $\alpha \in Z$  we define the linear map

$$\hat{\alpha} : H \otimes Z \rightarrow H, \quad h \otimes z \mapsto \langle \alpha, z \rangle_Z h.$$

It is an easy task to show that

$$\begin{aligned} \hat{\alpha} \circ (\otimes \alpha) &= \|\alpha\|_Z^2 \text{id}_H, \\ (\otimes \alpha) \circ \hat{\alpha} \Big|_{H \otimes \mathbb{C}\alpha} &= \|\alpha\|_Z^2 \text{id}_{H \otimes \mathbb{C}\alpha}. \end{aligned}$$

As a consequence, if  $\alpha$  is a unit vector, the restriction of  $\hat{\alpha}$  to  $H \otimes \mathbb{C}\alpha$  (the range of  $\otimes \alpha$ ) is the inverse of  $\otimes \alpha$ .

Proposition 3.  $\hat{\alpha}$  is bounded,  $\|\hat{\alpha}\| = \|\alpha\|_Z$ .

Proof. Let  $z_1, \dots, z_N$  be an orthonormal basis of  $Z$ .

Then

$$\begin{aligned} \left\| \hat{\alpha} \sum_{k=1}^N h_k \otimes z_k \right\|_H &\leq \sum_{k=1}^N \|h_k\|_H \left| \langle \alpha, z_k \rangle \right|_Z \leq \\ &\leq \sqrt{\sum_{k=1}^N \|h_k\|_H^2} \sqrt{\sum_{k=1}^N |\langle \alpha, z_k \rangle|_Z^2} = \\ &= \|\alpha\|_Z \left\| \sum_{k=1}^N h_k \otimes z_k \right\|_{H \otimes Z} \end{aligned}$$

and for all  $h \in H$

$$\|\hat{\alpha}(h \otimes \alpha)\|_H = \|\alpha\|_Z \|h \otimes \alpha\|_{H \otimes Z}.$$

Proposition 4. Let  $A$  be a bounded vector operator.

Then  $\|A \hat{\alpha}\| = \|A\| \|\hat{\alpha}\|$ .

Proof. It is well known that  $\|A \hat{\alpha}\| \leq \|A\| \|\hat{\alpha}\|$ . Conversely, if  $\alpha \neq 0$ ,

$$\begin{aligned} \|A \hat{\alpha}\| &\geq \sup \left\{ \|A \hat{\alpha}(h \otimes z)\|_{H \otimes Z} : \|h\|_H \leq 1, \|z\|_Z \leq 1 \right\} \geq \\ &\sup \left\{ \|A \hat{\alpha}(h \otimes \frac{\alpha}{\|\alpha\|_Z})\|_{H \otimes Z} : \|h\|_H \leq 1 \right\} = \|A\| \|\alpha\|_Z. \end{aligned}$$

Proposition 5. Let  $A$  be a bounded vector operator.

such that  $\|A\| < 1$ . Then for every unit vector  $\alpha$  of  $Z$  the vector operator  $A - \otimes \alpha$  has a bounded inverse (which is not necessarily defined on the whole  $H \otimes Z$ ), namely

$$(A - \otimes \alpha)^{-1} = -\hat{\alpha} \sum_{n=0}^{\infty} (A \hat{\alpha})^n \Big|_{\text{Ran}(A - \otimes \alpha)}.$$



Remark 2. The series on the right side is absolutely convergent, hence it defines a bounded operator on the whole  $H \otimes Z$ .

### 5. The spectrum of a vector operator

In the sequel  $A$  denotes a given vector operator.

Definition 3. A linear subspace  $D$  of  $\text{Dom } A$  is called invariant under  $A$  if  $A(D) \subset D \otimes Z$ .

Proposition 6.  $D$  is invariant under  $A$  if and only if  $D$  is invariant under  $(p|A)$  for all  $p \in Z^*$  which holds if and only if  $D$  is invariant under  $(p_k|A)$  ( $k = 1, \dots, N$ ) for an arbitrary basis  $p_1, \dots, p_N$  of  $Z^*$ .

Definition 4. An element  $\lambda$  of  $Z$  is called an eigenvalue of  $A$  if there is a non-zero  $h \in \text{Dom } A$  such that

$$Ah = h \otimes \lambda,$$

and then the linear subspace  $\{h \in H : Ah = h \otimes \lambda\}$  is the eigenspace of  $A$  corresponding to  $\lambda$ .

The set of the eigenvalues of  $A$  is denoted by  $\text{Eig } A$ .

Remark 3. The definition of eigenvalues would have a more familiar form if we considered  $Z \otimes H$  valued linear maps instead of  $H \otimes Z$  valued ones. However, for the sake of the simplicity of some other formulae we have not chosen this possibility.

Definition 5. A linear subspace  $T$  of  $H \otimes Z$  is called bulky if there is no proper closed linear subspace  $D$  of  $H$  such that  $T \subset D \otimes Z$ .

Proposition 7. A linear subspace  $T$  of  $H \otimes Z$  is bulky if and only if  $H$  is spanned by  $\bigcup_{p \in Z^*} \{ \langle p|a \rangle : a \in T \}$ .

Definition 6. An element  $\lambda$  of  $Z$  is a regular value of  $A$  if

- (i)  $A - \otimes \lambda$  is injective
- (ii)  $\text{Ran}(A - \otimes \lambda)$  is bulky,
- (iii)  $(A - \otimes \lambda)^{-1}$  is continuous.

The set

$$\text{Sp } A := \left\{ \lambda \in Z : \lambda \text{ is not a regular value of } A \right\}$$

is the spectrum of  $A$ .

Proposition 8. (i)  $\text{Eig } A \subset \text{Sp } A$   
and for all  $p \in Z^*$

- (ii)  $(p|\text{Eig } A) \subset \text{Eig } \langle p|A \rangle$ ,
- (iii)  $(p|\text{Sp } A) \subset \text{Sp } \langle p|A \rangle$ .

Proof. (i) and (ii) are evident. To prove (iii) suppose that  $\lambda \in \text{Sp } A$ ,  $S(\lambda) := A - \otimes \lambda$  is injective, and distinguish the following two cases.

Firstly, assume that  $\bigcup_{p \in Z^*} \langle p|\text{Ran } S(\lambda) \rangle$  does not span  $H$ . Then  $\text{Ran}(\langle p|S(\lambda) \rangle) = \langle p|\text{Ran } S(\lambda) \rangle$  cannot be dense in  $H$ , thus  $(p|\lambda) \in \text{Sp}(p|A)$  for all  $p \in Z^*$ .

Secondly, suppose that the inverse of  $S(\lambda)$  is not continuous. Then there are elements  $h_n$  ( $n \in \mathbb{N}$ ) of  $H$  such that the sequence  $n \mapsto h_n$  is not bounded but the sequence  $n \mapsto S(\lambda)h_n$  is bounded. Consequently, the sequence  $n \mapsto \langle p|S(\lambda)h_n \rangle$  is bounded, thus  $\langle p|S(\lambda) \rangle$  cannot have a continuous inverse (it can have no inverse at all), and  $(p|\lambda) \in \text{Sp } \langle p|A \rangle$  ( $p \in Z^*$ ).

Proposition 9. Let  $A$  be continuous. Equip  $Z$  with an inner product. Then the set  $\{\lambda \in Z : \|\lambda\|_Z > \|A\|\}$  is disjoint from  $\text{Sp } A$ .

Proof. We have a conjugate linear bijection  $Z \rightarrow Z^*$ ,  $x \mapsto x^\circ$  such that  $(x^\circ | z) := \langle x, z \rangle_Z$  ( $x, z \in Z$ ). Moreover,  $\|(x^\circ | A)\| = \hat{x} \circ A$  (see Proposition 3), thus  $\|(x^\circ | A)\| \leq \|x\|_Z \|A\|$ . If  $\lambda$  is an element of  $Z$  such that  $\|\lambda\|_Z > \|A\|$  then

$$A - \otimes \lambda = \|\lambda\|_Z \left( \frac{A}{\|\lambda\|_Z} - \otimes \frac{\lambda}{\|\lambda\|_Z} \right)$$

has a bounded inverse by Proposition 5. We have to show only that  $\text{Ran}(A - \otimes \lambda)$  is bulky.

Since  $\|(\lambda^\circ | A)\| < \|\lambda\|_Z^2$ ,  $\|\lambda\|_Z^2 = (\lambda^\circ | \lambda)$  is not in the spectrum of  $(\lambda^\circ | A)$  as it is well known from usual operator theory, consequently  $\text{Ran}[(\lambda^\circ | A) - (\lambda^\circ | \lambda)\text{id}_H] = (\lambda^\circ | \text{Ran}(A - \otimes \lambda))$  is dense in  $H$ . Apply Proposition 7 to and the proof.

Remark 4. (i) If  $Z = \mathbb{C}$ , Definition 6 gives back the usual definition of spectrum. If  $Z$  is one dimensional, the spectrum of a vector operator has the usual properties.

(ii) It is not known at present, whether the spectrum of a continuous vector operator is closed, yes or no.

(iii) In contradistinction to the case of usual operators, the spectrum of a continuous vector operator can be void. For instance, consider the following example. Let  $H$  be two dimensional and let  $h_1, h_2$  be an orthonormal basis of  $H$ , put

$$A_1 h_1 := h_1,$$

$$A_1 h_2 := 0,$$

$$A_2 h_1 := h_1 + h_2,$$

$$A_2 h_2 := h_1 + h_2,$$

and extend  $A_1$  and  $A_2$  to linear operators on  $H$ . Let  $Z$  be two dimensional and let  $z_1, z_2$  be a basis of  $Z$ . Then the vector operator

$$H \rightarrow H \otimes Z, \quad h \mapsto (A_1 h) \otimes z_1 + (A_2 h) \otimes z_2$$

has a void spectrum.

(iv) In contradistinction to the case of usual operators, the spectrum of a vector operator defined on a finite dimensional Hilbert space does not consist necessarily of eigenvalues. For instance, take the previous  $H$  and  $Z$ , put

$$\begin{aligned} B_1 h_1 &:= h_1 + h_2, & B_2 h_1 &:= 0, \\ B_1 h_2 &:= 0, & B_2 h_2 &:= h_1 + h_2 \end{aligned}$$

and extend  $B_1$  and  $B_2$  to linear operators on  $H$ . Then the zero is not an eigenvalue but it is in the spectrum of the vector operator

$$H \rightarrow H \otimes Z, \quad h \mapsto (B_1 h) \otimes z_1 + (B_2 h) \otimes z_2$$

(v) Observe that the norm of vector operators depends on the inner product on  $Z$ . It is interesting that even the set  $\{\lambda \in Z : \|\lambda\|_Z > |A|\}$  depends on it. To see this let  $H$  and  $Z$  as previously, let  $P_1$  and  $P_2$  be the projections onto the subspaces spanned by  $h_1$  and  $h_2$ , respectively. Then the vector operator

$$H \rightarrow H \otimes Z, \quad h \mapsto (P_1 h) \otimes z_1 + (P_2 h) \otimes z_2$$

has the same norm whatever be the inner product on  $Z$  such that  $\|z_1\|_Z = \|z_2\|_Z = 1$ .

6. Spectral theorem for vector operators

If  $T$  is a Hausdorff topological space,  $B(T)$  denotes the algebra of Borel subsets of  $T$ . A projection valued measure  $P$  on  $B(T)$  is a map assigning a projection  $P(E)$  on  $H$  to every Borel subset  $E$  of  $T$  such that

$$P(T) = \text{id}_H, \quad P(E \cap F) = P(E)P(F),$$

and if  $E_n$  ( $n \in \mathbb{N}$ ) are pairwise disjoint Borel subsets of  $T$  then

$$P\left(\bigcup_{n \in \mathbb{N}} E_n\right) = (\text{strong}) \sum_{n \in \mathbb{N}} P(E_n).$$

An element  $t$  of  $T$  is called a sharp value of  $P$  if  $P(\{t\}) \neq 0$ . The set of sharp values is denoted by  $\text{Sharp } P$ .

The support of  $P$  is the set

$$\text{Supp } P := \left\{ t \in T : P(G) \neq 0 \text{ for all open } G, t \in G \right\},$$

which is a non-void closed subset of  $T$ .

If  $h$  and  $g$  are elements of  $H$  then

$$E \mapsto P_{h,g}(E) := \langle h, P(E)g \rangle$$

is a complex measure on  $B(T)$ .

Definition 6. A  $\mathbb{Z}$  valued vector operator  $A$  in  $H$  is called

(i) partially normal if

-  $\overline{\langle p|A \rangle}$  is closable and its closure is normal for all  $p \in \mathbb{Z}^*$ ,

$$- \text{Dom } A = \bigcap_{p \in \mathbb{Z}^*} \text{Dom } \overline{\langle p|A \rangle};$$

(ii) totally normal if it is partially normal and  
 -  $\overline{(p|A)}$  and  $\overline{(q|A)}$  strongly commute for all  
 $p, q \in \mathcal{Z}^*$ .

Remark 5. A partially normal vector operator is necessarily densely defined and closed.

Proposition 10. Let  $A$  be a totally normal vector operator. Then there exists a unique projection valued measure  $R$  on  $B(\mathcal{Z})$  such that

$$\langle\langle h, Ag \rangle\rangle = \int_{\mathcal{Z}} \text{id}_{\mathcal{Z}} dR_{h,g} \quad (h \in H, g \in \text{Dom } A).$$

Proof. Let  $p_1, \dots, p_N$  be a basis in  $\mathcal{Z}^*$  and let the projection valued measure  $R_k$  be the spectral resolution of the normal operator  $\overline{(p_k|A)}$  i.e.

$$\langle h, \overline{(p_k|A)}g \rangle = \int_{\mathcal{C}} \text{id}_{\mathcal{C}} d(R_k)_{h,g} \quad (h \in H, g \in \text{Dom } \overline{(p_k|A)}).$$

for  $k = 1, \dots, N$ .

Then  $R_1, \dots, R_N$  are commuting projection valued measures, hence their product  $\bigotimes_{k=1}^N R_k$  exists, which is the unique projection valued measure on  $B(\mathcal{C}^N)$  determined by

$$\left( \bigotimes_{k=1}^N R_k \right) \left( \bigtimes_{k=1}^N E_k \right) = \prod_{k=1}^N R_k(E_k).$$

Let  $b$  denote the inverse of the linear bijection  $\mathcal{Z} \rightarrow \mathcal{C}^N$ ,  $z \mapsto \left( (p_k|z) : k = 1, \dots, N \right)$ , and put

$$R := \left( \bigotimes_{k=1}^N R_k \right) \circ b^{-1}.$$

Then for all  $k = 1, \dots, N$ ,  $h \in H$  and  $g \in \text{Dom } A$

$$\begin{aligned} \langle P_k | \langle h, Ag \rangle \rangle &= \langle h, (P_k | A) g \rangle = \int_{\mathbb{C}} \text{id}_{\mathbb{C}} d(R_k)_{h,g} = \\ &= \int_{\mathbb{C}^N} \text{pr}_k d\left(\bigotimes_{i=1}^N R_i\right)_{h,g} = \int_Z P_k dR_{h,g} = \\ &= \left\langle P_k \left| \int_Z \text{id}_Z dR_{h,g} \right. \right\rangle \end{aligned}$$

where  $\text{pr}_k : \mathbb{C}^N \rightarrow \mathbb{C}$  is the  $k$ -th canonical projection, and we used  $p_k \circ b = \text{pr}_k$  and the well known integral transformation formula.

The uniqueness of  $R$  follows from the uniqueness of  $R_k$ -s and from the equalities

$$R = \left[ \bigotimes_{k=1}^N (R \circ P_k^{-1}) \right] \circ b^{-1}, \quad R_k = R \circ P_k^{-1}.$$

Definition 7. The projection valued measure  $R$  is called the spectral resolution of the totally normal vector operator  $A$ .

Remark 6. We can define the integral of measurable functions  $T \rightarrow Z$  by projection valued measures defined on  $B(T)$  as a  $Z$  valued vector operator. It can be shown that such vector operators are totally normal. In other words, only the totally normal vector operators have spectral resolutions.

Proposition 11. A bounded operator  $L$  commutes with a totally normal vector operator  $A$  if and only if  $L$  commutes with the spectral resolution of  $A$ .

Proof. Use Proposition 2, the well known similar theorem for normal operators and the proof of the previous proposition.

The following assertion requires a number of notions and particular results from the theory of integration by projection valued measures. Who is familiar with them, can argue similarly as in the case of usual normal operators (see [1]), he must involve only one new step, a consideration on bulky subspaces. We omit the lengthy enumeration of notions and reasoning with them having referred to the literature.

Proposition 12. Let  $A$  be a totally normal vector operator having  $R$  as its spectral resolution. Then

$$\text{Eig } A = \text{Sharp } R, \quad \text{Sp } A = \text{Supp } R.$$

Remark 7. As a consequence, the spectrum of a totally normal vector operator has the properties of the spectrum of a usual operator (cf. Remark 4 (ii) - (iv)).

Definition 8. Let  $V$  be a finite dimensional real vector space. A  $V_{\mathbb{C}}$  valued vector operator  $A$  in  $H$  is called

(i) partially self-adjoint if

-  $(r|A)$  is closable and its closure is self-adjoint for all  $r \in V^*$ ;

$$- \text{Dom } A = \bigcap_{r \in V^*} \text{Dom } \overline{(r|A)};$$

(ii) totally self-adjoint if it is partially self-adjoint and

-  $(r|A)$  and  $(s|A)$  strongly commute for all  $r, s \in V^*$ .

Proposition 13. A totally self-adjoint vector operator is totally normal.

Proof. Use that  $V_{\mathbb{C}}^* = \{r + is : r, s \in V^*, i := \sqrt{-1}\}$ .



Proposition 14. The spectrum of a  $V_{\mathbb{C}}$  valued totally self-adjoint vector operator  $A$  is a subset of  $V$ ; the spectral resolution of  $A$  can be viewed as a projection valued measure given on  $B(V)$ .

Proof. Take a basis  $r_1, \dots, r_N$  in  $V^*$  (it is a in  $V_{\mathbb{C}}^*$ , too, with respect to the complex structure) and repeat the arguments of the proof of Proposition 11, taking  $(r_k|A)$  instead of  $(p_k|A)$ .

Remark 8. (i) A partially self-adjoint vector operator is necessarily densely defined and closed.

(ii) A partially self-adjoint vector operator need not be partially normal. For instance, the operator given in Remark 4 (iii), if  $Z = V_{\mathbb{C}}$ ,  $z_1, z_2 \in V$ , is partially self-adjoint without being partially normal.

(iii) In physical applications angular momentum (spin) is partially self-adjoint but it is neither partially normal nor totally self-adjoint. Position and momentum are totally self-adjoint, as it follows from the examples below.

Examples. (i) For  $z \in Z$ , the vector operator  $\otimes z$  is totally normal, its spectral resolution is the projection valued measure concentrated to  $z$ .

(ii) The identity multiplication operator in  $L^2(V)$  is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to  $E \in B(V)$  the operator of multiplication by the characteristic function of  $E$  (which is the projection onto  $L^2(E) \subset L^2(V)$ ).

(iii) The differentiation operator in  $L^2(V)$  is closable, its closure multiplied by the imaginary unit is totally self-adjoint. Its spectral resolution is the projection valued measure that assigns to  $S \in B(V^*)$  the projection  $F^{-1} K(S) F$ , where  $K(S)$  is the projection onto  $L^2(S) \subset L^2(V^*)$  and  $F : L^2(V) \rightarrow L^2(V^*)$  is the Fourier transformation defined by

$$(Ff)(x) := \int_V e^{i(x|v)} f(v) dv \quad \left( f \in L^2(V) \cap L^1(V), \text{ } \nu \in V^* \right)$$

and the translation invariant measure on  $B(V^*)$  is chosen such that  $F$  be unitary.

Reference:

[1] W. Rudin : Functional Analysis, McGraw-Hill, New York, 1973.