Workshop on higher-point correlation functions and integrable AdS/CFT

Trinity College Dublin, April 16th-20th 2018

Form factor approach to 3pt functions

Z. Bajnok

MTA Wigner Research Centre for Physics, Holographic QFT Group, Budapest

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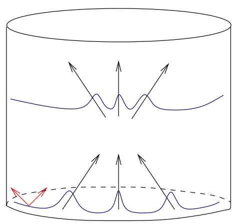
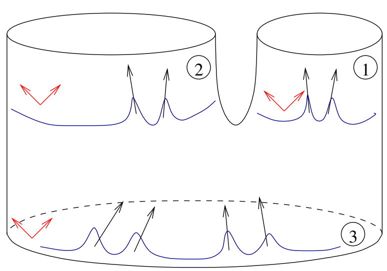
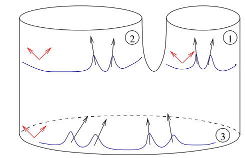
IIB strings on $AdS_5 \times S^5$	Integrability $\longleftrightarrow$	$N = 4$ SYM
	finite volume energy levels	$O(x) = \text{Tr}(\Phi(x)^J)$ J: length $\langle O(x)O(0)\rangle = x^{-2\Delta(\lambda)}$ scaling dimensions
	finite volume form factors	3pt functions $\langle O_1 O_2 O_3\rangle = C_{123}(\lambda)$

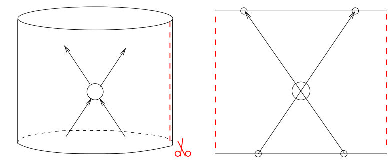
arXiv:1404.4556,1501.04533,1512.01471,1607.02830, 1704.03633, 1707.08027, 1802.04021: work done in collaboration with Romuald Janik, Andrzej Wereszczynski, Janos Balog, Marton Lajer, Chao Wu

Outline of the first part

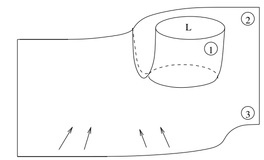
Setting: $AdS_5 \times S^5 /\ N = 4$ SYM corresondence and how integrability works



Bootstrap (S-matrix, Form factor) solution for the sinh-Gordon theory



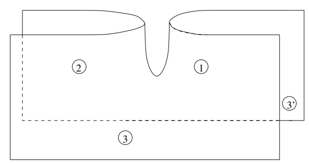
Bootstrap program for the string vertex, nonlocal operator insertion

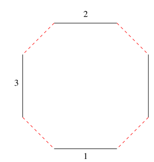


Exact solution in the pp-wave limit $N_L(\theta_1, \theta_2)$

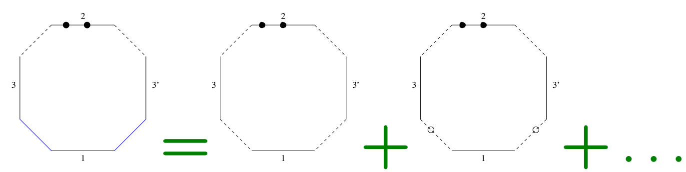
Factorizing ansatz: $F_L = FF \times\ N_L(z_1, z_2)$ the kinematical string vertex for $AdS_5 \times S^5$

The octagon form factors

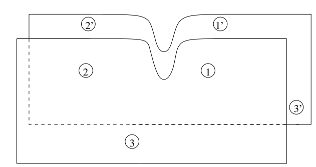


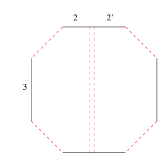


pp-wave octagon and gluing



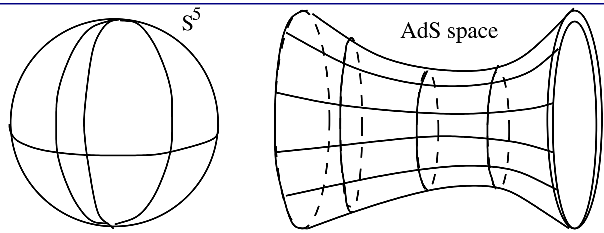
The hexagon form factors and comparision of the approaches





AdS/CFT correspondence [Maldacena]

II_B superstring on $AdS_5 \times S^5$



$\sum_1^6 Y_i^2 = R^2 \qquad - + + + + - = -R^2$

$\frac{R^2}{\alpha'} \int \frac{d\tau d\sigma}{4\pi} \left(\partial_a Y^M \partial^a Y_M + \partial_a X^M \partial^a X_M\right) + \ldots$

$\equiv$

$\mathcal{N} = 4$ D=4 $SU(N)$ SYM

$$\frac{2}{g_{YM}^2} \int d^4x \mathrm{Tr} \left[-\frac{1}{4}F^2 - \frac{1}{2}(D\Phi)^2 + i\overline{\Psi}\not{D}\Psi + V\right]$$

$$V(\Phi, \Psi) = \frac{1}{4}[\Phi, \Phi]^2 + \overline{\Psi}[\Phi, \Psi]$$

$\beta = 0$ superconformal $\frac{PSU(2,2|4)}{SO(5)\times SO(1,4)}$

gaugeinvariants: $\mathcal{O} = \mathrm{Tr}(\Phi^J)$, det( )

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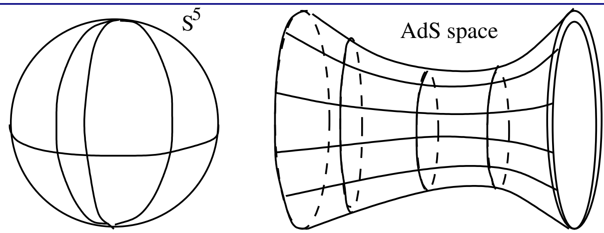

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Dictionary

Couplings: $\sqrt{\lambda} = \frac{R^2}{\alpha'}$, $g_s = \frac{\lambda}{N} \to 0$

2D QFT

String energy levels: $E(\lambda)$

$$E(\lambda) = E(\infty) + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{\lambda} + \dots$$

strong$\leftrightarrow$weak

$\Downarrow$

$\lambda = g_{YM}^2 N$, $N \to \infty$ planar limit

$$\langle \mathcal{O}_n(x) \mathcal{O}_m(0) \rangle = \frac{\delta_{nm}}{|x|^{2\Delta_n(\lambda)}}$$

Anomalous dim $\Delta(\lambda)$

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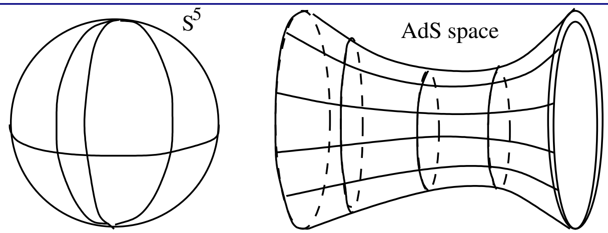

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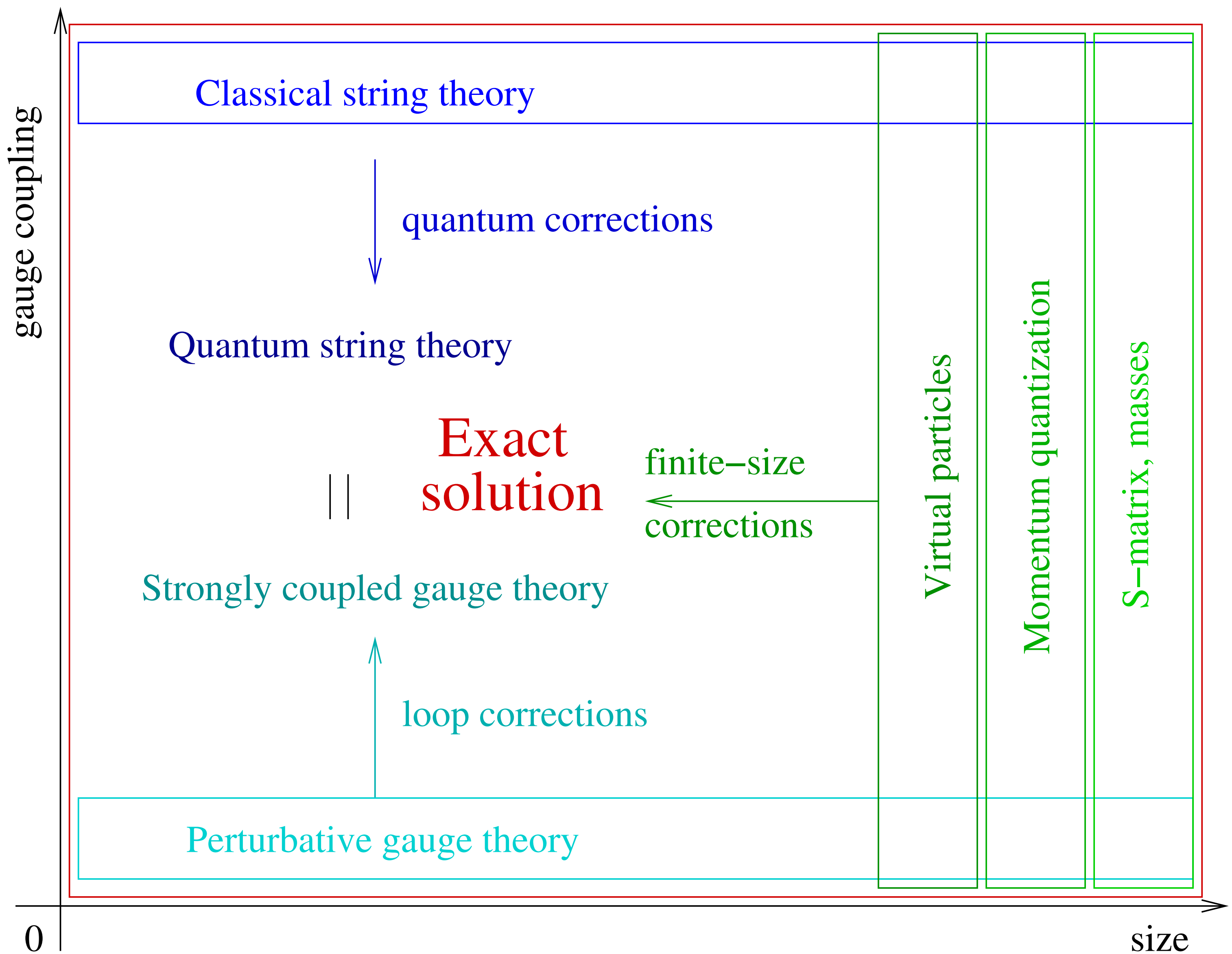
2D integrable QFT

spectrum: $Q = 1, 2, \dots, \infty$, $(\alpha, \dot{\alpha})$ dispersion: $\epsilon_Q(p) = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$

Exact scattering/reflection matrix: $S_{Q_1Q_2}(p_1, p_2, \lambda)$, $R_Q(p,\lambda)$

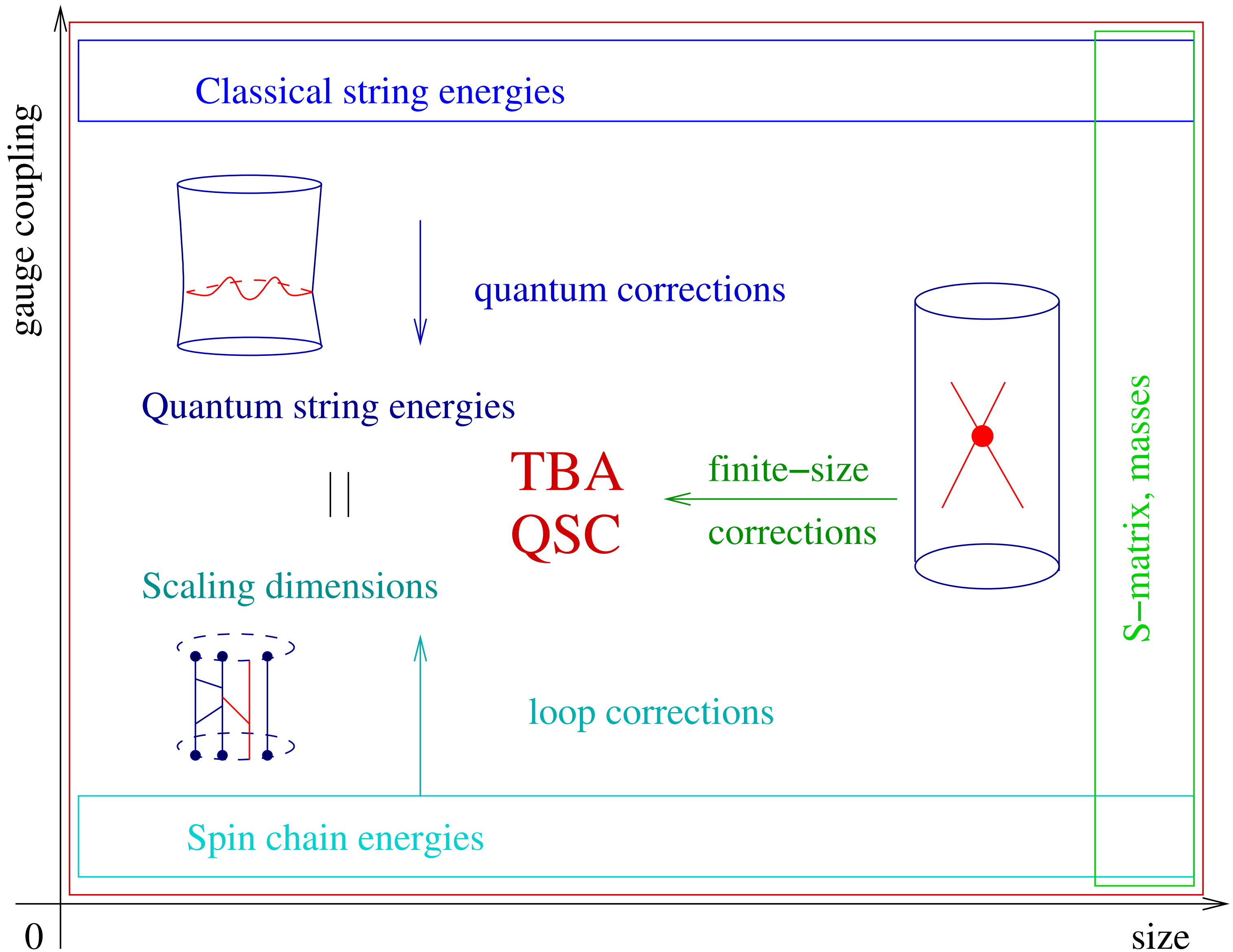
Finite size correction: Lüscher, TBA

Motivation:

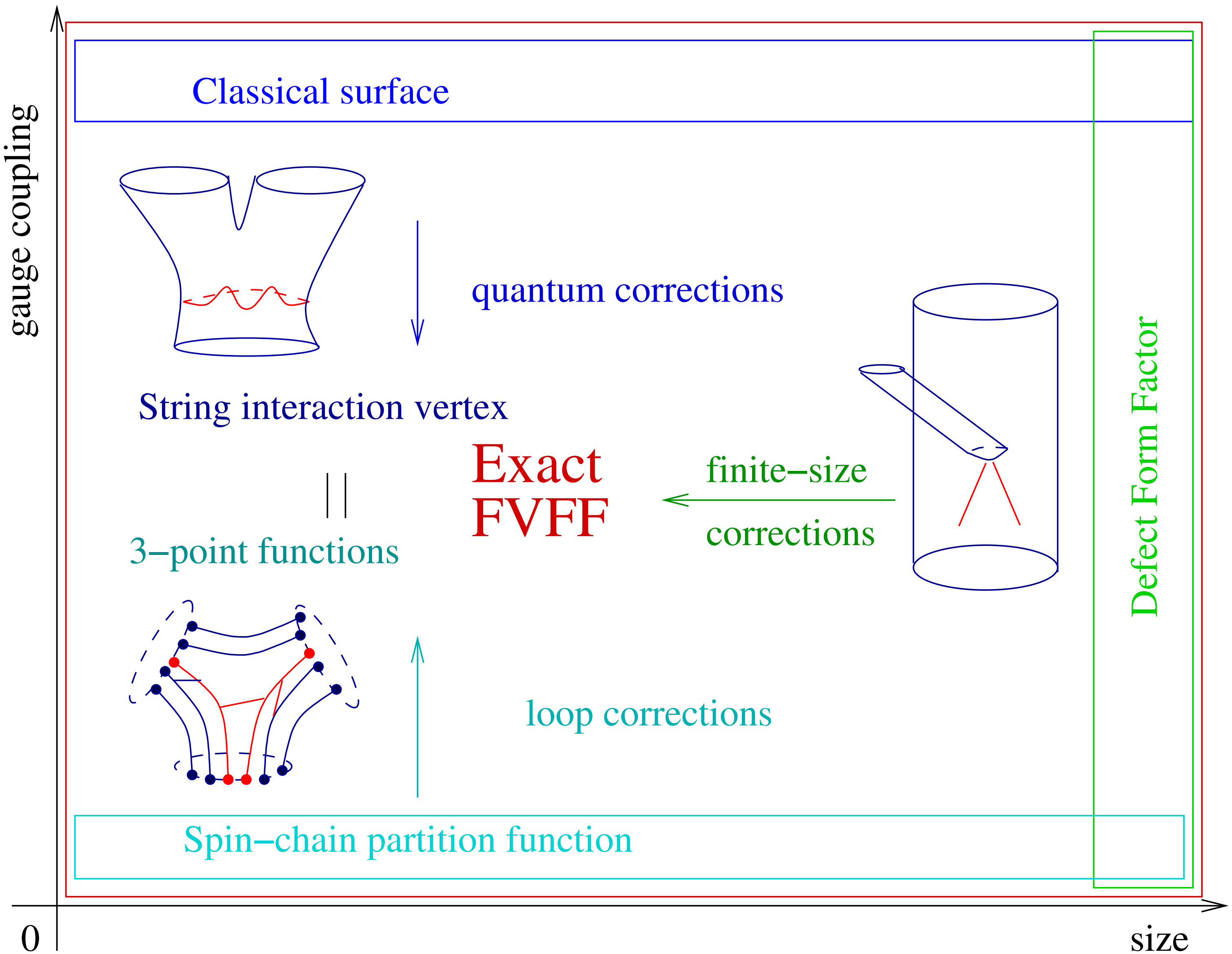


gauge coupling
Classical string theory
quantum corrections
Quantum string theory
Exact solution
||
finite−size corrections
Strongly coupled gauge theory
loop corrections
Perturbative gauge theory
Virtual particles
Momentum quantization
S−matrix, masses
0
size

Spectral problem: 2pt functions



String interaction, 3pt functions



A very useful toy model: sinh-Gordon theory

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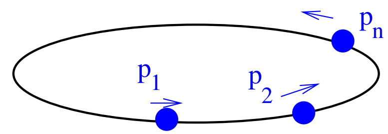
The simplest interacting QFT in 1+1 D: $\mathcal{L} = \frac{1}{2}(\partial_t \varphi)^2 - \frac{1}{2}(\partial_x \varphi)^2 - \frac{m^2}{b^2}(\cosh b\varphi - 1)$

interesting observables: finite size spectrum,



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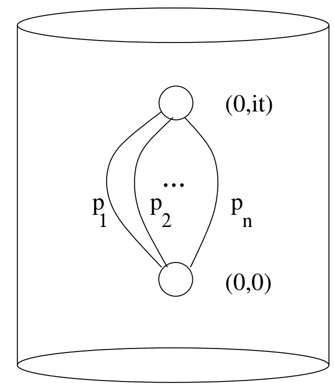

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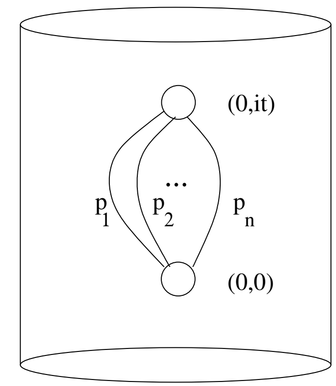

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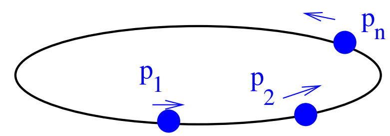

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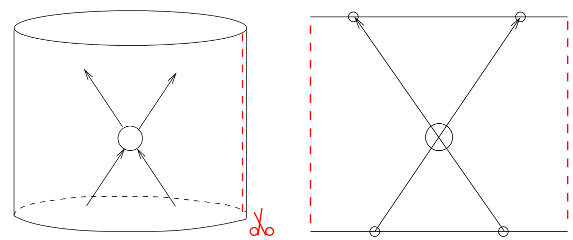

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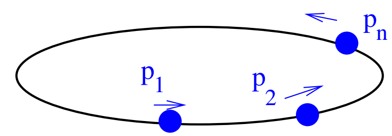

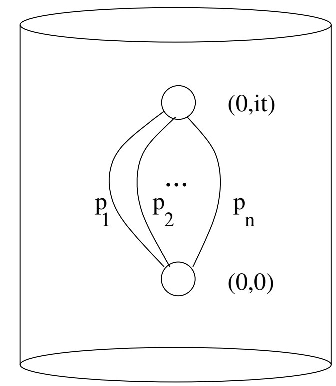

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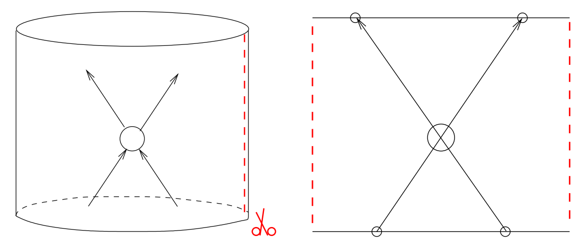

Infinite volume $\rightarrow$ LSZ reduction formula

$$\langle p_1', p_2'|\mathcal{O}|p_1, p_2\rangle = \bar{\mathcal{D}}_1' \bar{\mathcal{D}}_2' \mathcal{D}_1 \mathcal{D}_2 \langle 0|T(\mathcal{O}\varphi(1)\varphi(2)\varphi(3)\varphi(4))|0\rangle$$

$\mathcal{D}_j = i \int d^2 x_j e^{ip_j x - i\omega_j t} \left\{ \partial_t^2 - \partial_x^2 + m^2 \right\}$: amputates a leg + puts it onshell

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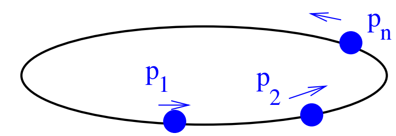

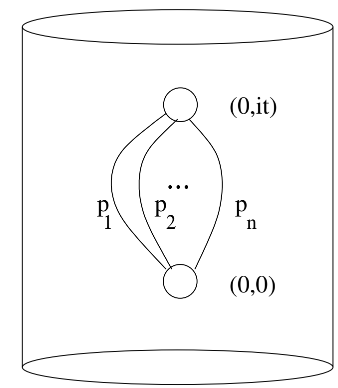

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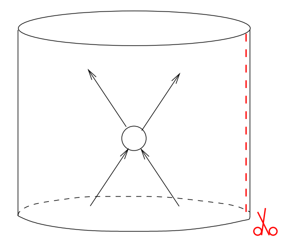

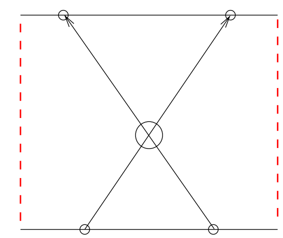

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$\mathcal{D}_j = i\int d^2x_j e^{ip_jx - i\omega_jt}\left\{\partial_t^2 - \partial_x^2 + m^2\right\}$: amputates a leg + puts it onshell

Observables:

S-matrix	Form factor (FF)	correlator
on-shell	on-shell/off-shell	off-shell

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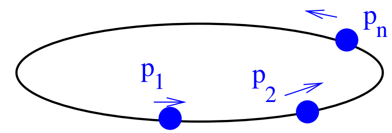

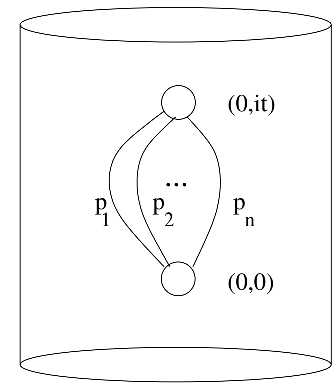

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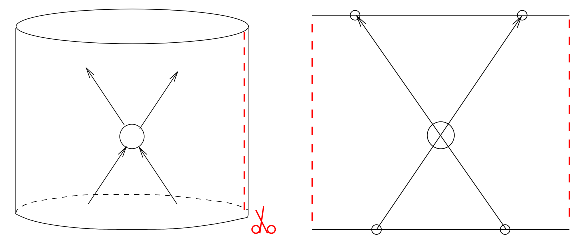

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Perturbative definition, calculational tool: [Arefyeva et al]

$$S(\theta) = 1 - \frac{1}{4}ib^2 \text{csch}\theta - \frac{b^4(\text{csch}\theta(\pi\text{csch}\theta - i))}{32\pi} + \frac{ib^6 \text{csch}\theta(\pi\text{csch}\theta - i)^2}{256\pi^2} + O\left(b^8\right)$$

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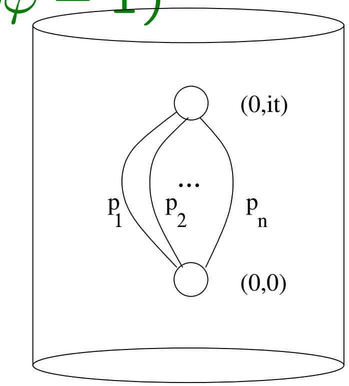

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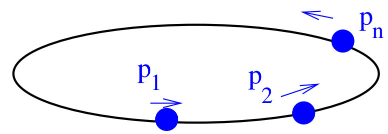

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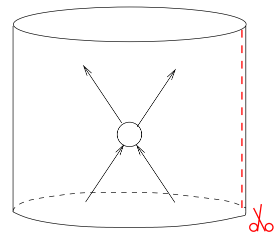

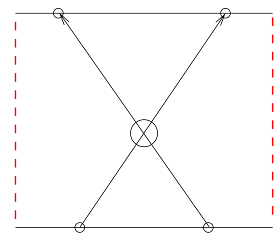

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Analytical properties: unitarity, crossing $S(\theta) = S(-\theta)^{-1} = S(i\pi - \theta)$ similar for FF

S-matrix bootstrap

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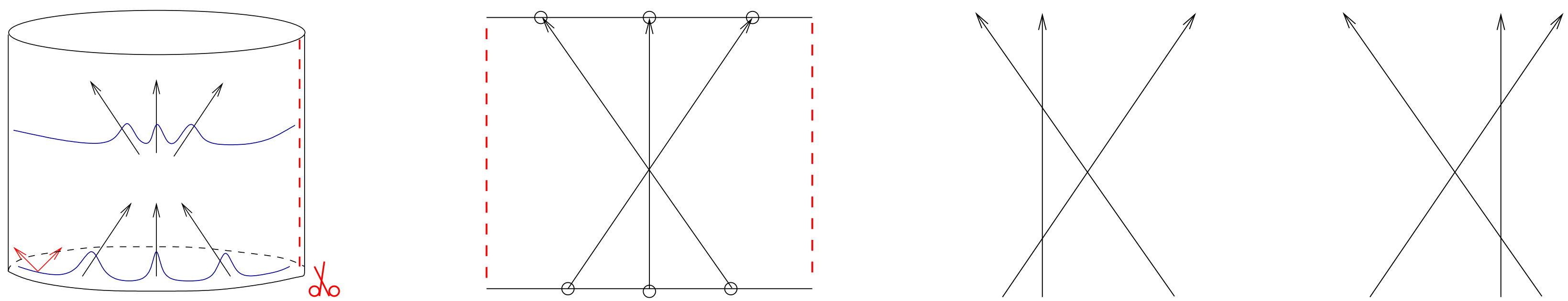
$Q_s|\theta_1, \ldots, \theta_n\rangle = \sum_j q_s e^{\theta_j s}|\theta_1, \ldots, \theta_n\rangle \quad \rightarrow \quad$ factorization $S = \prod_{i<j} S(\theta_i - \theta_j)$

S-matrix bootstrap: fundamental object is the two particle S-matrix [Zamolodchikov^2 '79]



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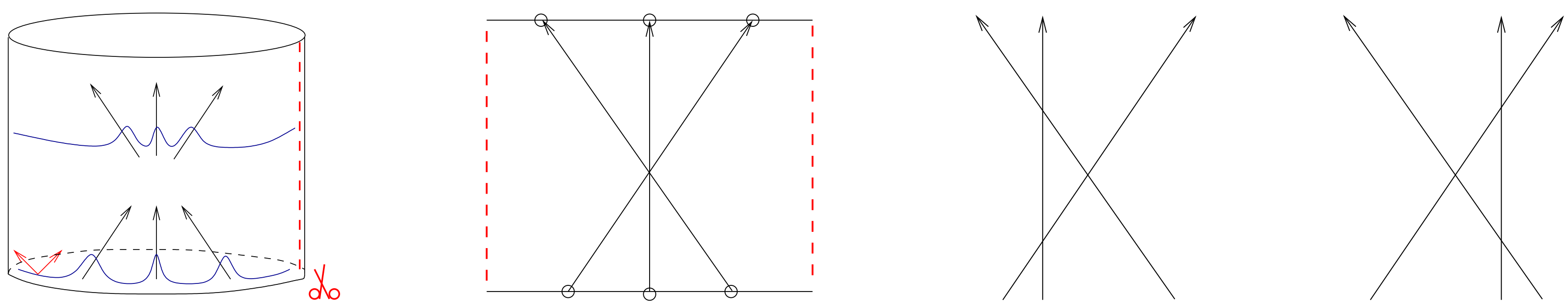
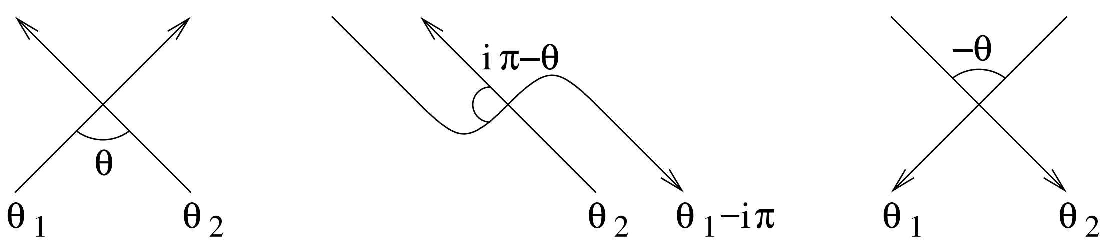


Infinite volume $\rightarrow$ crossing symmetry, $\theta \rightarrow i\pi - \theta$ in rapidity $(E(\theta), p(\theta)) = m(\cosh\theta, \sinh\theta)$



$$S(\theta_1 - \theta_2) = S(\theta) = S(i\pi - \theta) = S(-\theta)^{-1} :$$

S-matrix bootstrap

Integrability $\equiv$ Infinite number of higher spin conserved charges Q_s:

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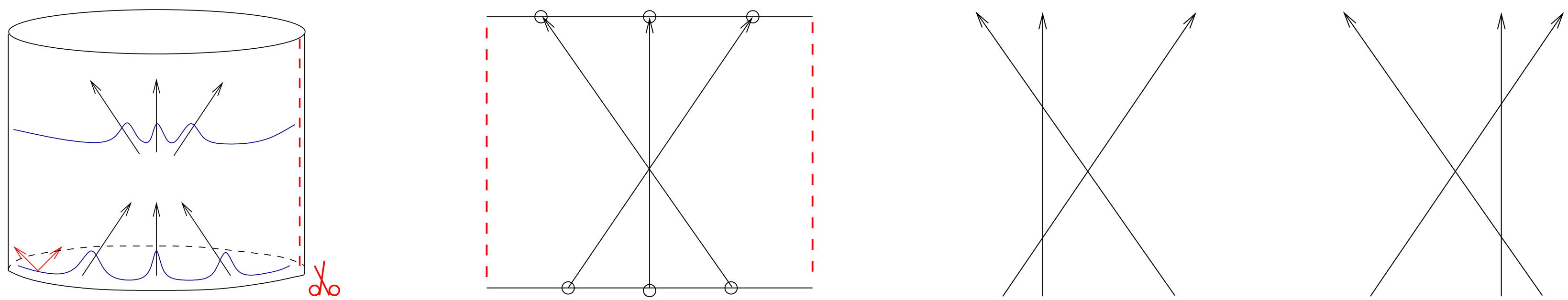

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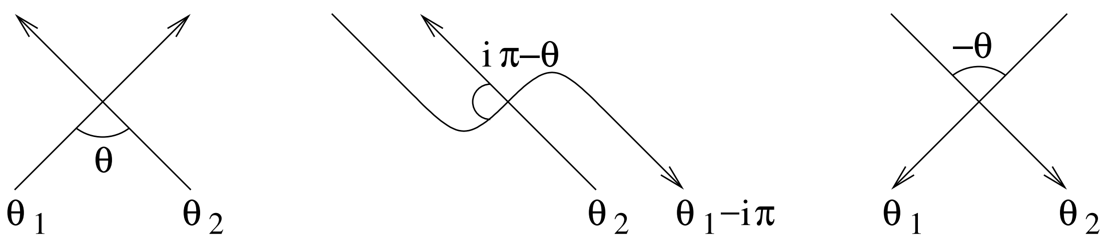

Simple solution:
sinh-Gordon

$$S(\theta) = \frac{\sinh\theta - i\sin a}{\sinh\theta + i\sin a}$$

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agrees with 4-loop perturbative calculation if $a = \frac{\pi b^2}{8\pi + b^2}$

Form factor bootstrap

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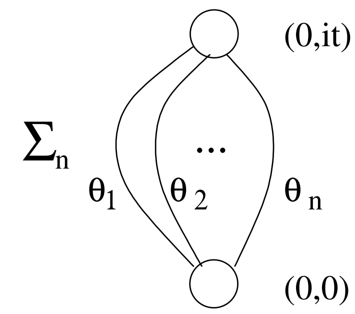
Correlation functions: [Smirnov, Karowszki, Weisz] $\langle 0|\mathcal{O}(it)\mathcal{O}(0)|0\rangle =$

$\sum_n \frac{1}{n!}\int\frac{d\theta_1}{2\pi}\cdots\int\frac{d\theta_n}{2\pi}|\langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle|^2 e^{-m(\sum_i \cosh\theta_i)t}$



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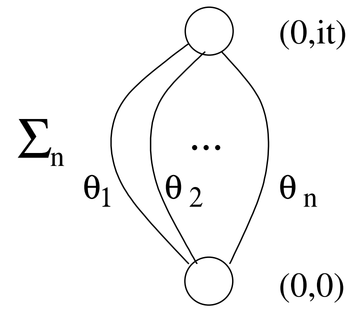

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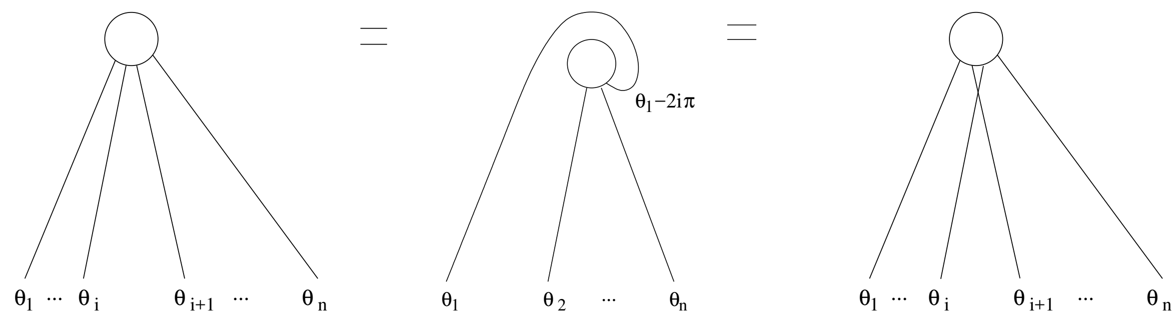

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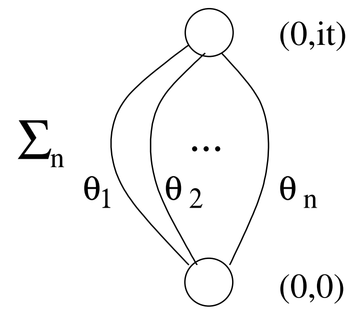

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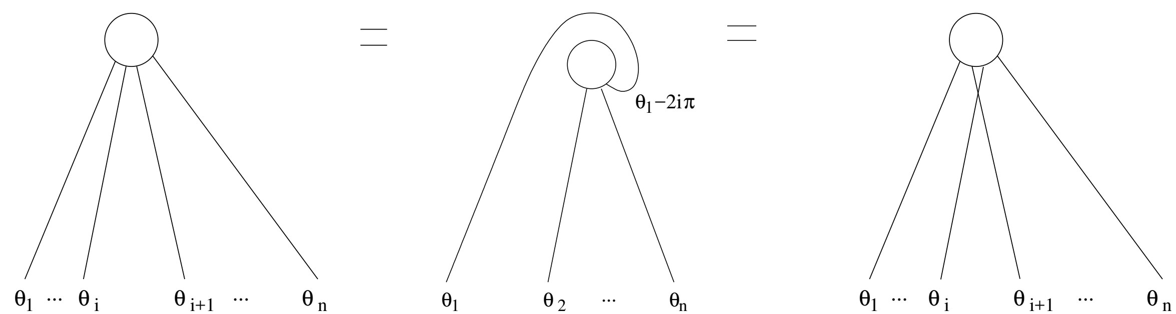

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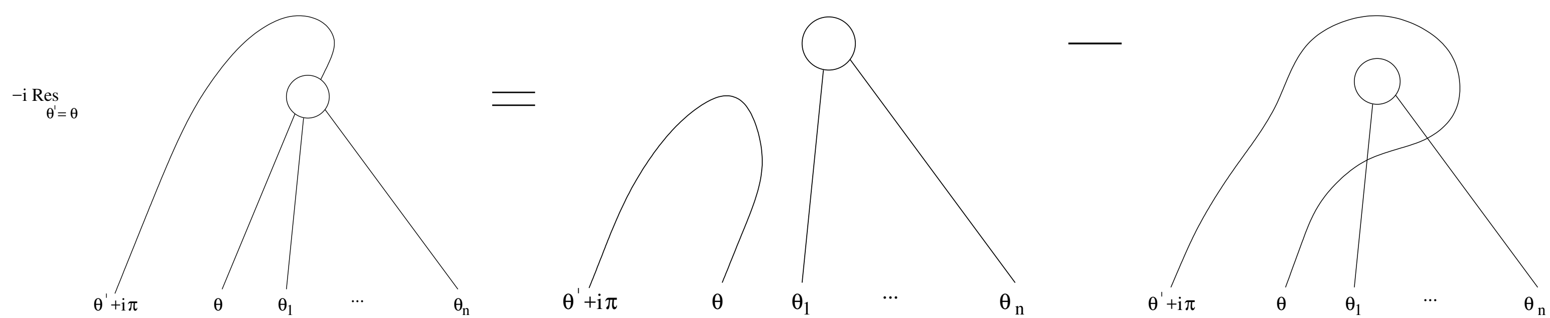
Singularity stucture



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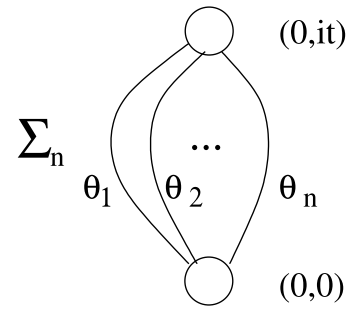

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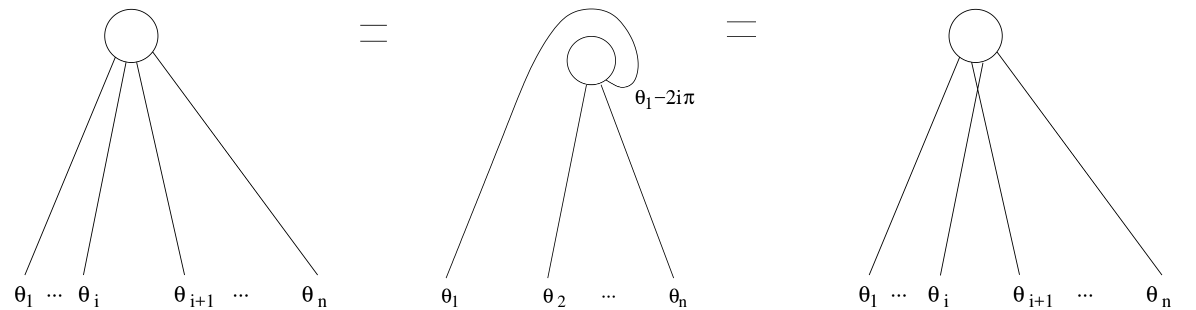

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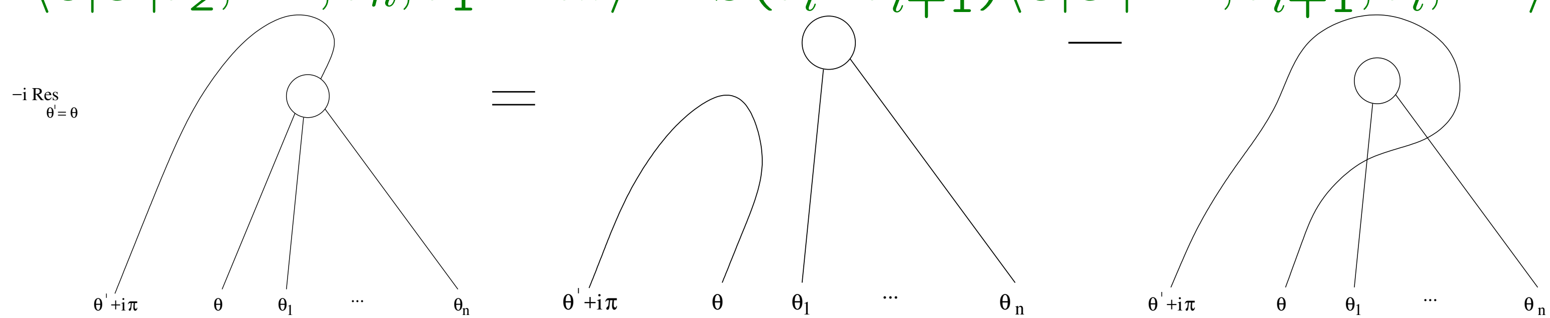

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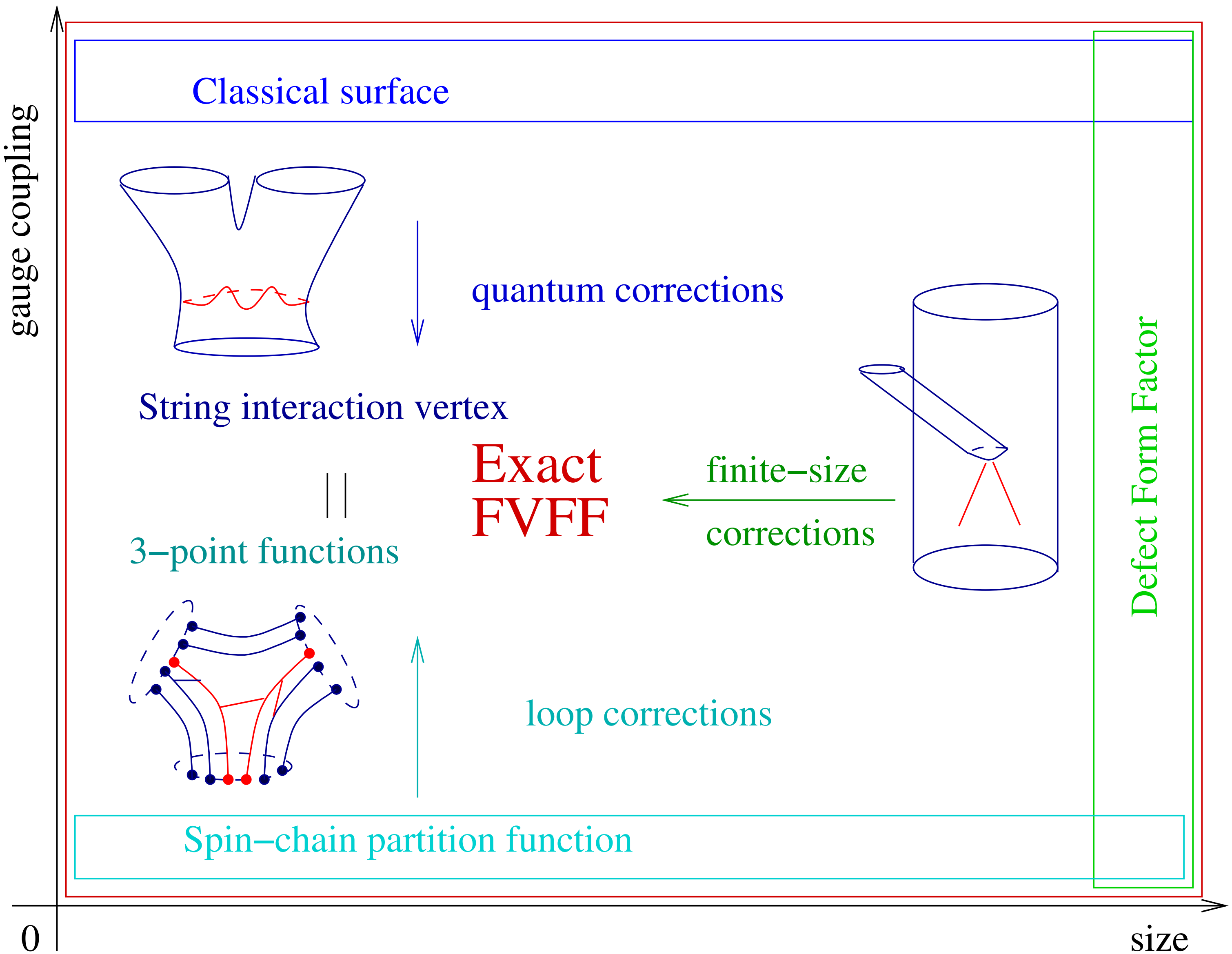
Minimal solution for sinh-Gordon: $\langle 0|\mathcal{O}|\theta_1,\theta_2\rangle = e^{(D+D^{-1})^{-1}\log S(\theta)}$; $Df(\theta)=f(\theta+i\pi)$

[Fring, Mussardo, Simonetti]

Generic solution: $F_n^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = H_n\prod_{i<j}\frac{f(\theta_i-\theta_j)}{e^{\theta_i}+e^{\theta_j}}Q_n^{\mathcal{O}}(e^{\theta_1},\ldots,e^{\theta_n})$

String interaction vertex, 3pt functions



Decompactification limit of the string vertex

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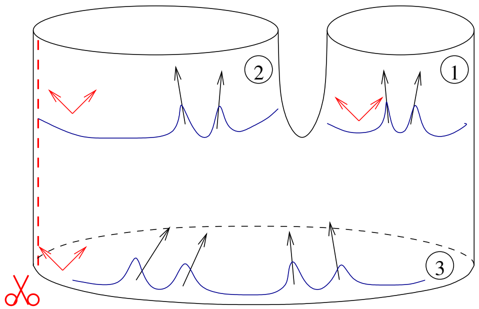
Decompactify string 2 & 3:



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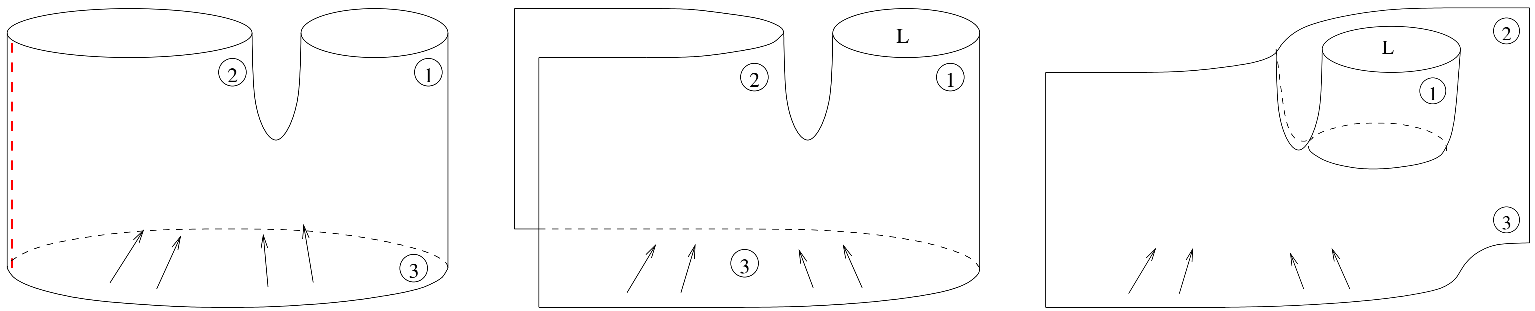

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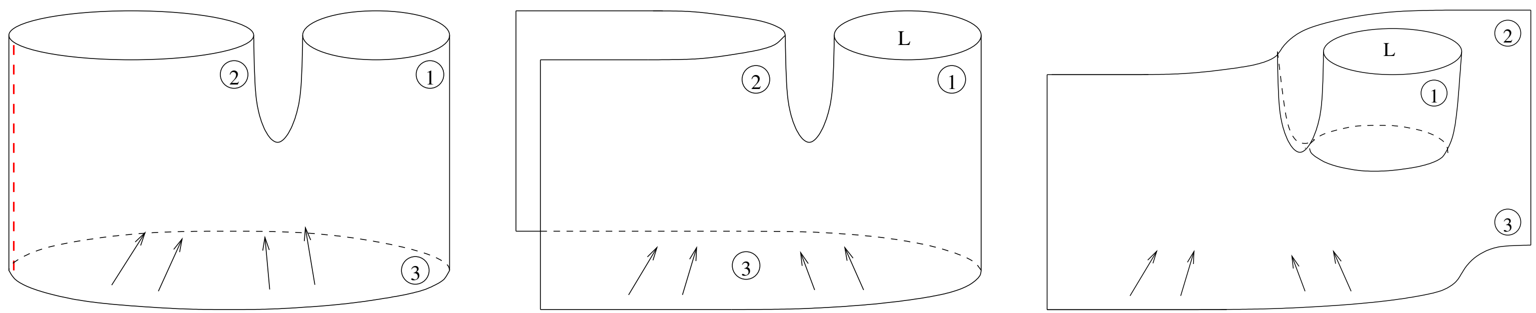
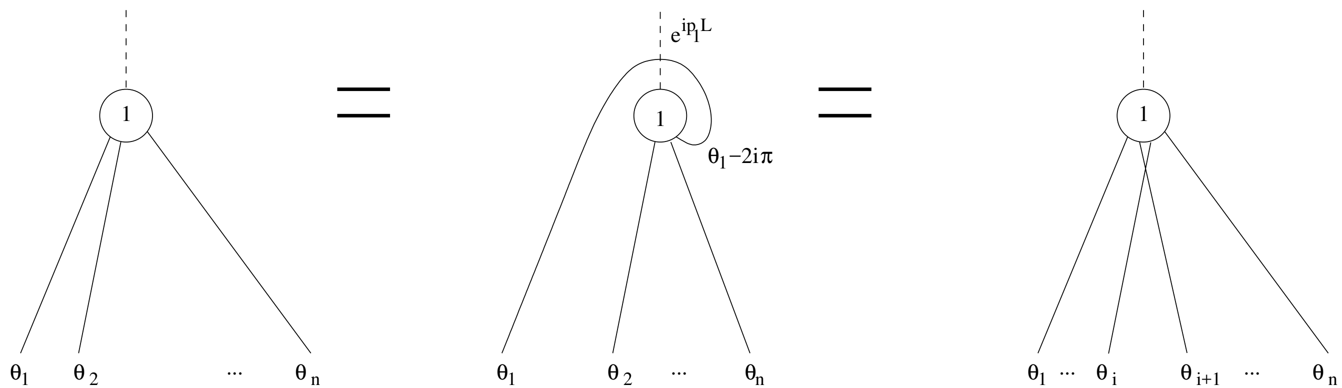


Form factor equations:



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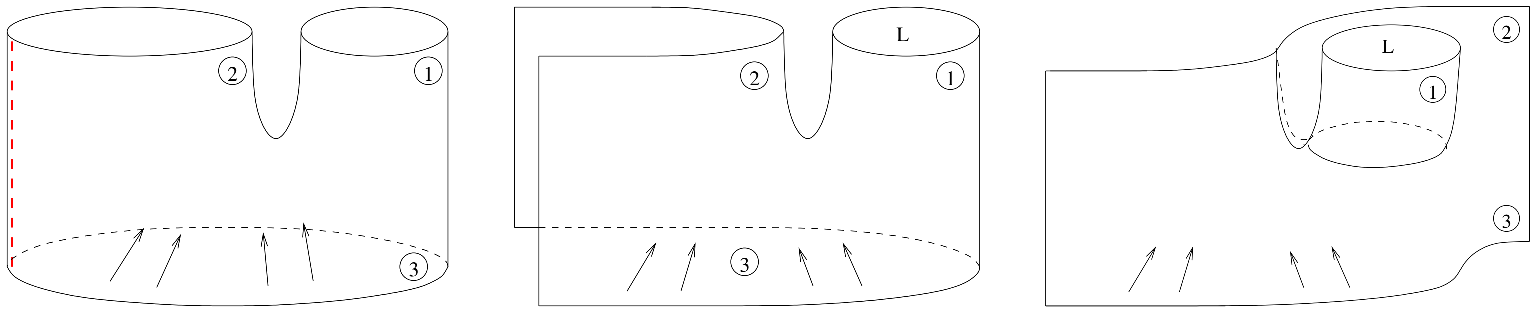

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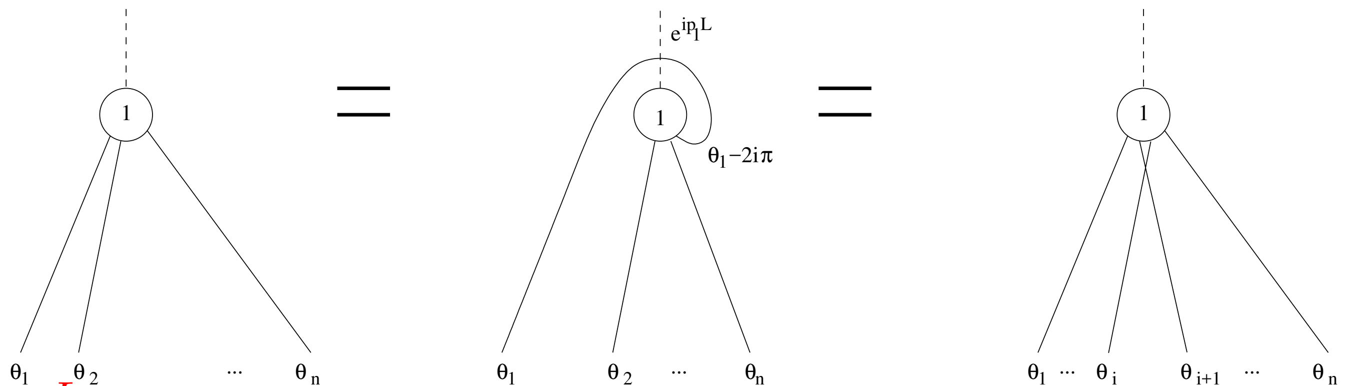

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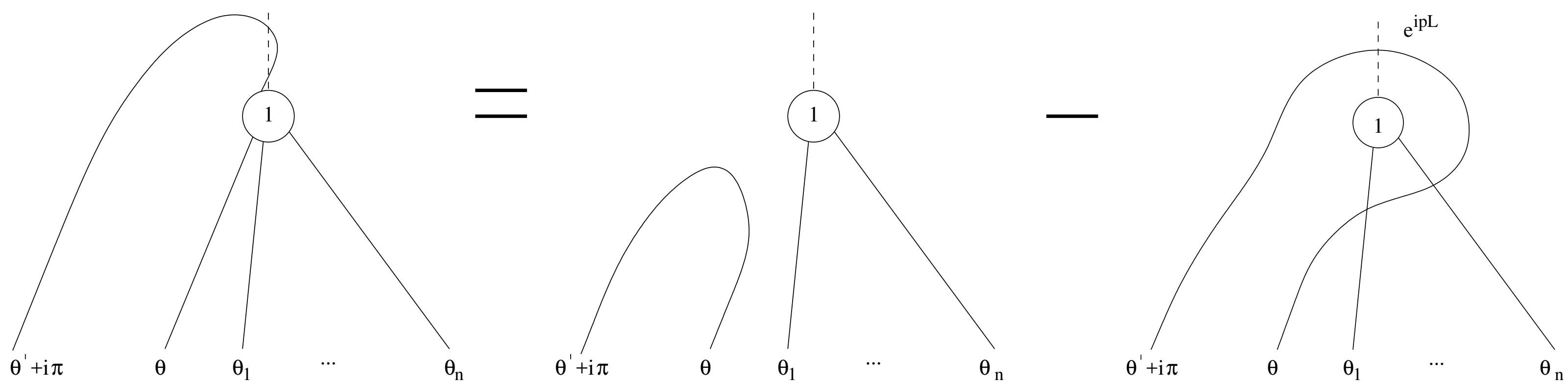

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Free massive boson: pp-wave limit of strings on $AdS_5 \times S^5$

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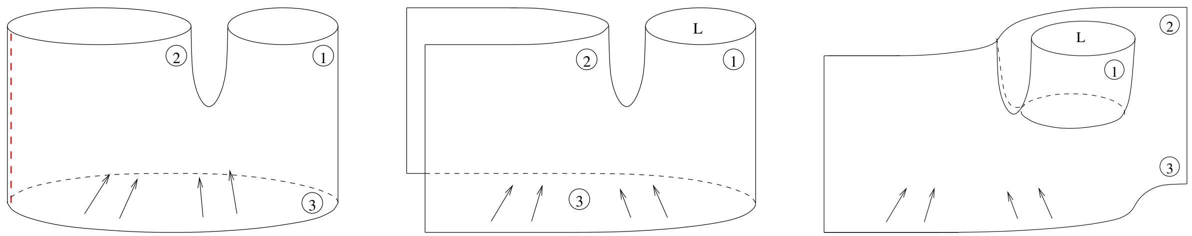

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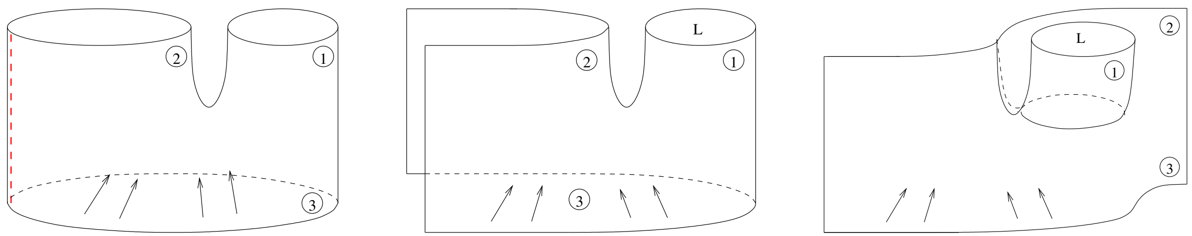

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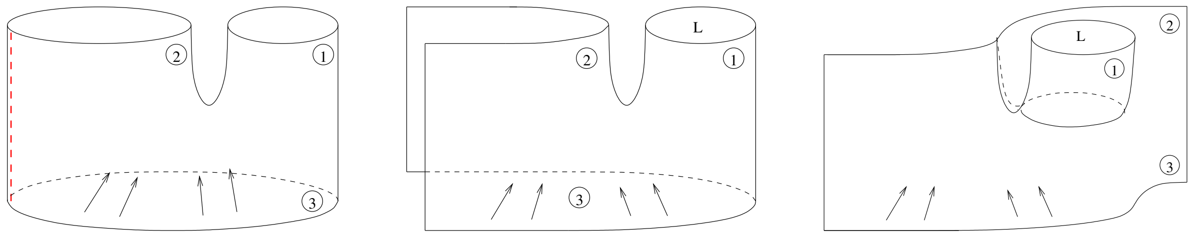

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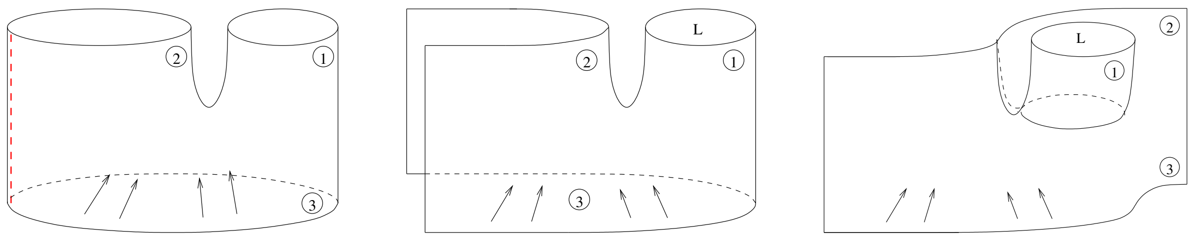

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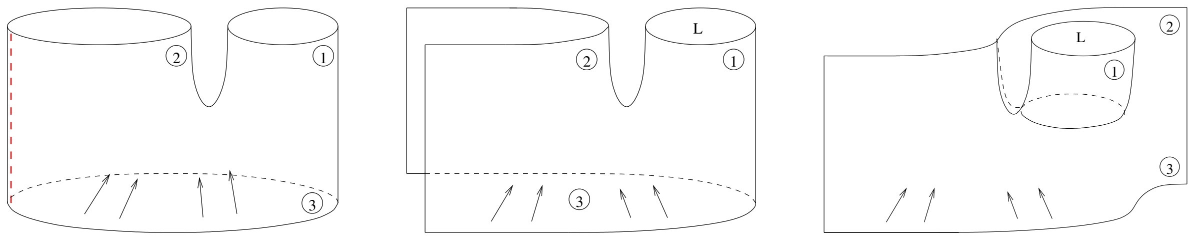

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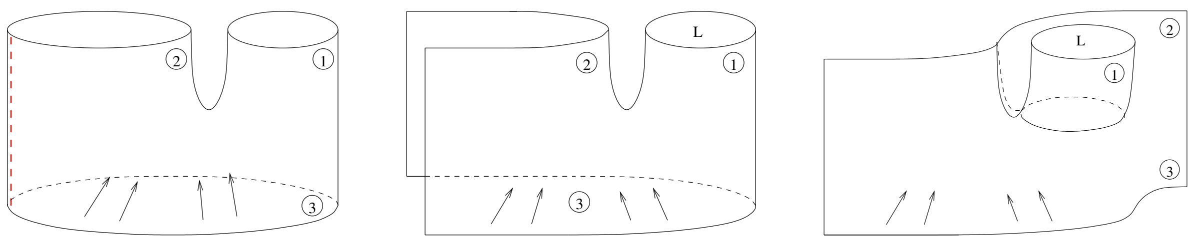

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$n(\theta) = \sinh\frac{\theta}{2} \sin\frac{pL}{2}\, \Gamma_{\frac{mL}{2\pi}}(m\sinh\theta)$ where Γ_μ removes zeros at $\theta = \frac{2\pi n}{L} + i\pi$

$\Gamma_\mu(z) = z^{-1} e^{-\omega_z(\gamma + \log\frac{\mu}{2e})} \prod \frac{n}{\omega_n + \omega_z} e^{-\frac{\omega_n}{z}}$ and $\omega_z = \sqrt{\mu^2 + z^2}$

[Spradlin et al '02,Lucietti et al '03]

Nontrivial scattering: Factorizing ansatz

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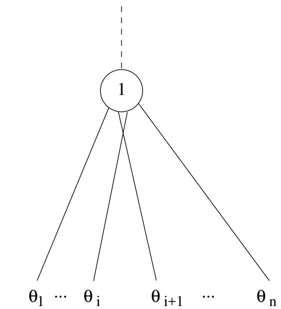

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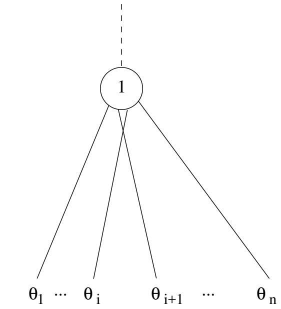

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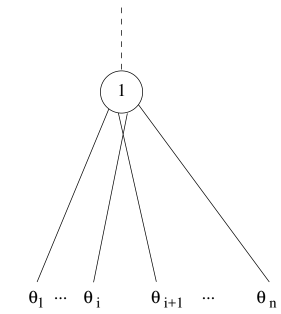

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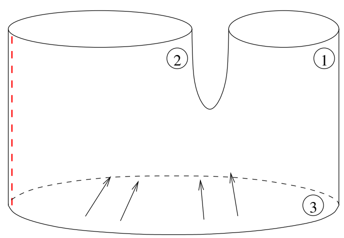

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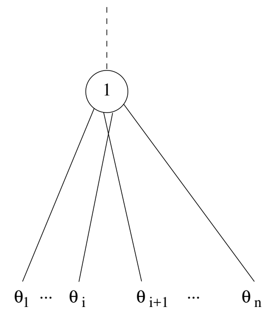

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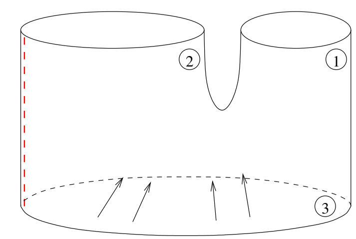

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Try to generalize $N_L(p_1,p_2)$ for $AdS_5 \times S^5$. (pp-wave limit: $g \to \infty$, $p \to 0$)

Relativistic dispersion $E(p) = \sqrt{p^2 + m^2}$ rapidity $E(\theta) = m\cosh\theta$; $p(\theta) = m\sinh\theta$

AdS/CFT: $E(p) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}$ elliptic torus: $E = \text{dn}(w, -16g^2)$; $\frac{p}{2} = \text{am}(w)$

The kinematical $AdS_5 \times S^5$ Neumann coefficient

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monodromy equations: $N_L(z_1, z_2) = N_L(z_2, z_1) = e^{-ip(z_1)L} N_L(z_2, z_1 - \tau)$

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The kinematical $AdS_5 \times S^5$ Neumann coefficient

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remove zeros + ensure periodicity (use elliptic gamma functions):

$$n(z) = \frac{2g\sqrt{L}}{\pi} \sin p \, G_L(z) h(z) \; ; \; G_{L=2n}(z) = \sqrt{\frac{L}{2}} \prod_{k=1}^{n-1} \frac{\sqrt{1 + 16g^2 \sin^2\frac{\pi k}{L}} - E(z)}{4g \sin\frac{\pi k}{L}}$$

$h(z) \propto e^{-i\frac{p}{2}n} e^{-ipL} H(z)^L$ with $H(z) = e^{i\frac{\pi}{2}z} \frac{\Gamma_{ell}(z - \frac{1}{2} + \frac{\tau}{4})\Gamma_{ell}(z - \frac{1}{2} - \frac{3\tau}{4})}{\Gamma_{ell}(z - \frac{1}{2} - \frac{\tau}{2})^2}$ and $\Gamma_{ell} = \prod_{k=0}^{\infty} \left(\frac{1 - e^{2\pi i\tau(k+2)} e^{-2\pi i z}}{1 - e^{2i\pi\tau k} e^{2i\pi z}}\right)^{k+1}$

The kinematical $AdS_5 \times S^5$ Neumann coefficient

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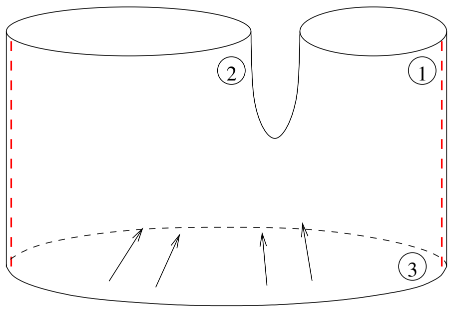
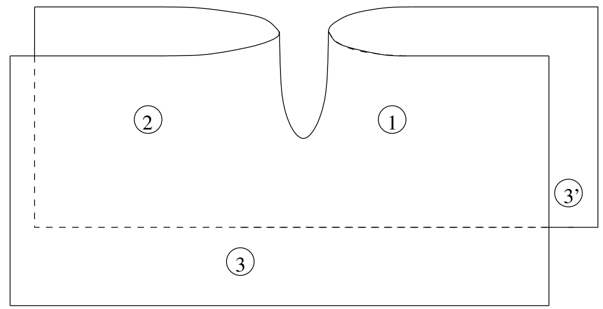
Checked in the pp-wave, weak coupling and large L limit. Strong coupling check against [Kazama, Komatsu] is on the way... **But we need form factors now!**

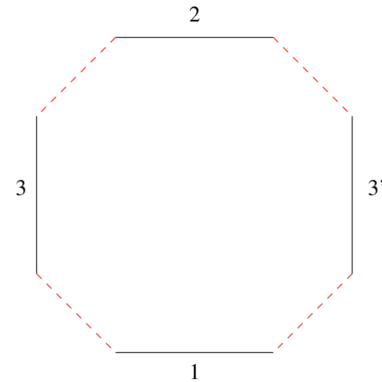
Decompactifying all volumes $L_1 = \infty$: octagon

Decompactifying all volumes $L_1 = \infty$: octagon

Decompactify all volumes

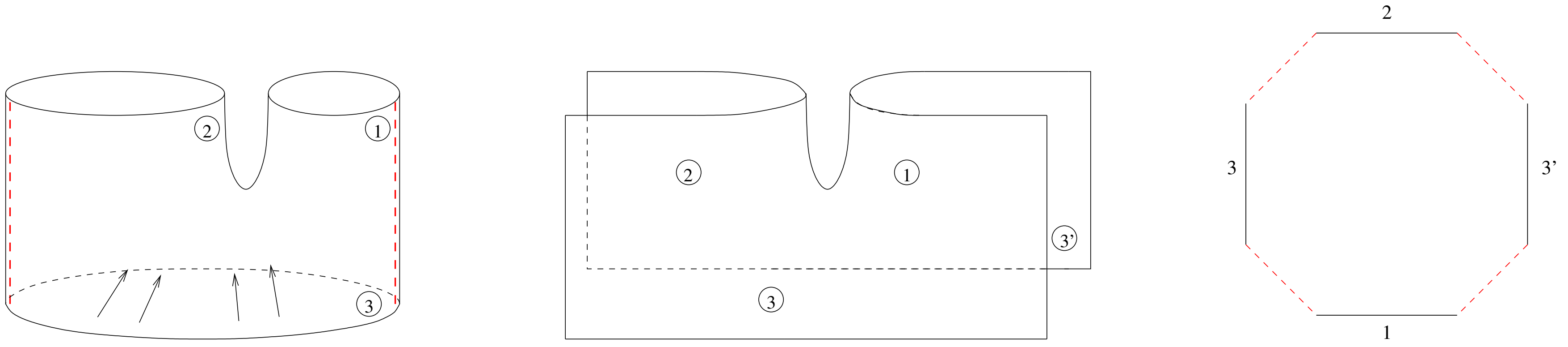




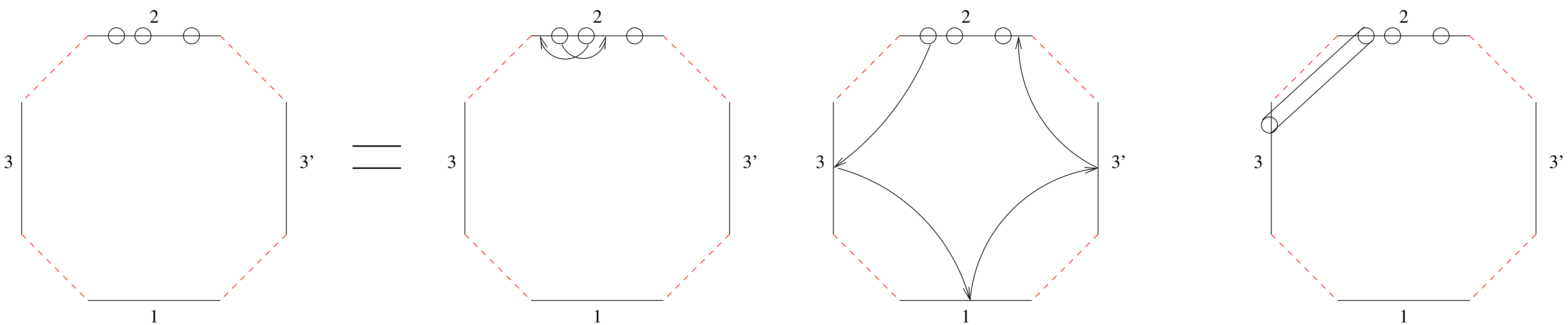


Decompactifying all volumes $L_1 = \infty$: octagon

Decompactify all volumes



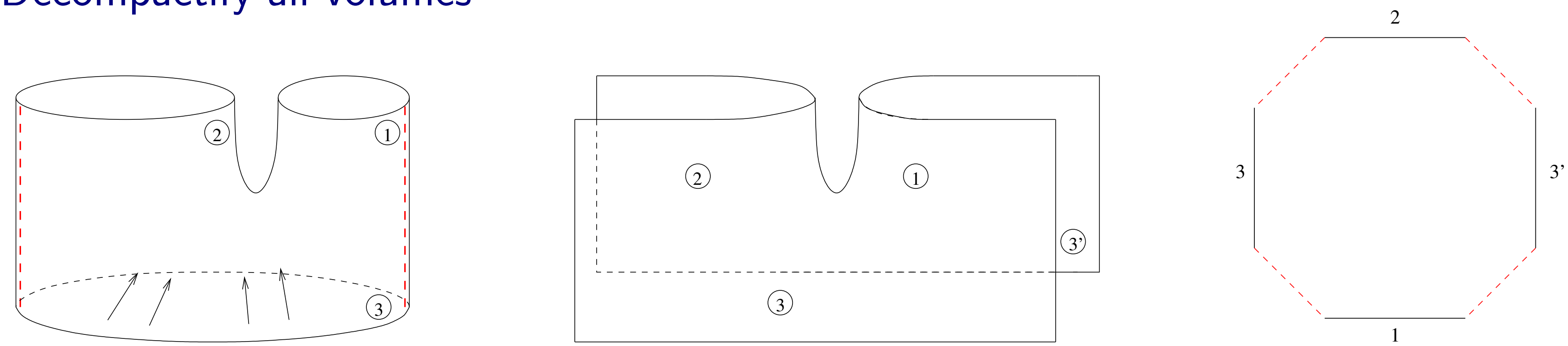
Octagon axioms:



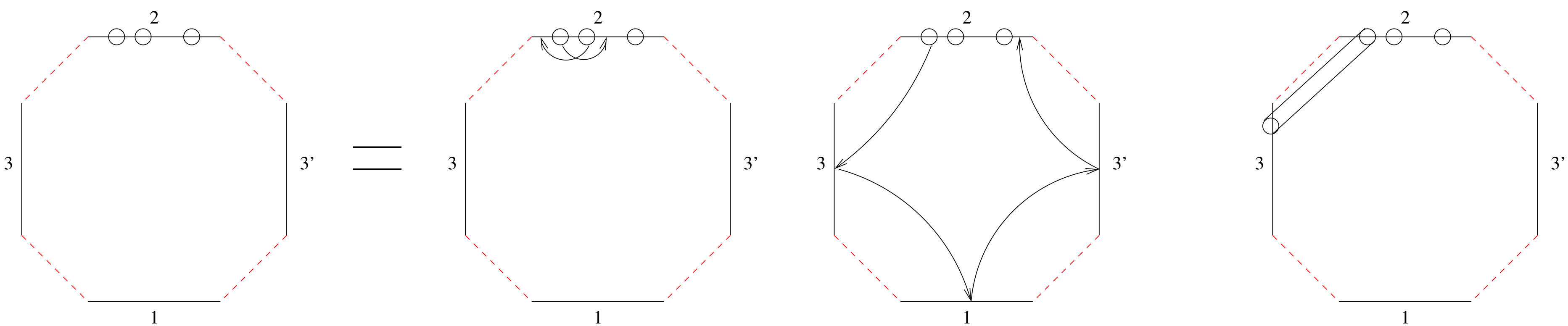
$$O(\theta_i, \ldots, \theta_n) = S(\theta_i, \theta_{i+1}) O(\ldots, \theta_{i+1}, \theta_i, \ldots) = O(\theta_2, \ldots, \theta_n, \theta_1 - 4i\pi)$$

Decompactifying all volumes $L_1 = \infty$: octagon

Decompactify all volumes



Octagon axioms:



$$O(\theta_i, \dots, \theta_n) = S(\theta_i, \theta_{i+1}) O(\dots, \theta_{i+1}, \theta_i, \dots) = O(\theta_2, \dots, \theta_n, \theta_1 - 4i\pi)$$

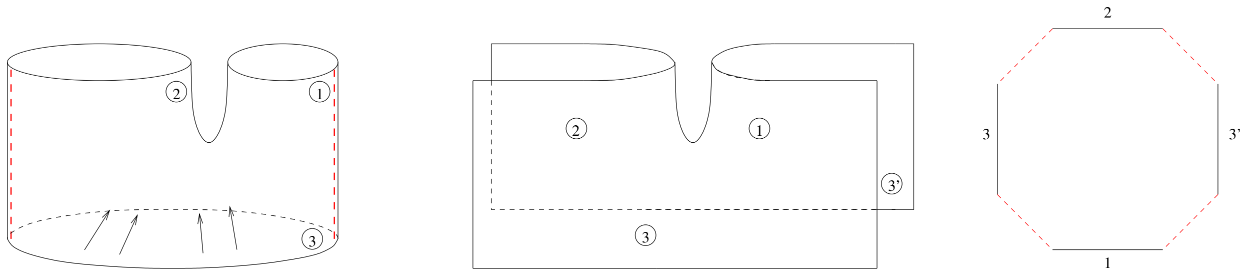
Kinematical singularity $-i\mathrm{Res}_{\theta'=\theta} O(\theta' + i\pi, \theta, \theta_1, \dots, \theta_n) = O(\theta_1, \dots, \theta_n)$

pp-wave octagons

pp-wave octagons

2 cuts

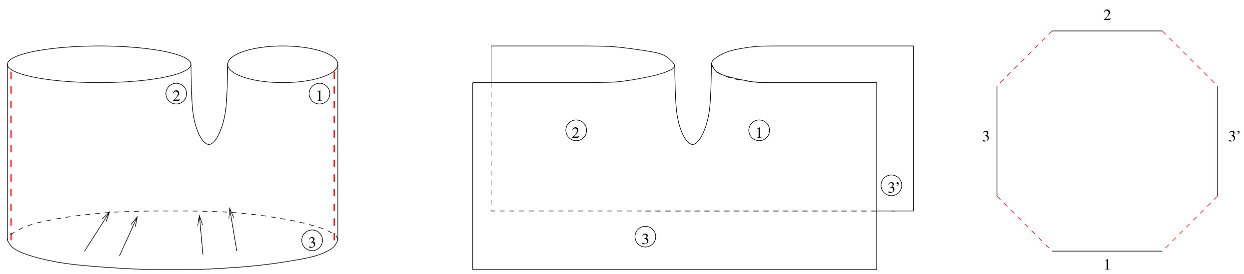
octagon



pp-wave octagons

2 cuts

octagon



2 particles: $O(\theta_1, \theta_2) = O(\theta_2, \theta_1) = O(\theta_2, \theta_1 - 4i\pi)$

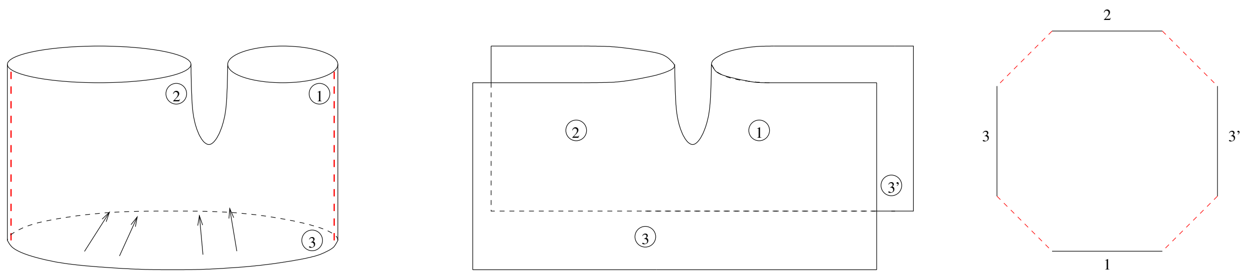
kinematical singularity: $-i\mathrm{Res}_\epsilon O(\theta + i\pi + \epsilon, \theta) = 1$

$$\rightarrow \quad O(\theta_1, \theta_2) = -\frac{1}{2\cosh\frac{\theta_1 - \theta_2}{2}}$$

pp-wave octagons

2 cuts

octagon



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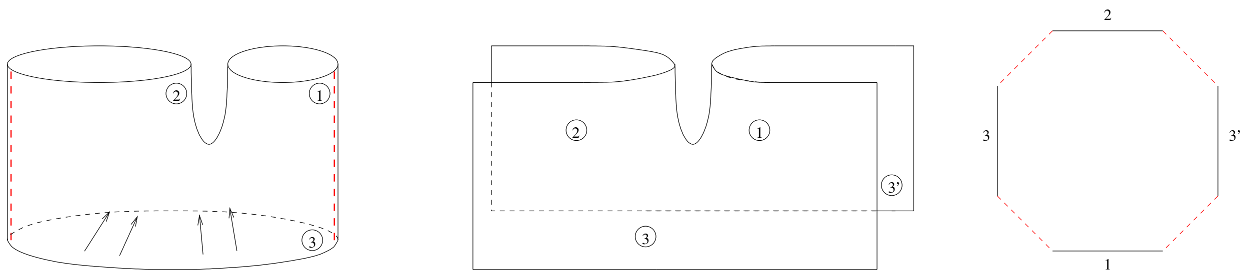
Multiparticle solution: $O(1,2,3,4) = O(1,2)O(3,4) + O(1,3)O(2,4) + O(1,4)O(2,3)$

(Wick theorem)

pp-wave octagons

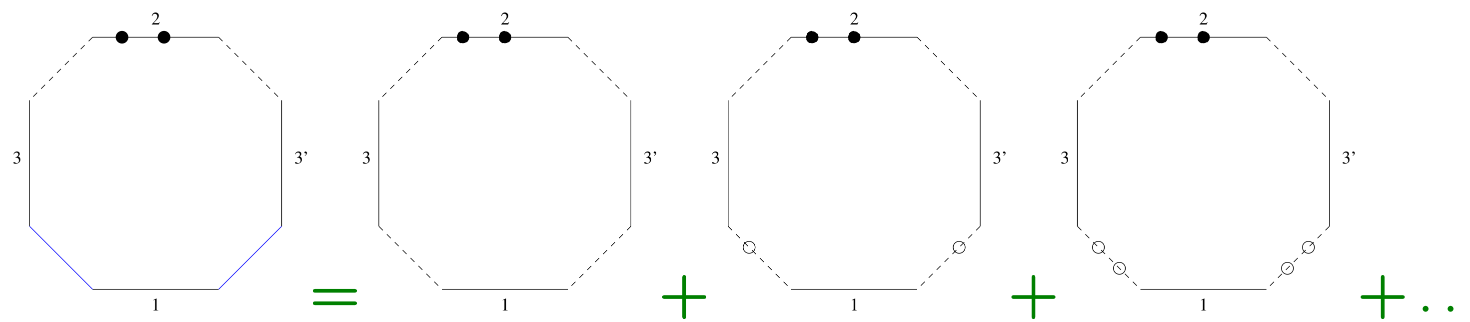
2 cuts
octagon



2 particles: $O(\theta_1,\theta_2) = O(\theta_2,\theta_1) = O(\theta_2,\theta_1-4i\pi)$ $\quad\rightarrow\quad$ $O(\theta_1,\theta_2) = -\dfrac{1}{2\cosh\frac{\theta_1-\theta_2}{2}}$
kinematical singularity: $-i\mathrm{Res}_\epsilon O(\theta+i\pi+\epsilon,\theta) = 1$

Multiparticle solution: $O(1,2,3,4) = O(1,2)O(3,4)+O(1,3)O(2,4)+O(1,4)O(2,3)$
(Wick theorem)

Summing up
virtual corrections?
Gluing back?



[Basso, Komatsu, Vieira] $\sum_{n=0}^{\infty}\frac{1}{n!}\prod_{i=1}^{n}\int_{-\infty}^{\infty}\frac{du_i}{2\pi}\mu(\{u\})e^{-\sum_i E(u_i)L}|n\rangle\langle n|$

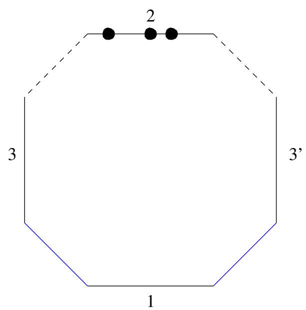
$$N_L(\theta_1,\theta_2) = O(\theta_1,\theta_2) + \int\frac{du}{2\pi}O(\theta_1,\theta_2,u-3i\tfrac{\pi}{2},u+3i\tfrac{\pi}{2})_c\, e^{-m\cosh uL} + \dots$$

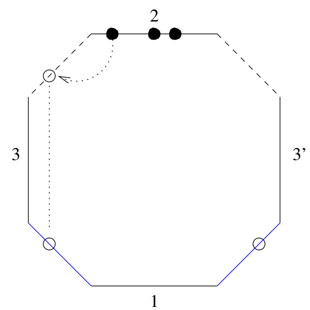
Kinematically singular: adhoc regularization (connected part) agrees with NLO $N_L(\theta_1,\theta_2)$!

FF Axioms from gluing octagons

FF Axioms from gluing octagons

Analytical continuation: teleportation





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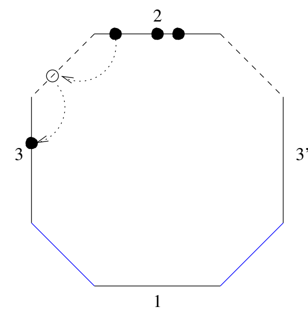
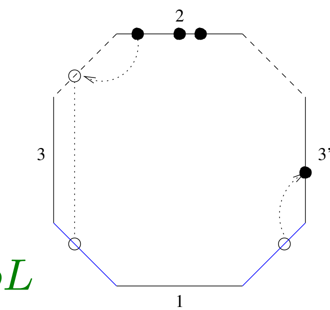


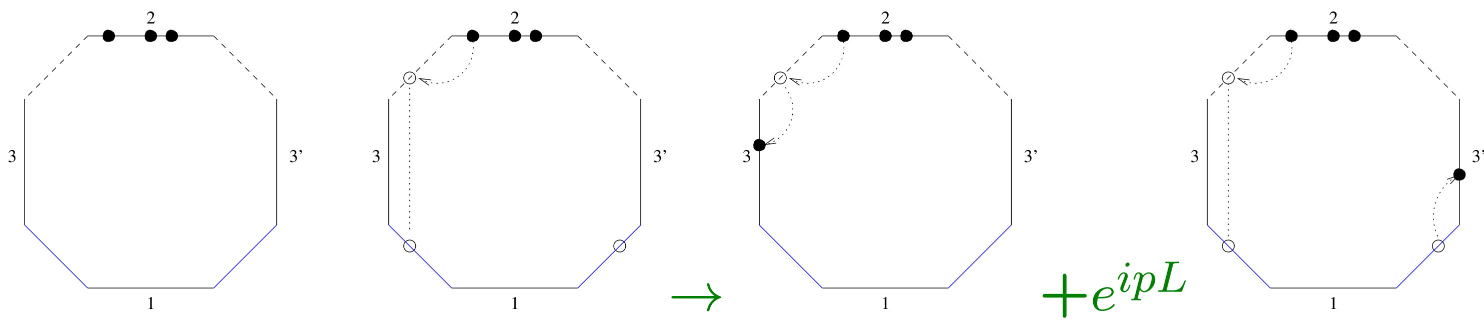
$+e^{ipL}$



kinematical singularity: two contributions $1 - e^{ipL}$

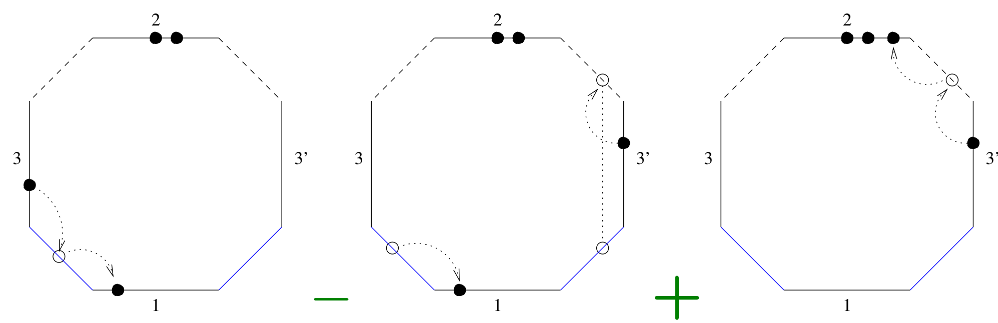
FF Axioms from gluing octagons

Analytical continuation: teleportation



kinematical singularity: two contributions $1 - e^{ipL}$

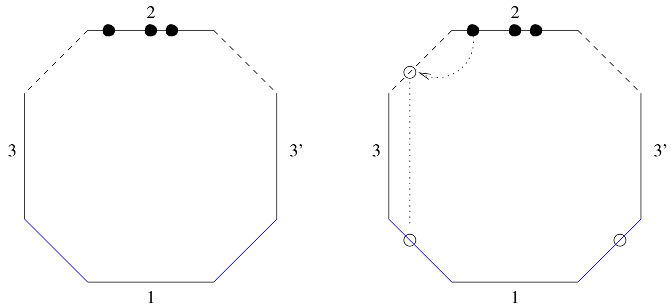
continue further, cross terms cancel, $4i\pi$ periodicity



Naive summation : $O_L(\theta_1, \theta_2) = O(\theta_1, \theta_2) + \int_{-\infty}^{\infty} \frac{du}{2\pi} O(\theta_1, \theta_2, u^-, u^+) e^{-LE(u)} + \dots$

FF Axioms from gluing octagons

Analytical continuation: teleportation



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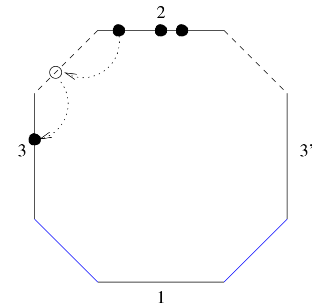
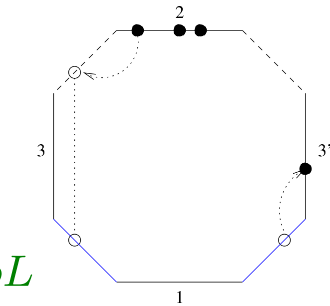


$+e^{ipL}$



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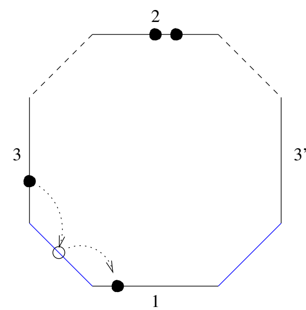

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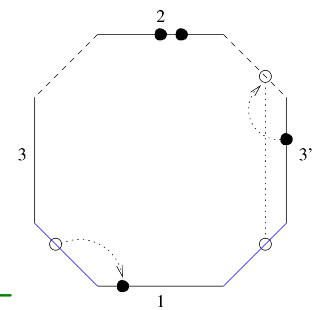

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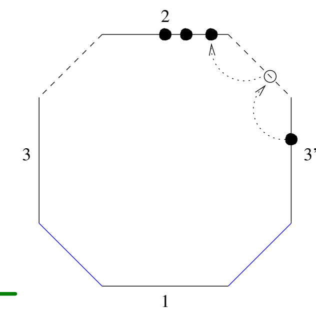
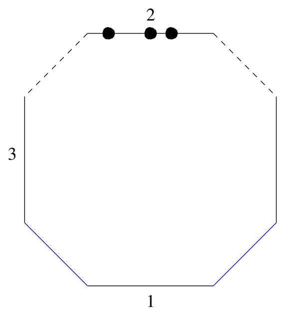


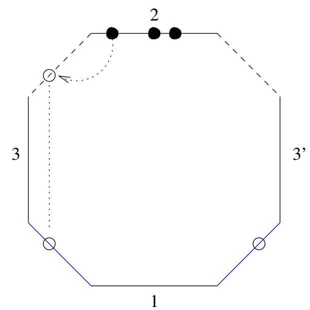
Naive summation : $O_L(\theta_1, \theta_2) = O(\theta_1, \theta_2) + \int_{-\infty}^{\infty} \frac{du}{2\pi} O(\theta_1, \theta_2, u^-, u^+) e^{-LE(u)} + \ldots$

connected part: $O(\theta_1, u^-) O(\theta_2, u^+) + (\theta_1 \leftrightarrow \theta_2) = -O(\theta_1, \theta_2) \left(\frac{1}{\cosh(u-\theta_1)} + \frac{1}{\cosh(u-\theta_2)} \right)$

FF Axioms from gluing octagons

Analytical continuation: teleportation





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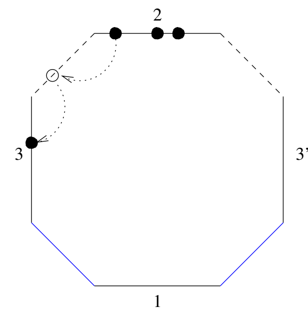
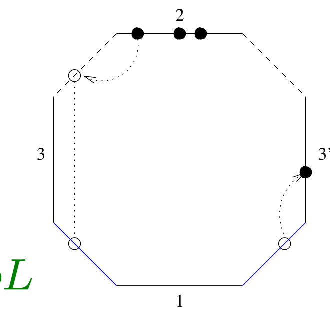
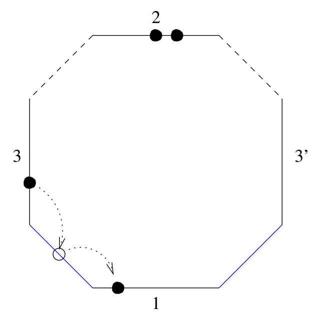


$+e^{ipL}$



kinematical singularity: two contributions $1 - e^{ipL}$

continue further,
cross terms cancel,
$4i\pi$ periodicity



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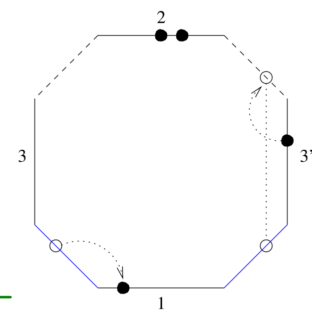

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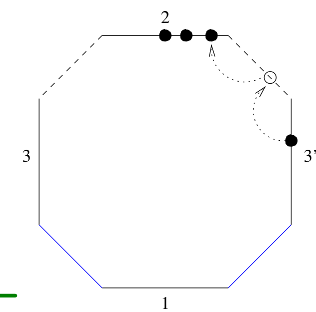

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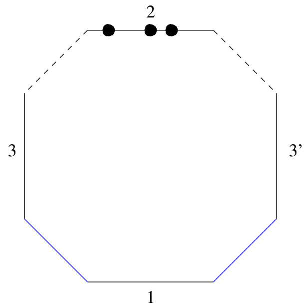
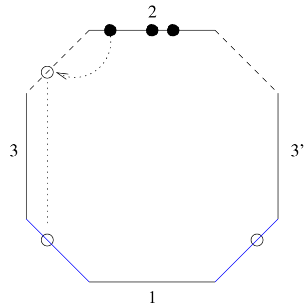
in general $O(\theta_1,\theta_2,u_1^-,...,u_n^-,u_1^+,...,u_n^+)_c = O(\theta_1,\theta_2)\prod_{i=1}^{3}\left(\frac{-1}{\cosh(u_i-\theta_1)} + \frac{-1}{\cosh(u_i-\theta_2)}\right)$

FF Axioms from gluing octagons

Analytical continuation: teleportation





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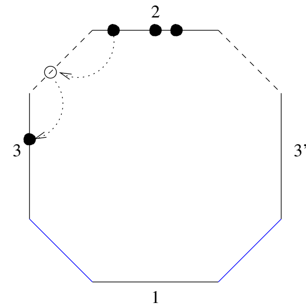
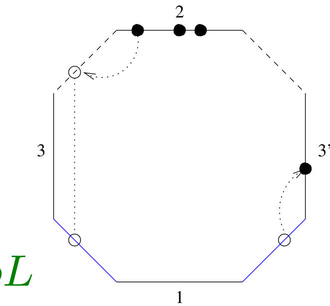
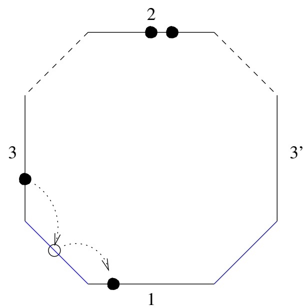


$+e^{ipL}$



kinematical singularity: two contributions $1 - e^{ipL}$

continue further,
cross terms cancel,
$4i\pi$ periodicity



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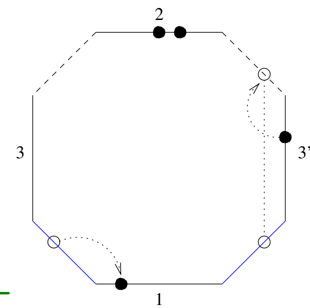

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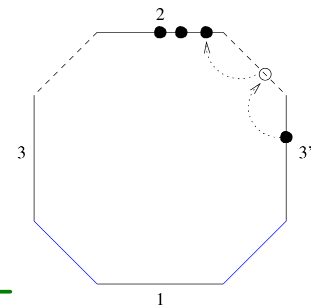


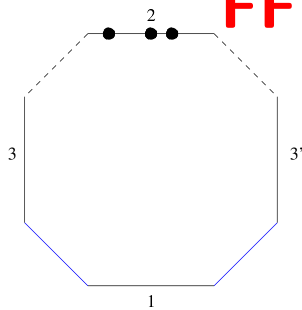
Naive summation : $O_L(\theta_1,\theta_2) = O(\theta_1,\theta_2) + \int_{-\infty}^{\infty} \frac{du}{2\pi} O(\theta_1,\theta_2,u^-,u^+)e^{-LE(u)} + \dots$

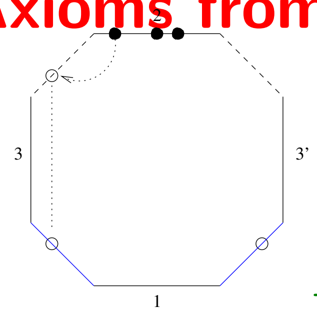
in general $O(\theta_1,\theta_2,u_1^-,...,u_n^-,u_1^+,...,u_n^+)_c = O(\theta_1,\theta_2) \prod_{i=1}^3 \left(\frac{-1}{\cosh(u_i-\theta_1)} + \frac{-1}{\cosh(u_i-\theta_2)} \right)$

Wick $\rightarrow$ exponentiation: $O(\theta_1,\theta_2)d(\theta_1)d(\theta_2)$ with: $\log d(\theta) = -\int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{e^{-mL\cosh u}}{\cosh(u-\theta)}$

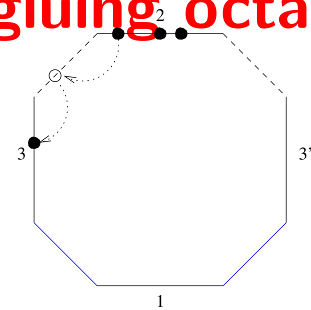
FF Axioms from gluing octagons

Analytical continuation: teleportation

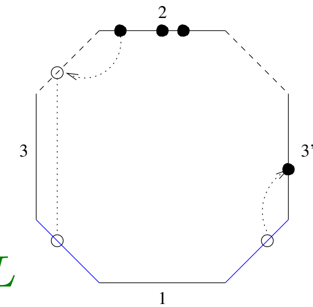




$\rightarrow$



$+e^{ipL}$



kinematical singularity: two contributions $1-e^{ipL}$

continue further,
cross terms cancel,
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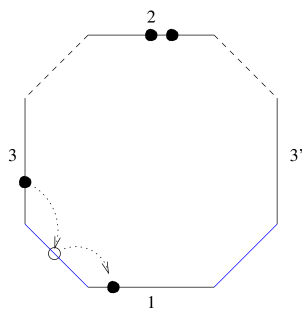

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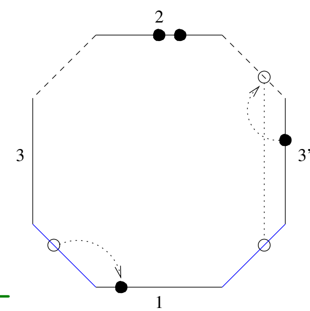

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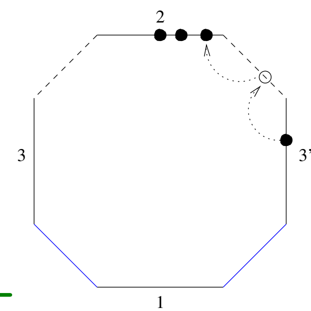

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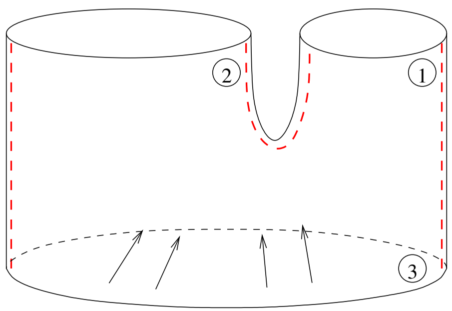
exact result is similar to ground-state energy in volume L: $\log n(\theta) = \int_{-\infty}^{\infty} \frac{du}{2\pi} \frac{\log(1-e^{-mL\cosh u})}{\cosh(u-\theta)}$

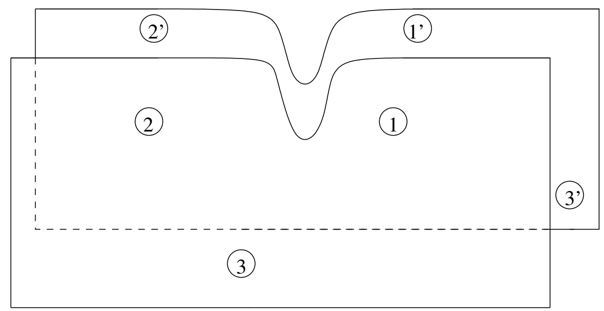
Problem with the finite part of the singular contributions $\rightarrow$ regulate by sums (finite volume) and pay attention on the weight of the (partially) diagonal terms: correct undecompactified vertex

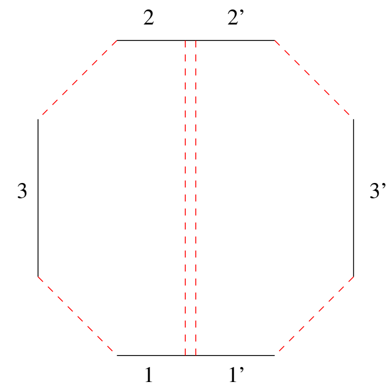
Cutting one more: two hexagons

Cutting one more: two hexagons

Decompactify all volumes

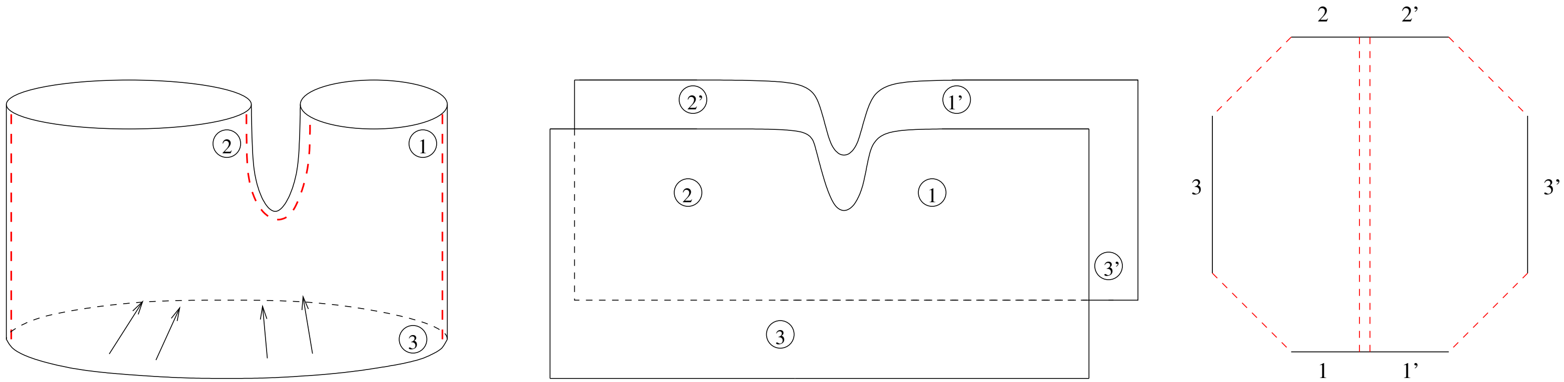




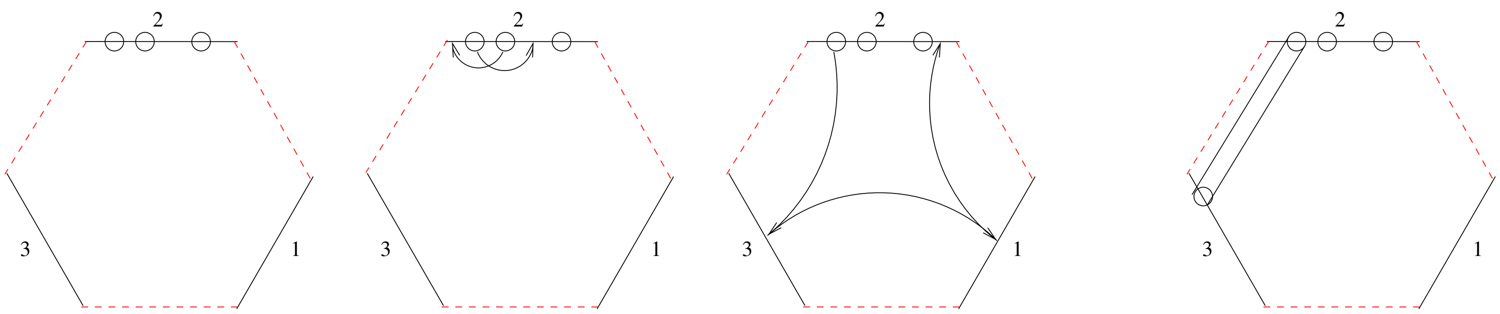


Cutting one more: two hexagons

Decompactify all volumes



Hexagon axioms: [Basso,Komatsu,Vieira '15]

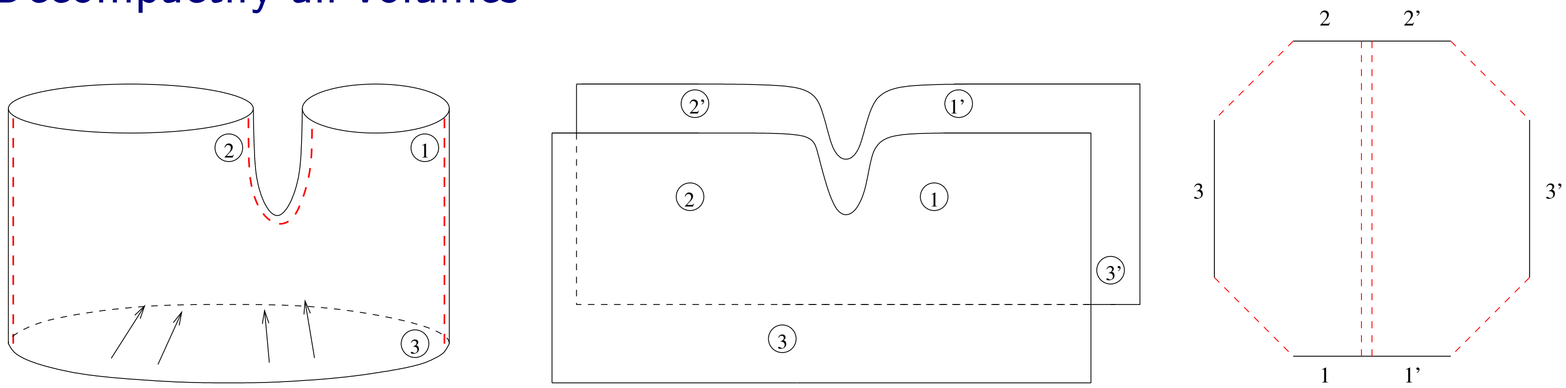


$$h(\theta_i,\ldots,\theta_n)=S(\theta_i,\theta_{i+1})h(\ldots,\theta_{i+1},\theta_i,\ldots)=h(\theta_2,\ldots,\theta_n,\theta_1-3i\pi)$$

Kinematical singularity $-i\mathrm{Res}_{\theta'=\theta}h(\theta'+i\pi,\theta,\theta_1,\ldots,\theta_n)=h(\theta_1,\ldots,\theta_n)$

Cutting one more: two hexagons

Decompactify all volumes



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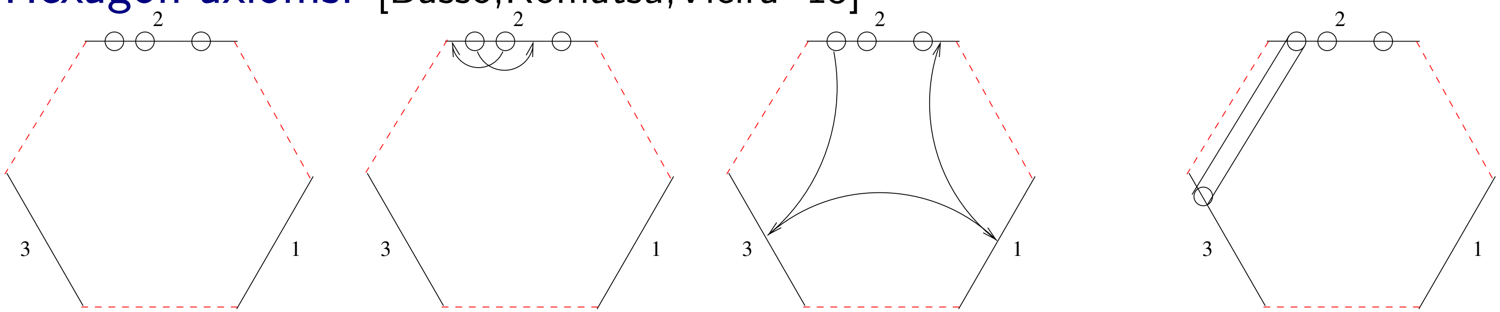


$$h(\theta_i,\dots,\theta_n)=S(\theta_i,\theta_{i+1})h(\dots,\theta_{i+1},\theta_i,\dots)=h(\theta_2,\dots,\theta_n,\theta_1-3i\pi)$$

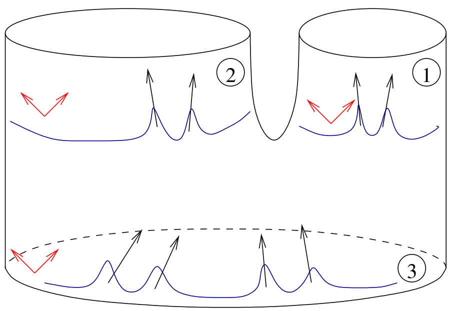
Kinematical singularity $-i\mathrm{Res}_{\theta'=\theta}h(\theta'+i\pi,\theta,\theta_1,\dots,\theta_n)=h(\theta_1,\dots,\theta_n)$

Complete solution: $h(\theta_1,\theta_2)\propto\sigma(\theta_1,\theta_2)S_{\mathrm{Beisert}}(\theta_1,\theta_2)$ What makes it unique?

Comparision of the different approaches

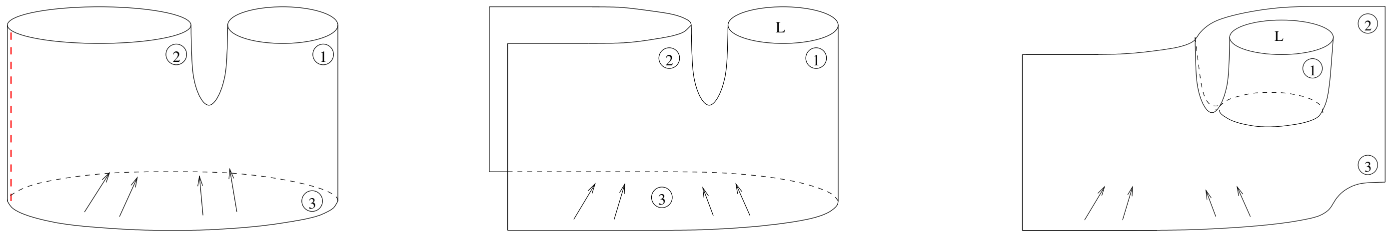
Comparision of the different approaches

Ultimate goal:



Comparision of the different approaches

1 cut:
nonlocal
form factors



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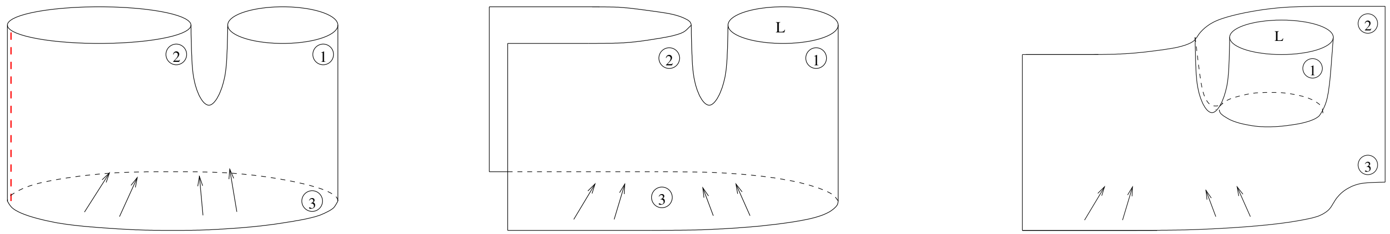
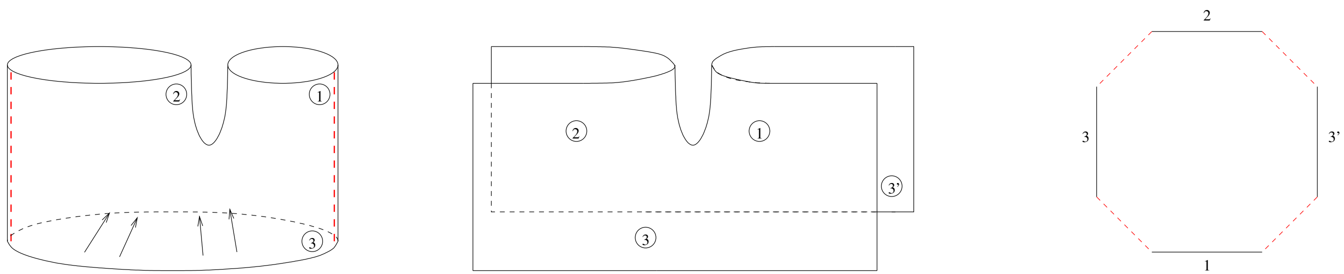


2 cuts
octagon



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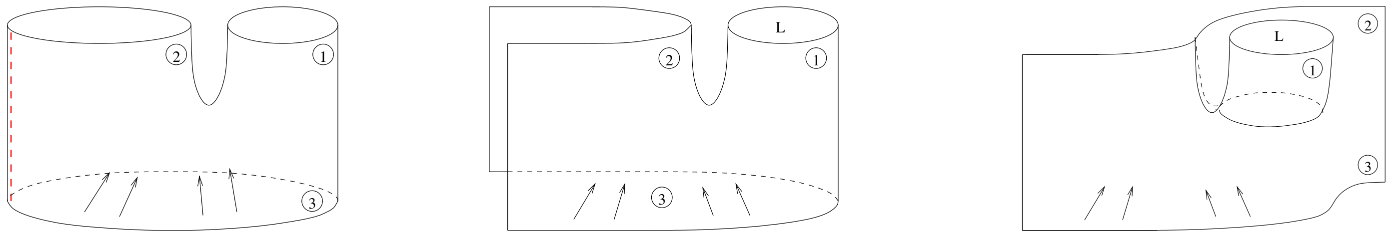

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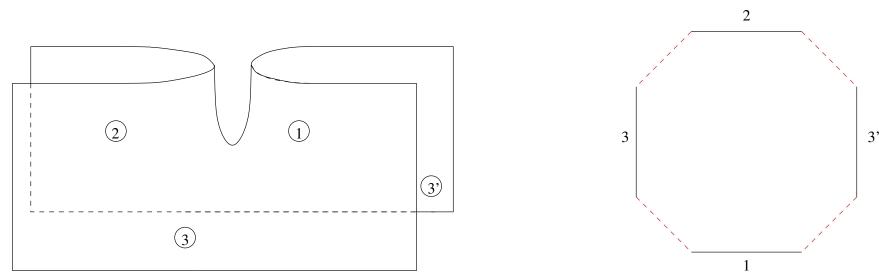
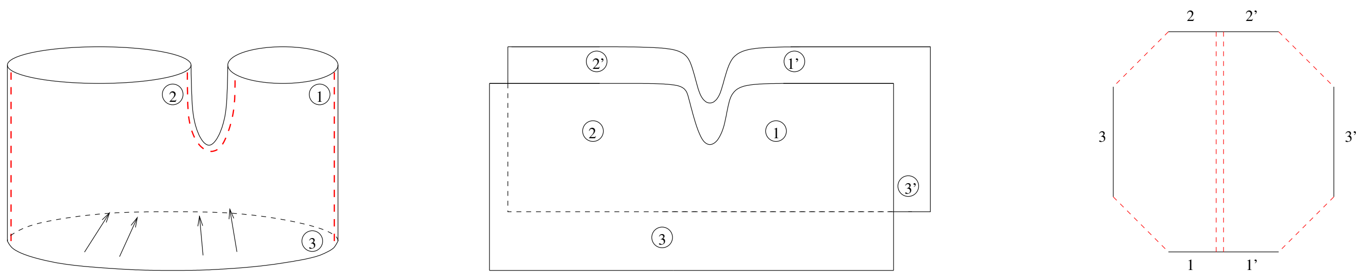


3 cuts
hexagon



Comparision of the different approaches

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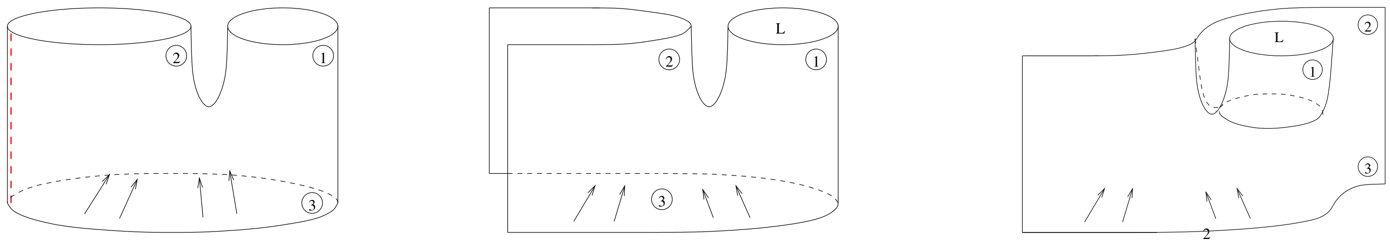
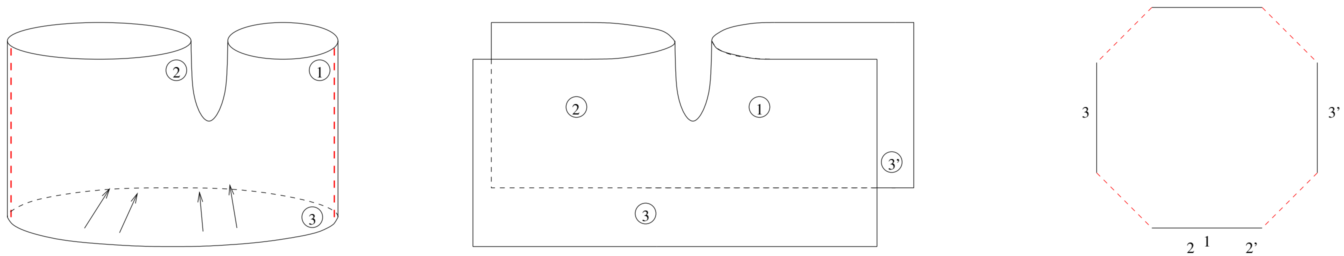


2 cuts
octagon



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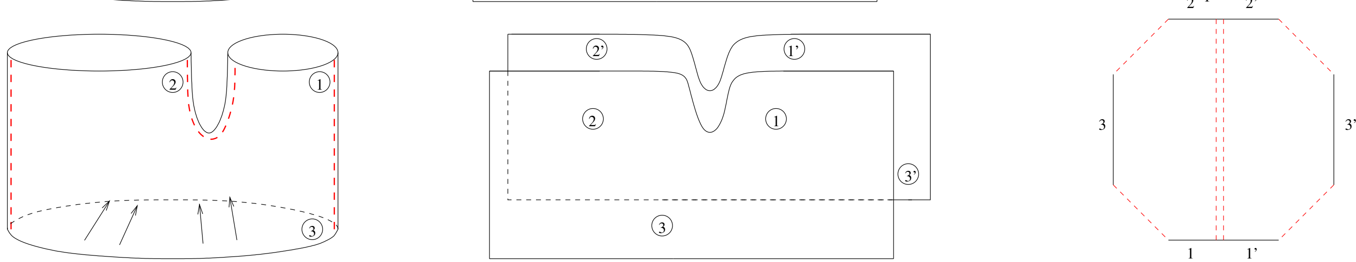
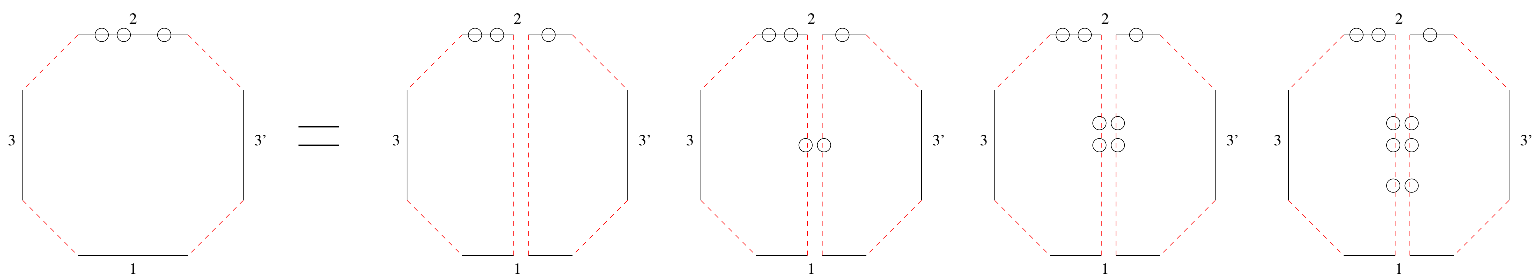
hexagon





sewing back

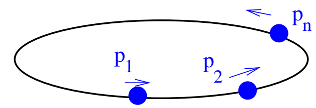
$$O(\theta_1,\theta_2,\theta_3)=h(\theta_1,\theta_2)h(\theta_3)+\ldots+\int\frac{du}{2\pi}\mu(u)h(\theta_1,\theta_2,u-i\tfrac{\pi}{2})h(u+i\tfrac{\pi}{2},\theta_3)e^{-E(u)l}+$$

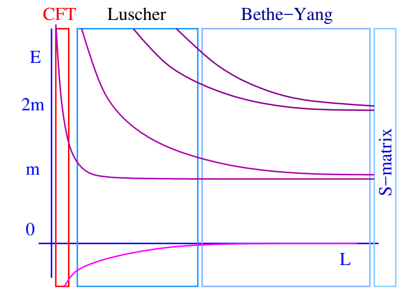
Octagon axioms from hexagon axioms via teleportation. What to do with singular contributions?

Understand finite size effects!

Outline of the second part

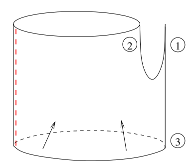
Finite size effects in the spectral problem



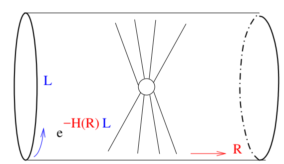


Finite volume form factors: polynomial corrections

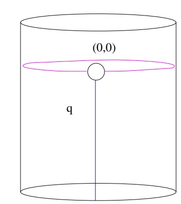
Diagonal form factors and HHL correlators



Exact one-point function: the LM series



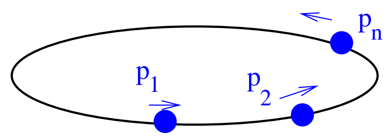
Lüscher correction for non-diagonal form factors



Conclusions

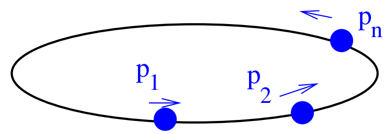
Finite size effects: Spectral problem

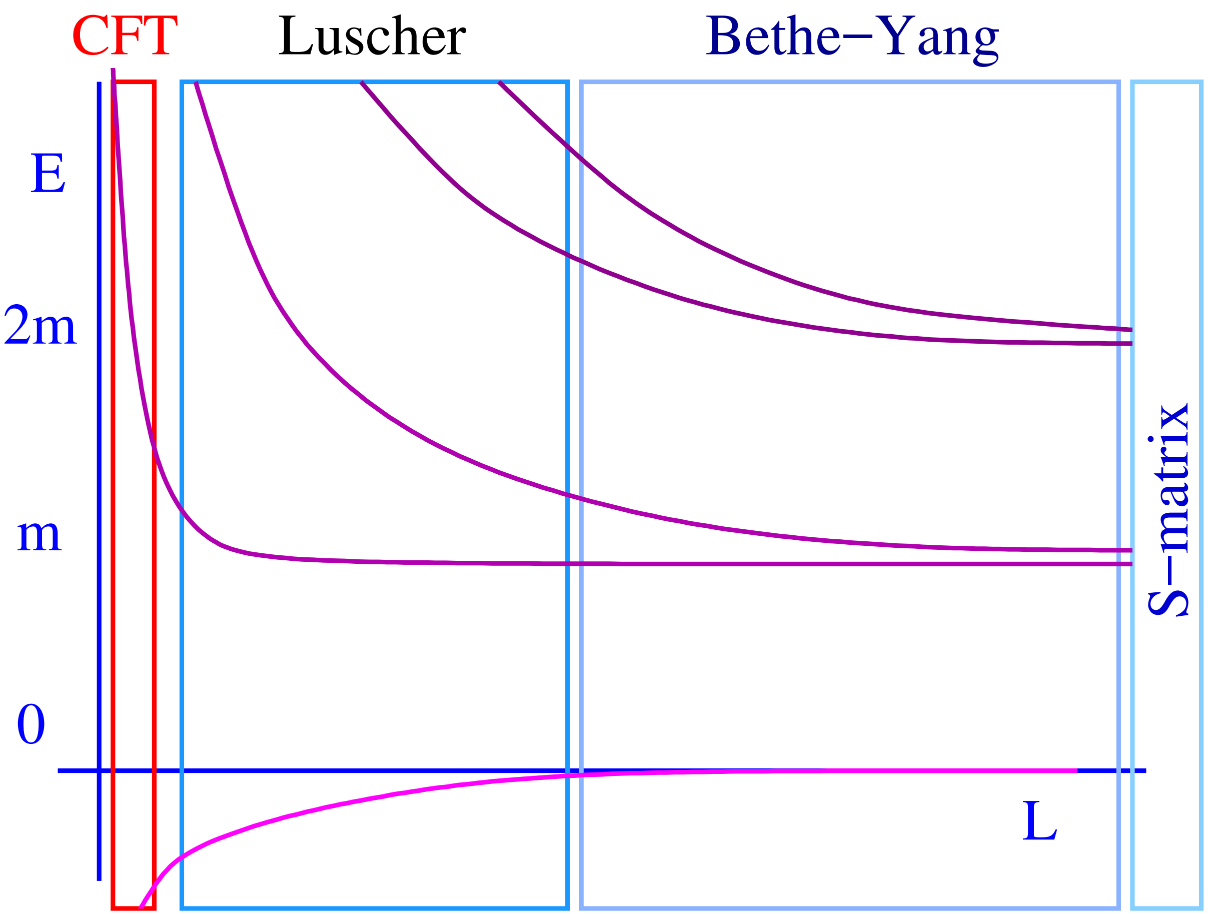
Finite volume spectrum



Finite size effects: Spectral problem

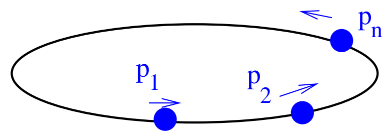
Finite volume spectrum

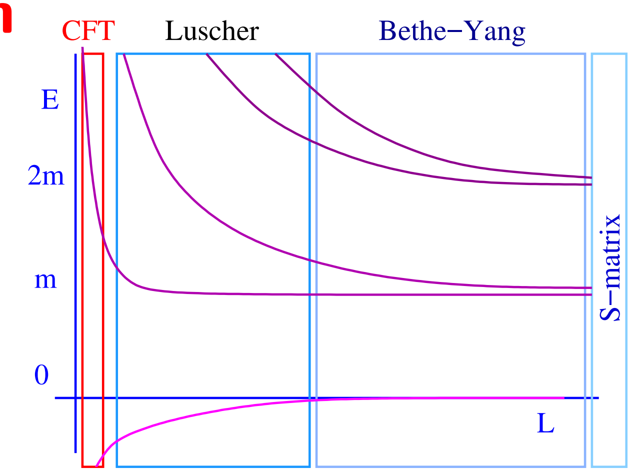




Finite size effects: Spectral problem

Finite volume spectrum



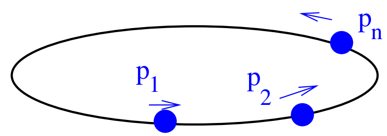


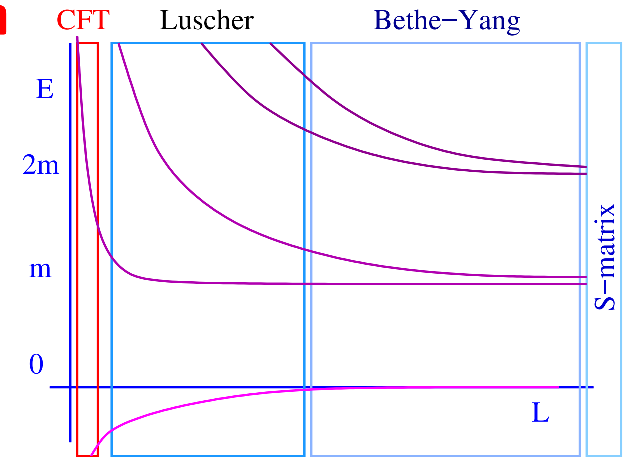
Polynomial volume corrections:

$$E(\theta_1,\dots,\theta_n) = \sum_i E(\theta_i)$$

Finite size effects: Spectral problem

Finite volume spectrum





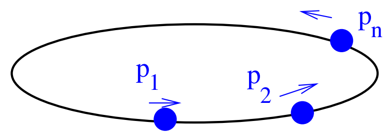
Polynomial volume corrections:

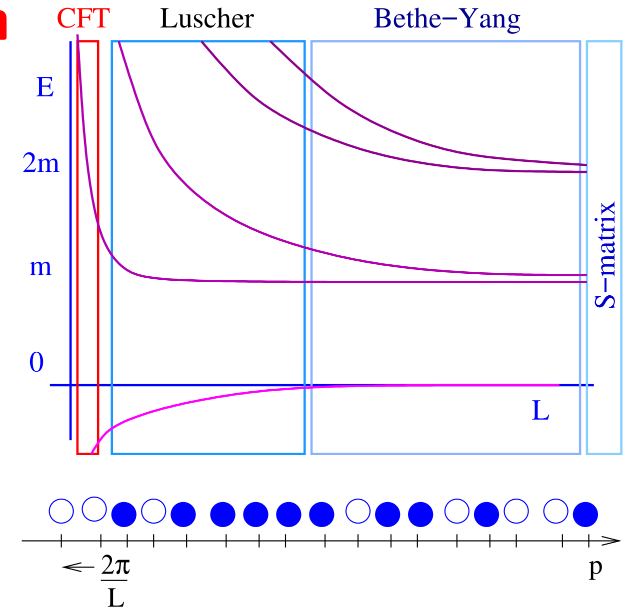
$E(\theta_1, \ldots, \theta_n) = \sum_i E(\theta_i)$

Bethe-Yang; p_i quantized: $e^{ip_jL} \prod_k S(\theta_j - \theta_k) = -1$

Finite size effects: Spectral problem

Finite volume spectrum





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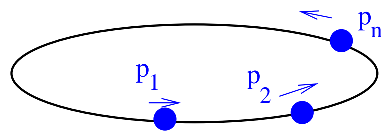
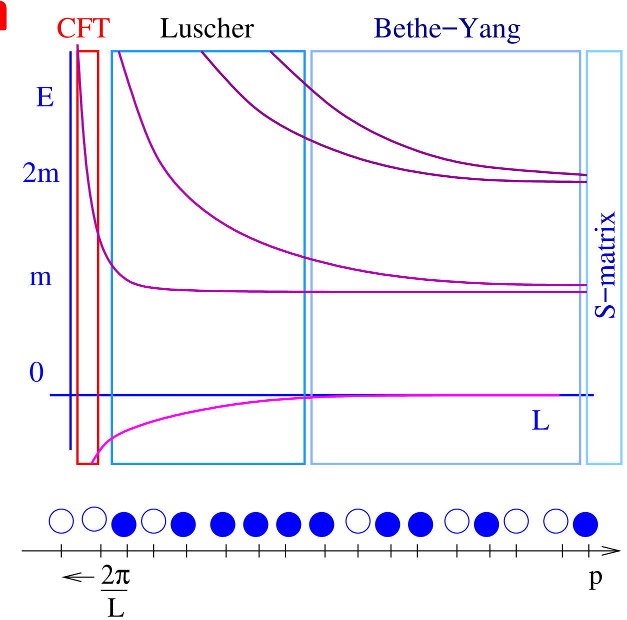
Bethe-Yang; p_i quantized: $e^{ip_jL} \prod_k S(\theta_j - \theta_k) = -1$

$p_j L + \sum_k \frac{1}{i} \log S(\theta_j - \theta_k) = (2n+1)\pi$

Finite size effects: Spectral problem

Finite volume spectrum





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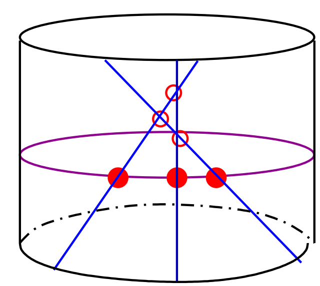
Lüscher-type corrections:



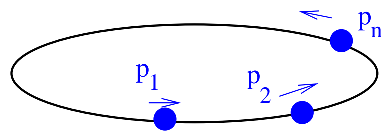
$E(\theta_1, \ldots, \theta_n) = \sum_i E(\theta_i) - \int \frac{d\theta}{2\pi} \prod_k S(\theta + i\frac{\pi}{2} - \theta_k) e^{-mL\cosh\theta}$

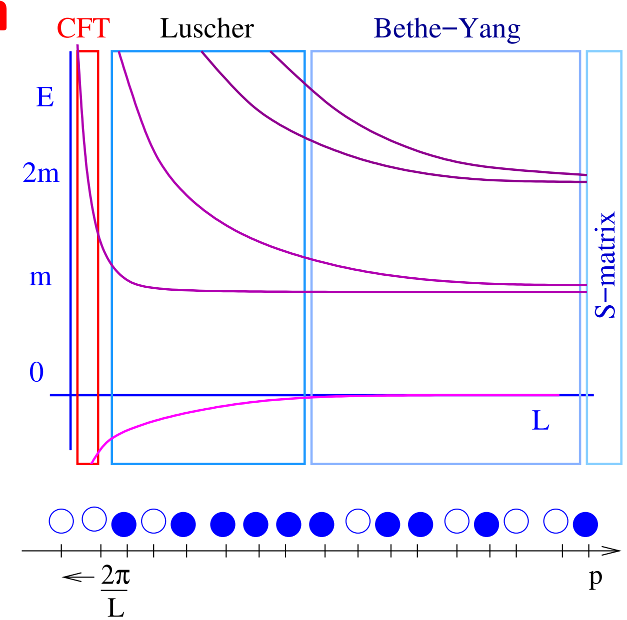
BY modified as $p_j L + \sum_k \frac{1}{i} \log S(\theta_j - \theta_k) + \delta = (2n+1)\pi$

where $\delta = i \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \log' S(\theta_j - \theta') \prod_k S(i\frac{\pi}{2} + \theta_k - \theta') e^{-mL\cosh\theta'}$

Finite size effects: Spectral problem

Finite volume spectrum





Polynomial volume corrections:

$$E(\theta_1,\ldots,\theta_n)=\sum_i E(\theta_i)$$

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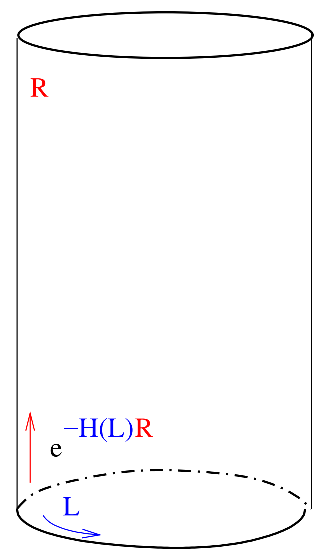
$$p_jL+\sum_k \frac{1}{i}\log S(\theta_j-\theta_k)=(2n+1)\pi$$



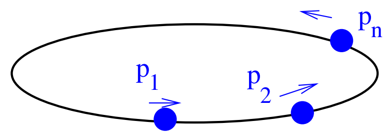
Ground-state energy from

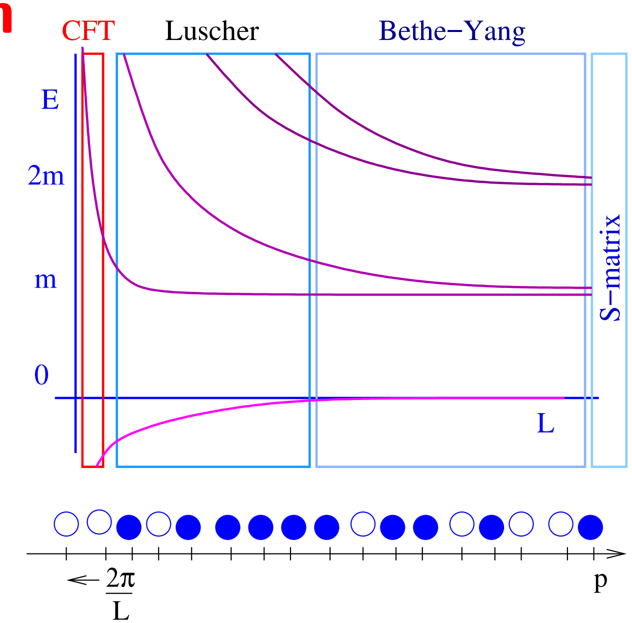
Euclidean partition function:

$$Z(L,R)=_{R\to\infty}\mathrm{Tr}(e^{-H(L)R})=e^{-E_0(L)R}+\ldots$$

Finite size effects: Spectral problem

Finite volume spectrum





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Ground-state energy from

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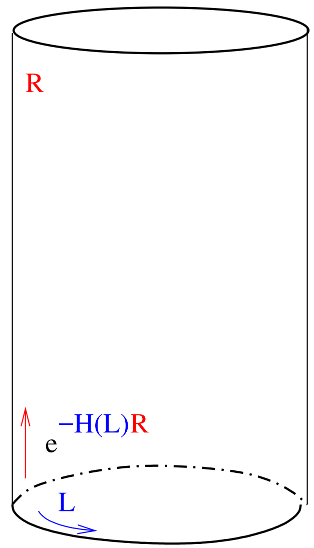
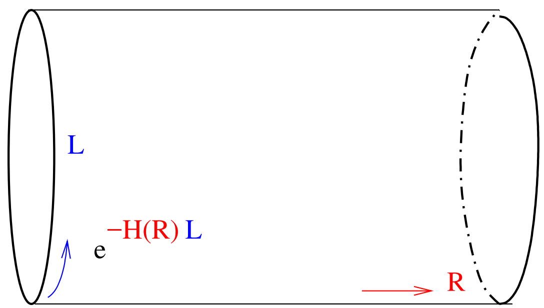
$Z(L,R) =_{R\to\infty} \mathrm{Tr}(e^{-H(L)R}) = e^{-E_0(L)R} + \dots$

Exchange space and Euclidean time

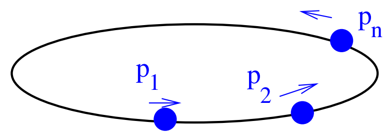
$Z(L,R) =_{R\to\infty} \mathrm{Tr}(e^{-H(R)L}) =_{R\to\infty} \sum_n e^{-E_n(L)R}$

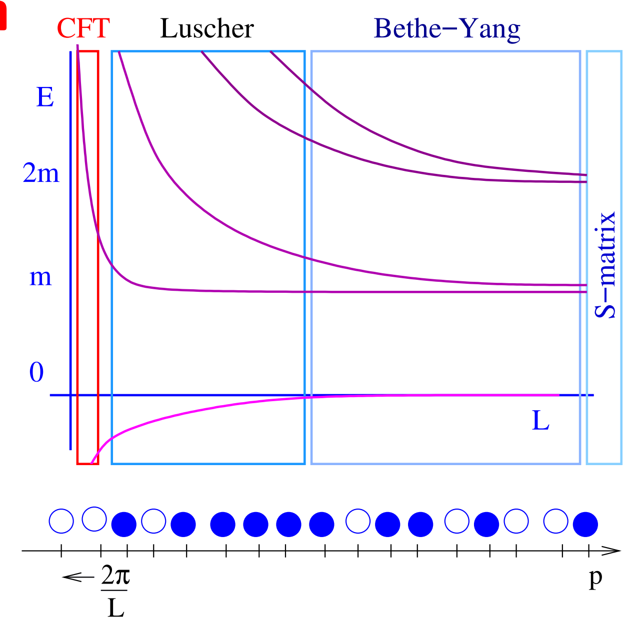




Finite size effects: Spectral problem

Finite volume spectrum





Polynomial volume corrections:

$E(\theta_1, \dots, \theta_n) = \sum_i E(\theta_i)$

Bethe-Yang; p_i quantized: $e^{ip_jL} \prod_k S(\theta_j - \theta_k) = -1$

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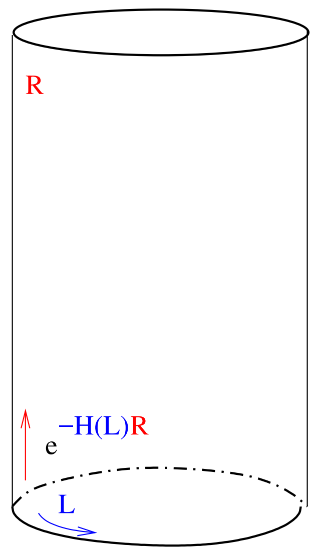
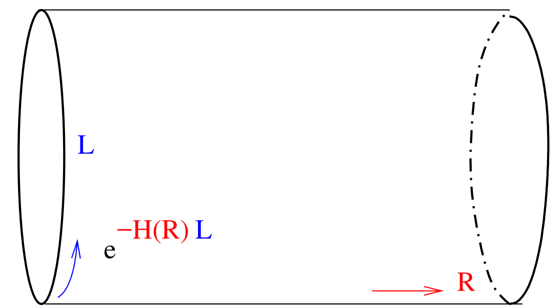


Ground-state energy from

Euclidean partition function:

$Z(L,R) =_{R\to\infty} \mathrm{Tr}(e^{-H(L)R}) = e^{-E_0(L)R} + \dots$



Exchange space and Euclidean time

$Z(L,R) =_{R\to\infty} \mathrm{Tr}(e^{-H(R)L}) =_{R\to\infty} \sum_n e^{-E_n(L)R}$

Main contribution:
finite density ρ, ρ_h

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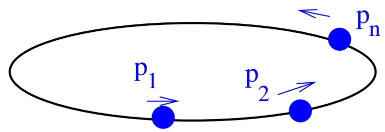

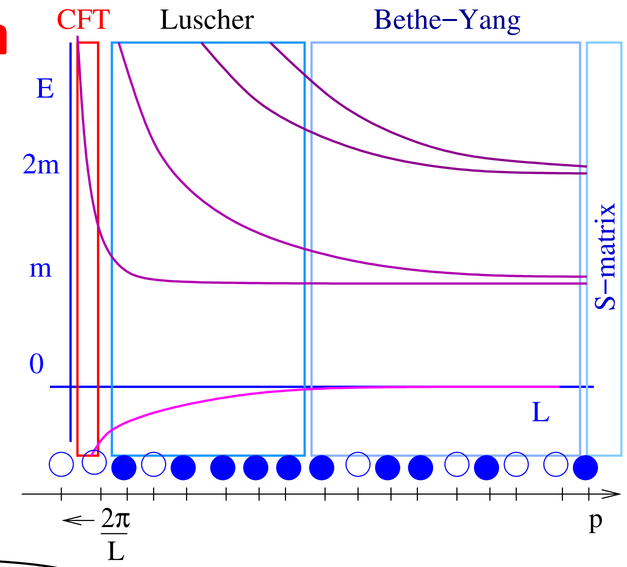

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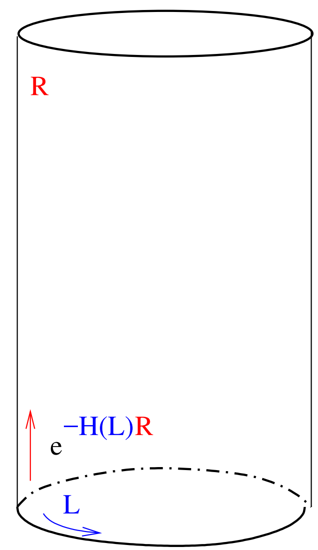

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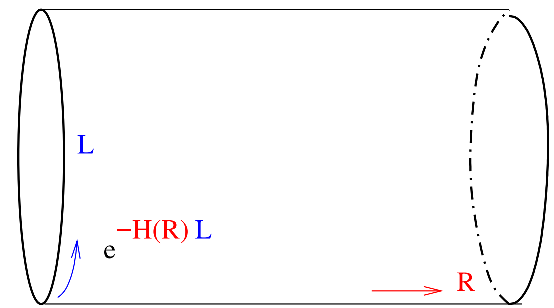

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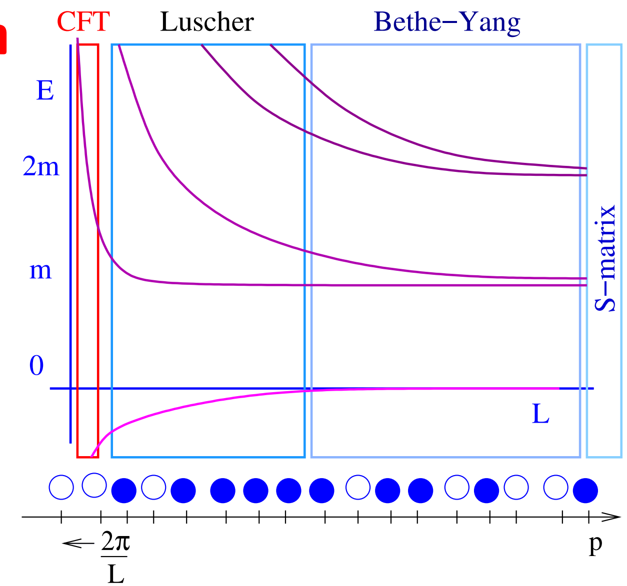
Main contribution: finite density ρ, ρ_h

Large volume: Bethe-Yang can be used

$p_j R + \sum_k \frac{1}{i} \log S(\theta_j - \theta_k) = (2n+1)\pi \quad \longrightarrow R + \int(-i d_p \log S(p,p'))\rho(p')dp' = 2\pi(\rho + \rho_h)$

$Z(L,R) = \int d[\rho, \rho_h] e^{-LE(R) - \int((\rho+\rho_h)\ln(\rho+\rho_h) - \rho\ln\rho - \rho_h \ln\rho_h)dp}$

Finite size effects: Spectral problem



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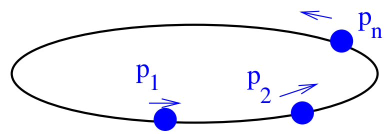

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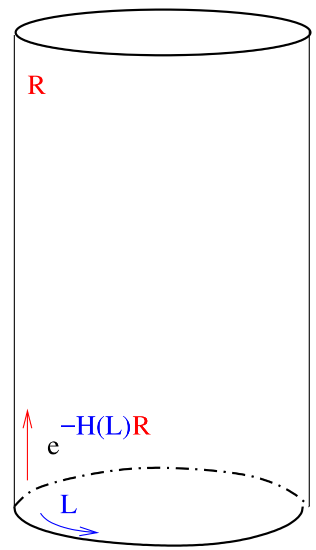
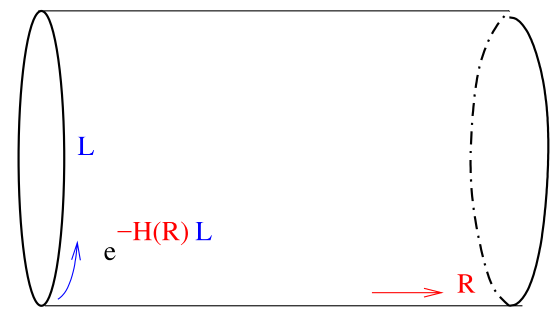




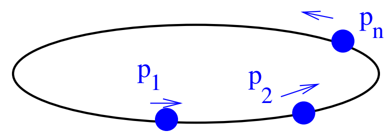
Main contribution: finite density ρ,ρ_h

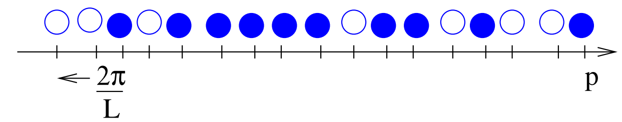
Saddle point : $\epsilon(p)=\ln\frac{\rho_h(p)}{\rho(p)}$ $\boxed{\epsilon(p)=\ E(p)L+\int\frac{dp}{2\pi}id_p\log S(p',p)\log(1+e^{-\epsilon(p')})}$

Exact ground state energy: $\boxed{E_0(L)=-\int\frac{dp}{2\pi}\log(1+e^{-\epsilon(p)})}$ [Zamolodchikov]

Finite volume form factors

Finite volume state $|\{\theta_i\}\rangle_L \equiv |\{n_i\}\rangle$





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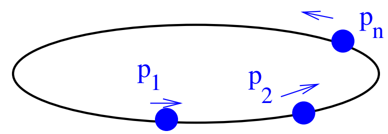

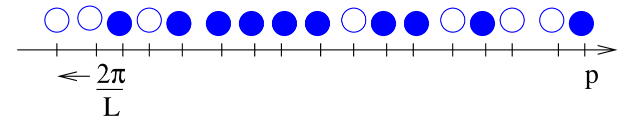

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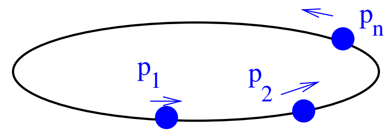

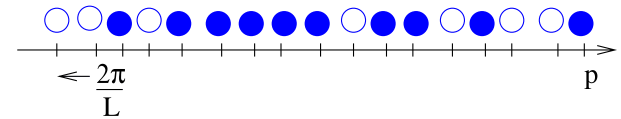

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[proved Pozsgay,Takacs]

crossing $\bar\theta = \theta + i\pi$

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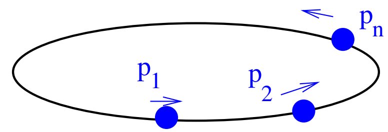

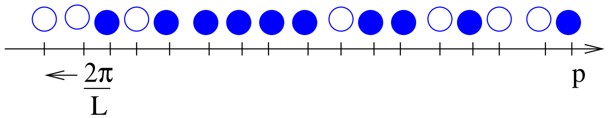

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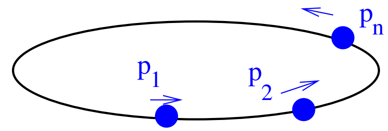

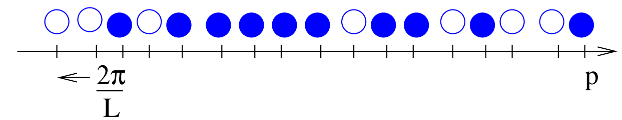

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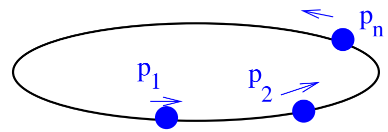

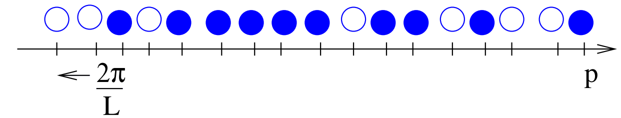

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BY: $Q_1 = p_1L - i\log S(\theta_1-\theta_2) = 2\pi n_1$; $Q_2 = p_2L - i\log S(\theta_2-\theta_1) = 2\pi n_2$

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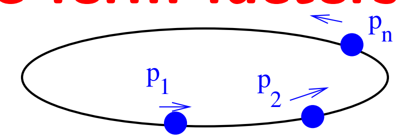

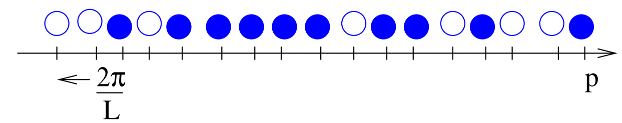

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$$\rho_2(\theta_1,\theta_2) = \begin{vmatrix} E_1L+\phi & -\phi \\ -\phi & E_2L+\phi \end{vmatrix} = E_1E_2L^2 + \phi(E_1+E_2)L \qquad \phi(\theta) = -i\partial_\theta \log S(\theta)$$

and $\rho_1(\theta_1) = E_1L+\phi \quad ; \qquad \rho_1(\theta_2) = E_2L+\phi$

Connected form factors

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$\boxed{\text{Connected form factor: } F^c(\theta_1,\dots,\theta_k)=n!a_{12\dots n}}$ ϵ-independent part

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[Pozsgay,Takacs]

graphs: oriented, tree-like, at each vertex only at most one outgoing edge

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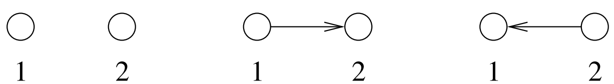

which gives $F_4(\bar{\theta}_1+\epsilon_1,\bar{\theta}_2+\epsilon_2,\theta_2,\theta_1)=F_4^c(\theta_1,\theta_2)+\frac{\epsilon_1}{\epsilon_2}\phi_{12}F_2^c(\theta_1)+\frac{\epsilon_2}{\epsilon_1}\phi_{21}F_2^c(\theta_2)$

The string vertex for $L_1 = 0$: diagonal form factor

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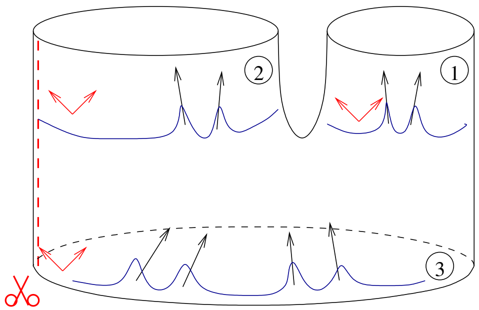
Decompactify string 2 & 3 but $L_1 = 0$:



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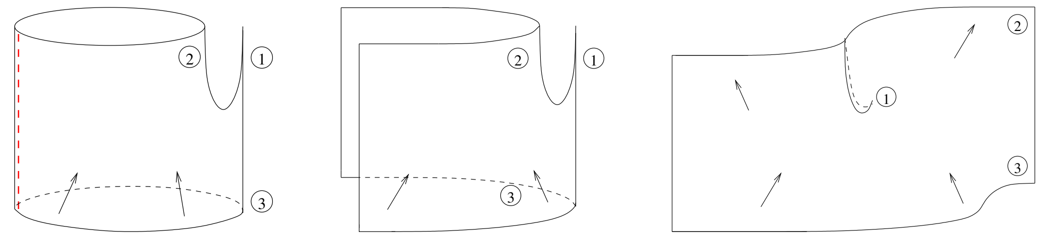
Decompactify string 2 & 3 and $L_1 = 0$:



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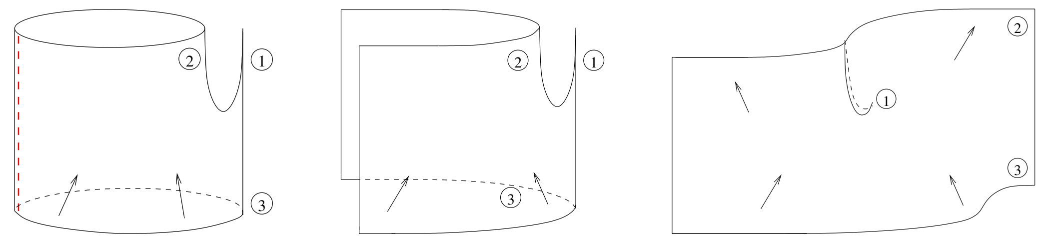

Local operator form factor equations:

$$N_0(\theta_1, \dots, \theta_n\rangle = N_0(\theta_2, \dots, \theta_n, \theta_1 - 2i\pi\rangle = S(\theta_i - \theta_{i+1})N_0(\dots, \theta_{i+1}, \theta_i, \dots)$$

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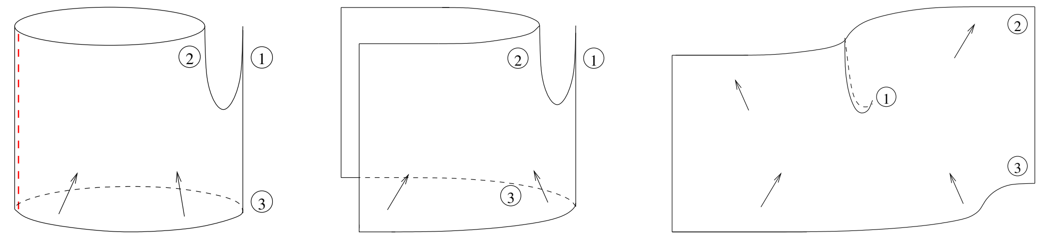

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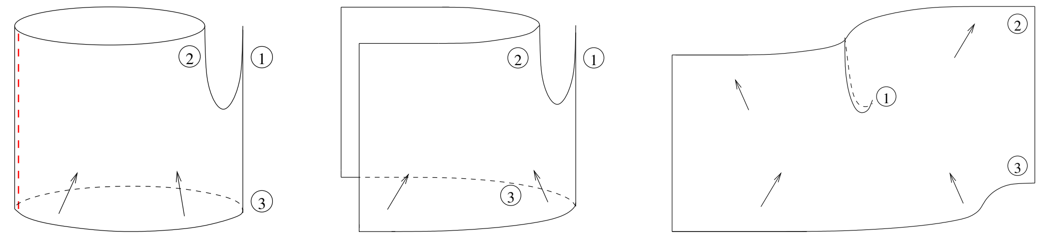

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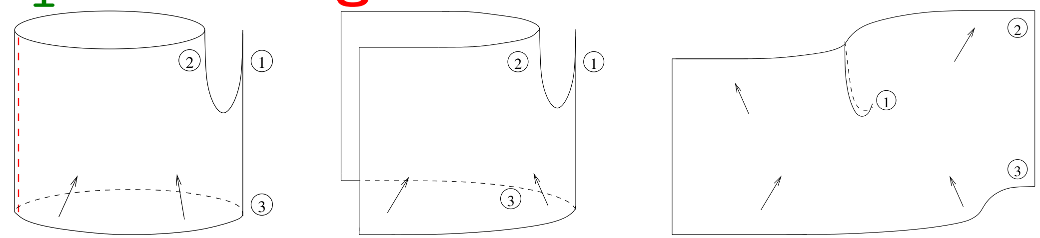

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(classical) diagonal form factors: ${}_L\langle\theta_2, \theta_1|\mathcal{V}|\theta_1, \theta_2\rangle_L = \frac{F_2^s(\theta_1,\theta_2) + \rho_1(\theta_1) F_1^s(\theta_2) + \rho_1(\theta_2) F_1^s(\theta_1)}{\rho_2(\theta_1,\theta_2)}$

$$e^{i\Phi_k} = 1 \quad ; \quad \Phi_k = p_k L - i\sum_{j:j\neq k} \log S(\theta_k, \theta_j) \quad ; \qquad \rho_n(\theta_1, ..., \theta_n) = \det\left[\frac{\partial \Phi_j}{\partial \theta_i}\right]$$

diagonal form factor $F_2^s(\theta_1, \theta_2) = \lim_{\epsilon\to 0} N_0(\bar\theta_2, \bar\theta_1, \theta_1 + \epsilon, \theta_2 + \epsilon)$

Explicitly checked at weak coupling [Hollo, Jiang, Petrovskii] and from hexagons asymptotically [Jiang, Petrovskii]

Exact finite volume 1-point function

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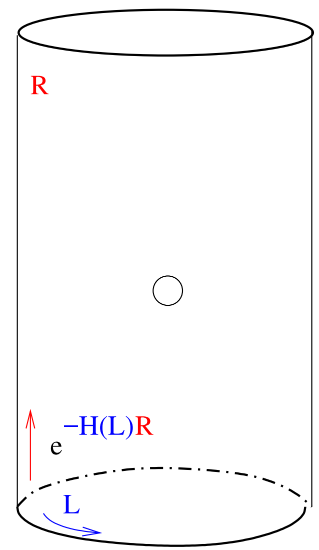
LeClair-Mussardo formula

from thermal evaluation:

$$\langle 0|\mathcal{O}|0\rangle_L =_{R\to\infty} \mathrm{Tr}(\mathcal{O}e^{-H(L)R})/Z(L,R) + \ldots$$



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Exchange space and Euclidean time

$$_{R\to\infty}\mathrm{Tr}(\mathcal{O}e^{-H(L)R})/Z =_{R\to\infty} \mathrm{Tr}(e^{-H(R)L})/Z$$

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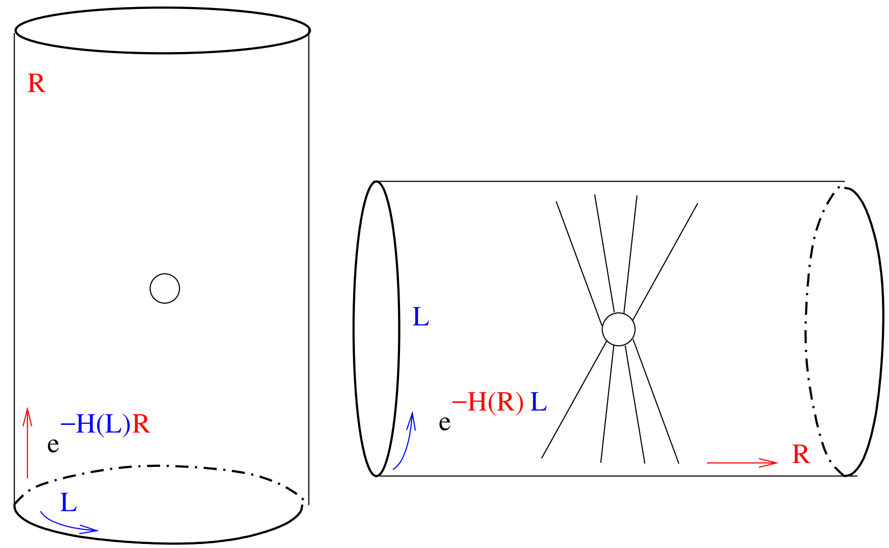

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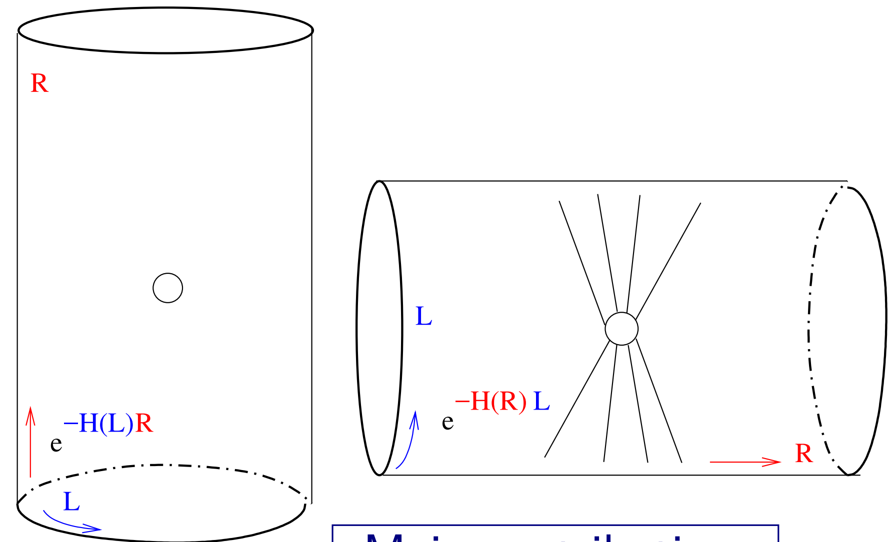

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finite density ρ, ρ_h

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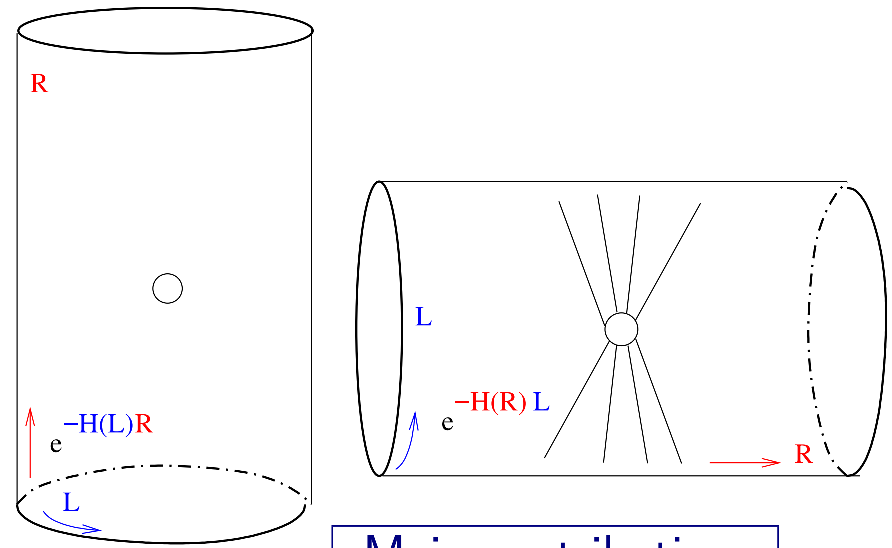

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we need $\langle \rho, \rho_n|\mathcal{O}|\rho, \rho_n\rangle$ in a highly excited Bethe state [Pozsgay]

Large volume: asymptotic formula $\frac{\sum_{\alpha\cup\bar{\alpha}} F^c_\alpha \rho_{\bar{\alpha}}}{\rho_n}$ can be used as

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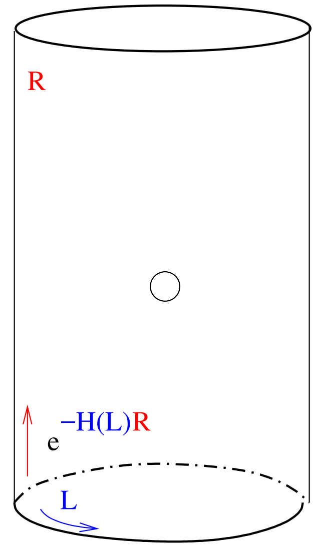

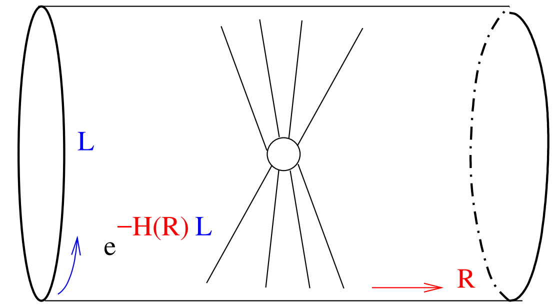

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Saddle point : $\epsilon(p) = \ln\frac{\rho_h(p)}{\rho(p)}$ $\boxed{\epsilon(\theta) = E(\theta)L - \int\frac{d\theta}{2\pi}\phi(\theta-\theta')\log(1+e^{-\epsilon(\theta')})}$

Finite volume expectation value: $\boxed{\langle\mathcal{O}\rangle_L = \sum_n \frac{1}{n!}\prod_{j=1}^n \int\frac{d\theta_i}{2\pi}\frac{e^{-\epsilon(\theta)}}{1+e^{-\epsilon(\theta)}}F^c(\theta_1,\dots,\theta_n)}$

[LeClair-Mussardo] diagonal FF: excited states [Pozsgay]
What about non-diagonal theories/form factors? Sine Gordon theory from lattice [Hegedűs]

Lüscher correction for nondiagonal FFs: the method

Finite volume 2-point function: $\langle \mathcal{O}(x,t)\mathcal{O}\rangle_L = \frac{\int [\mathcal{D}\phi]\mathcal{O}(x,t)\mathcal{O}(0,0)e^{-S[\phi]}}{\int [\mathcal{D}\phi]e^{-S[\phi]}}$

in Fourier space: $\Gamma(\omega,q) = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i(\omega t+qx)}\langle \mathcal{O}(x,t)\mathcal{O}\rangle_L$

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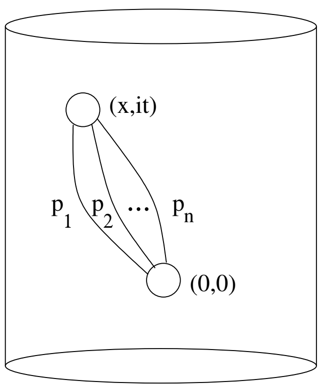

evaluating in the finite volume channel

$$\Gamma(\omega,q) = \sum_N |\langle 0|\mathcal{O}|\theta_1,\dots,\theta_N\rangle_L|^2\left\{\frac{\delta_{q-P_N(L)}}{E_N(L)-i\omega}+\frac{\delta_{q+P_N(L)}}{E_N(L)+i\omega}\right\}$$

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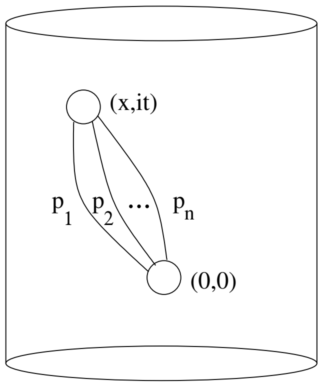

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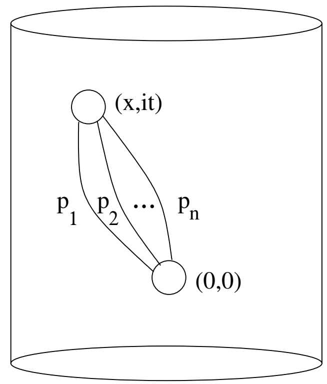

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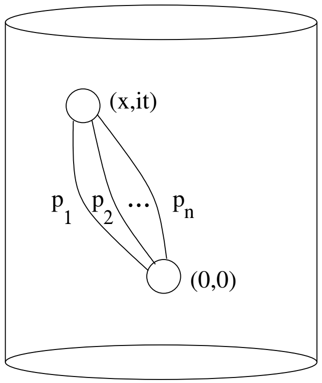

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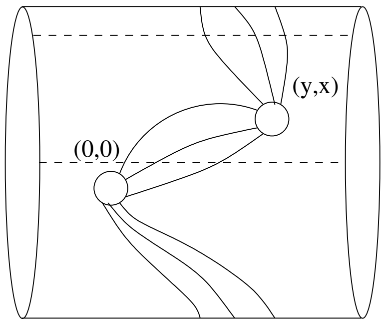

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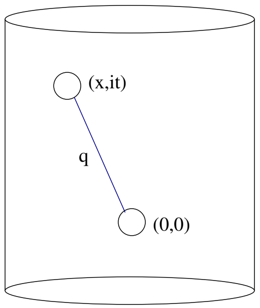
Insert two complete systems of states:

$$Z\Gamma(\omega,q) = \frac{2\pi}{L}\sum_{\mu,\nu}|\langle\nu|\mathcal{O}|\mu\rangle|^2 e^{-E_\nu L}\delta(P_\mu-P_\nu+\omega)\left\{\frac{1}{E_\mu-E_\nu-iq} + \frac{1}{E_\mu-E_\nu+iq}\right\}$$

Use asymptotic expressions. Do analytical continuation as $\omega\to iE_N(L)$

1-particle energy and form factor from 2-pt function

Specify to a 1-particle pole $\quad \Gamma(\omega, q) = \frac{\mathcal{F}(q)^2}{E(q)+i\omega} + \ldots$



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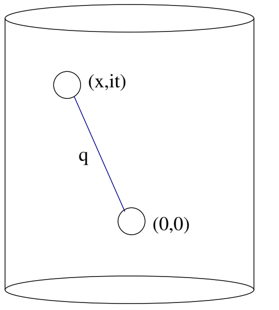

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Lüscher correction: Expand around the BY pole $\mathcal{E}(q) = \sqrt{q^2+m^2}$

$$\Gamma(\omega, q) = \frac{2\pi F_1^2(q)}{L\mathcal{E}(q)} \frac{-i}{\omega - i\mathcal{E}(q)} + \frac{\mathcal{L}_0(q)}{(\omega - i\mathcal{E}(q))^2} + \frac{\mathcal{L}_1(q)}{\omega - i\mathcal{E}(q)} + \text{regular}$$

Energy correction: $E(q) = \mathcal{E}(q)\left\{1 + \frac{L}{2\pi F_1^2}\mathcal{L}_0(q) + \ldots\right\}$

FF correction $\mathcal{F}(q) = \frac{\sqrt{2\pi}F_1}{\sqrt{L\mathcal{E}(q)}}\left\{1 + \frac{iL\mathcal{E}(q)}{4\pi F_1^2}\mathcal{L}_1(q) + \ldots\right\}$

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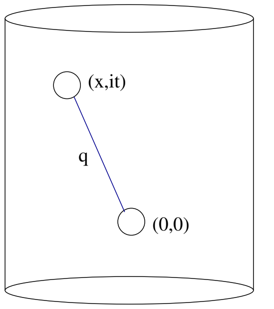

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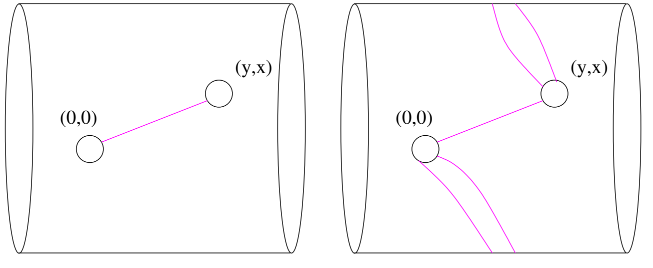
Leading Lüscher correction from ν 1-particle state,
relevant pole: μ vacuum or 2-particle state
similar to: [Pozsgay,Szecsenyi]



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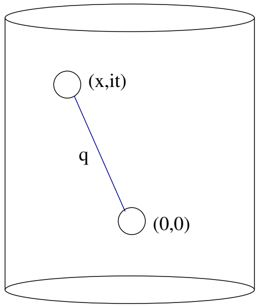

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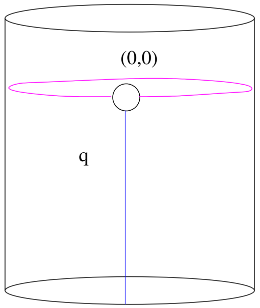
FF correction $\mathcal{F}(q) = \frac{\sqrt{2\pi}F_1}{\sqrt{L\mathcal{E}(q)}}\left\{1 + \frac{iL\mathcal{E}(q)}{4\pi F_1^2}\mathcal{L}_1(q) + \ldots\right\}$



The main result: energy correction reproduced, form factor

$$\boxed{\mathcal{F}(q) = \frac{\sqrt{2\pi}}{\sqrt{\rho_1^{(1)}}}\left\{F_1 + \int_{-\infty}^{\infty} d\theta\, F_3^{\text{reg}}(\theta + i\pi, \theta, \theta_1 - i\tfrac{\pi}{2}) e^{-mL\cosh\theta} + \ldots\right\}}$$

$$F_3^{\text{reg}}(\theta, \theta_1, \theta_2) = F_3(\theta, \theta_1, \theta_2) - \frac{iF_1}{\theta - \theta_1 - i\pi}\left[1 - S(\theta_1 - \theta_2)\right] + i\frac{F_1}{2} S'(\theta_1 - \theta_2)$$

density of states at Lüscher order: $\rho_1^{(1)}$ from Lüscher quantization

Conclusion

The more cutting the simpler the equations are but the more gluing

We need to solve the $AdS_5 \times S^5$ form factor equations, or the octagon equations as they would sum up virtual particle contributions (achieved in the pp-wave only)

We need to understand the gluing (relevant also for 4pt functions)

They are related to finite size corrections to form factors

We recently calculated the μ terms for generic non-diagonal form factors

Calculate the F-terms for generic non-diagonal form factors

Develop a LM type expansion for non-diagonal form factors (XXZ and sine-Gordon correspondence might help [Smirnov])

Other decompactification: the bootstrap program

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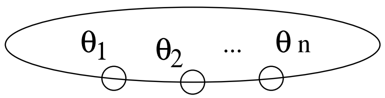
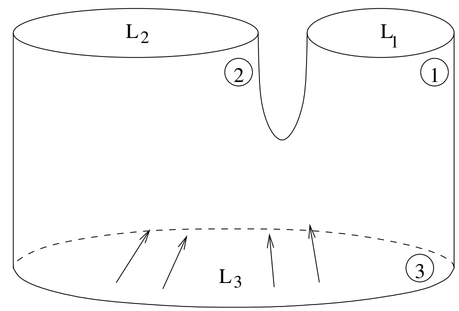
The full large volume amplitude $O(e^{-mL})$

$$e^{ip_1 L} S(\theta_1 - \theta_2) \ldots S(\theta_1 - \theta_n) = 1$$





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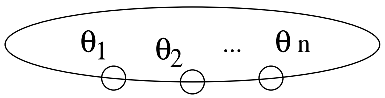
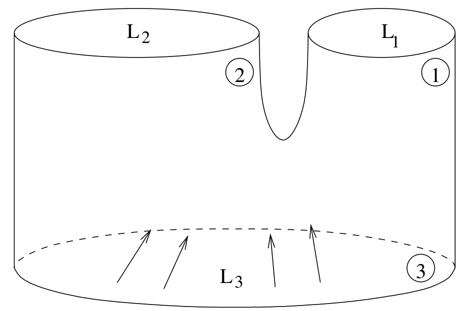
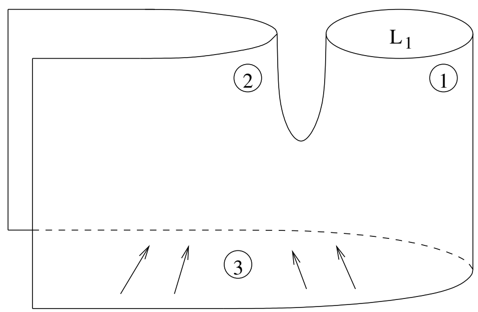




Decompactify
string 2 & 3:
$N_{L_1}(\theta_1, \dots, \theta_n)$



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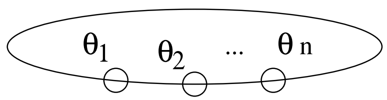

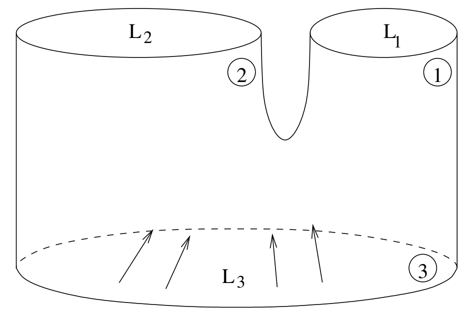

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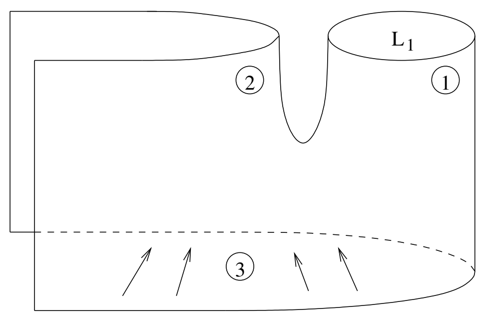
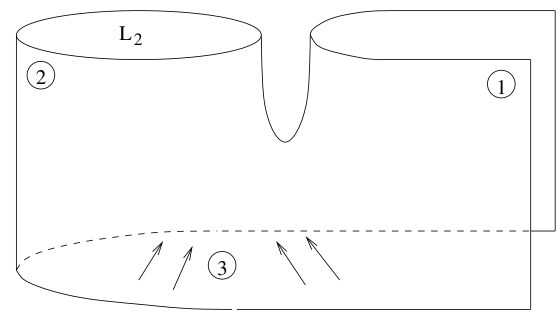


Decompactify

string 1 & 3:

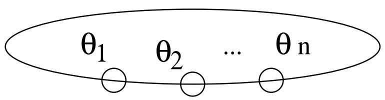
$N_{L_2}(\theta_1, \dots, \theta_n)$

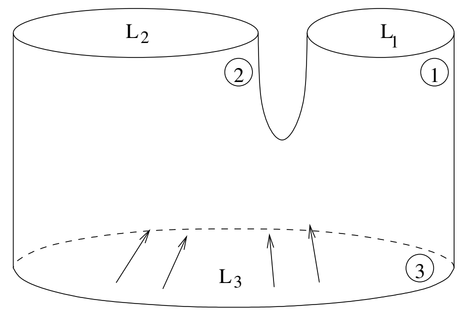


Other decompactification: the bootstrap program

The full large volume amplitude $O(e^{-mL})$

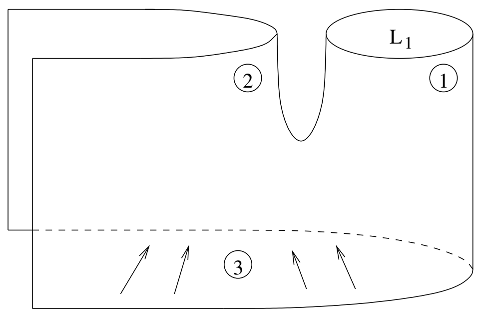
$$e^{ip_1 L} S(\theta_1-\theta_2)\dots S(\theta_1-\theta_n)=1$$





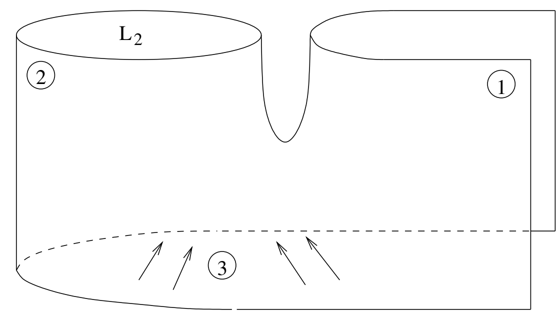
Decompactify string 2 & 3:

$N_{L_1}(\theta_1,\dots,\theta_n)$



Decompactify string 1 & 3:

$N_{L_2}(\theta_1,\dots,\theta_n)$



Finite (large volume) and infinite volume amplitudes are the same (upto normalization).

Find the relevant solutions by matching the two in the large L_1, L_2 limit:

$$N_{L_1}(\theta_1,\dots,\theta_n)\propto N_{L_1,L_2}(\theta_1,\dots,\theta_{n)})\propto N_{L_2}(\theta_1,\dots,\theta_n)$$

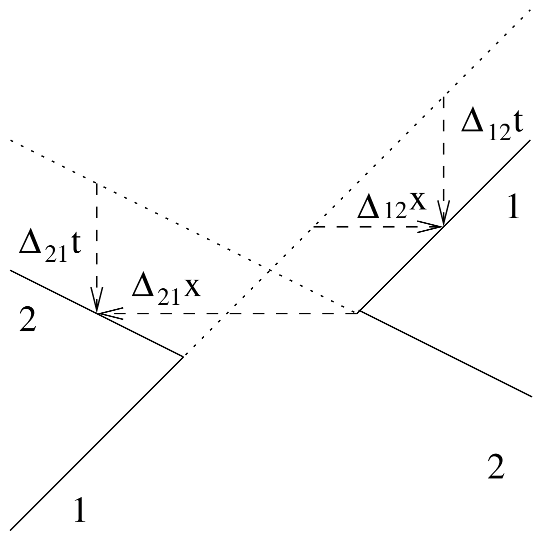
Classical two particle solution

sine Gordon:

$$\tan\frac{\beta\varphi_{12}}{4} = e_{12} = \frac{e_1+e_2}{1-u_{12}^2 e_1 e_2} \quad ; \qquad u_{12} = \tanh\frac{\theta_1-\theta_2}{2} \quad ; \quad e_i = e^{E_i x - p_i t + y_i}$$

Before scattering: $x_1(t) = v_1 t + x_1^- = v_1(t - t_1^-) \quad ; \quad x_2(t) = v_2 t + x_2^- = v_2(t - t_2^-)$

after scattering: $x_1(t) = v_1 t + x_1^+ = v_1(t - t_1^+) \quad ; \quad x_2(t) = v_2 t + x_2^+ = v_2(t - t_2^+)$



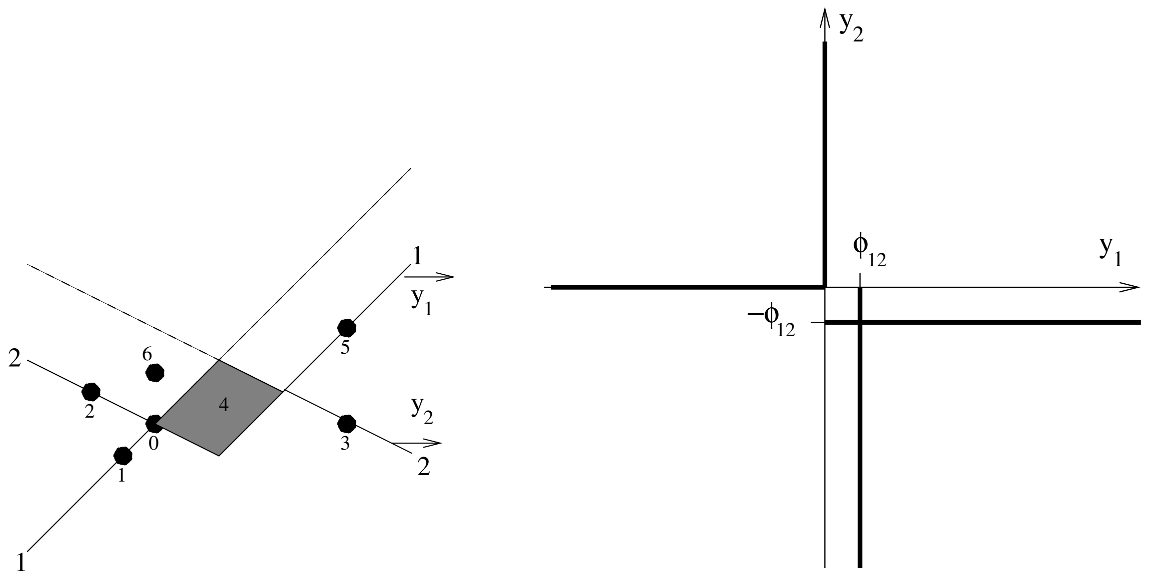
$$\partial_{E_1}\delta(E_1,E_2) = \Delta_{12}t \qquad ; \qquad S = e^{i\delta(\theta_1-\theta_2)}$$

$$\phi(\theta_1-\theta_2) = \partial_{\theta_1}\delta(\theta_1-\theta_2) = \frac{\partial E_1}{\partial\theta_1}\Delta_{12}t = p_1\Delta_{12}t = \phi_{12} = \log\tanh^2\left(\frac{\theta_1-\theta_2}{2}\right)$$

Infinite volume 2 particle form factor

The moduli space

$$y_1 = E_1 x_1^- = -p_1 t_1^- \quad ; \quad y_2 = E_2 x_2^- = -p_2 t_2^- \quad ; \quad \Delta_{12} y = \phi_{12} = -\Delta_{21} y$$

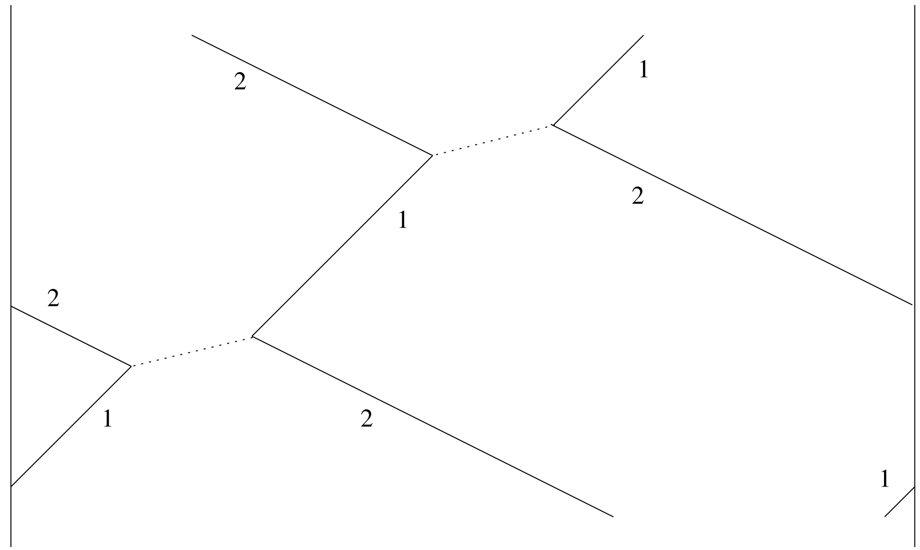


The connected 2pt form factor is defined as

$$F_{2,c}^{\mathcal{O}} = \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \Big[\mathcal{O}[\varphi_{12}(y_1, y_2)] - \Theta(-y_1)\mathcal{O}[\varphi_{12}(-\infty, y_2)] \tag{1}$$
$$-\Theta(y_1)\mathcal{O}[\varphi_{12}(\infty, y_2)] - \Theta(-y_2)\mathcal{O}[\varphi_{12}(y_1, -\infty)] - \Theta(y_2)\mathcal{O}[\varphi_{12}(y_1, \infty)]$$

Finite volume two particle state

approximate two particle finite volume states (up to exponentially small corrections)

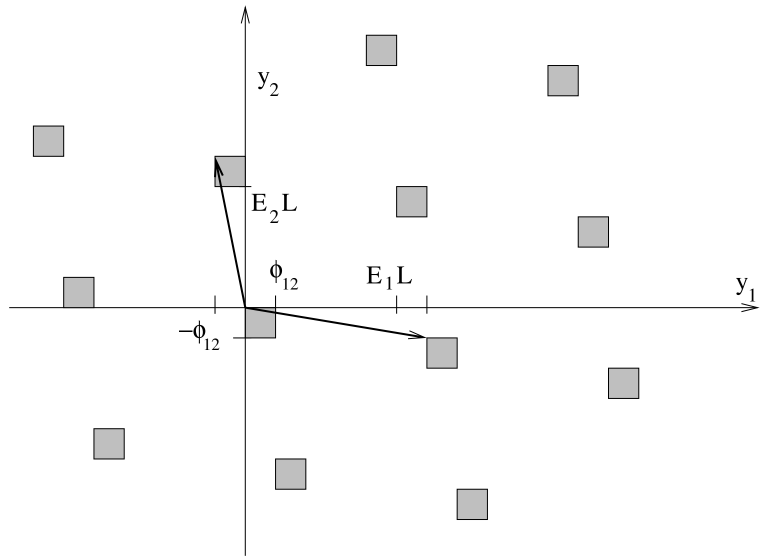


Classical 'Bethe Ansatz'

$$(y_1, y_2) \to (y_1 + Y_1, y_2 - \phi_{21}) \quad ; \qquad Y_1 = E_1 L - E_1 \Delta_{12} x = E_1 L + \phi_{12}$$

$$(y_1, y_2) \to (y_1 - \phi_{12}, y_2 + Y_2) \quad ; \qquad Y_2 = E_2 L - E_2 \Delta_{21} x = E_2 L + \phi_{12}$$

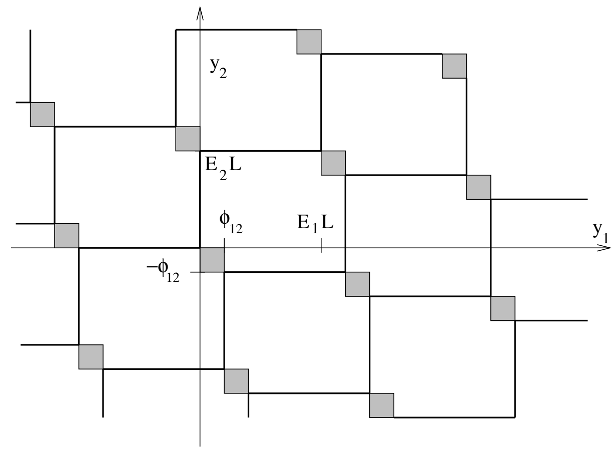
Finite volume moduli and form factor



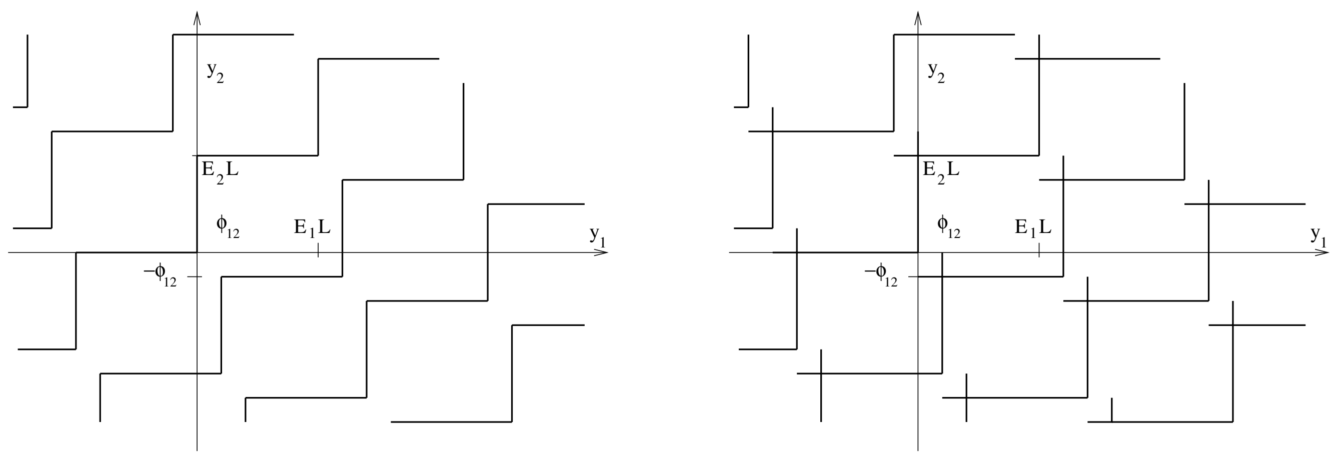
Finite volume diagonal matrix element:

$$F_2(L) = \frac{1}{Vol} \int_{Vol} dy_1 dy_2 \mathcal{O}[\varphi(y_1, y_2)] \quad ; \qquad Vol = \det \begin{bmatrix} Y_1 & -\phi_{21} \\ -\phi_{12} & Y_2 \end{bmatrix}$$

Finite diagonal matrix elements in terms of form factors



elementary cell: $|(E_1L+\phi_{12},-\phi_{12})\times(-\phi_{12},E_2L+\phi_{12})| = \rho_2 = E_1L(E_2L+\phi_{12})$



$$\rho_2F_2(L) = F_{2,s}(\theta_1,\theta_2)+E_1LF_1(\theta_2)+E_2LF_1(\theta_1) = F_{2,c}(\theta_1,\theta_2)+Y_1F_1(\theta_2)+Y_2F_1(\theta_1)$$