Eötvös Loránd University
Faculty of Science
PhD School of Physics
Particle Physics and Astronomy Programme

Numerical investigation of linear perturbations and solutions to constraint equations in general relativity

PhD doctoral thesis

Károly Zoltán Csukás
PhD student
Wigner Research Centre for Physics
Institute for Particle and Nuclear Physics

Supervisor
Dr. István Rácz
scientific adviser, Wigner RCP
doctor of HAS

Head of Doctoral School
Dr. Jenő Gubicza
professor, ELTE
doctor of HAS

Head of Doctoral Program
Dr. Kristóf Petrovay
professor, ELTE
doctor of HAS

2021

1 Introduction

General relativity is the most widely accepted theory of dynamical processes dominated by strong gravitational fields–such that the large scale structure of the universe, gravitational collapse of stars, black holes interacting with their surroundings–which is confirmed by experiments to a high precision. However the Einstein field equations describing the dynamics form a system of coupled strongly nonlinear equations and as such there are no methods to generate all the solutions. Even among the known analytic solutions only a small subset has a clear physical meaning while solutions corresponding to various dynamical processes supposedly occuring in nature are unknown.

Therefore numerical methods are among the most valuable tools on our way to understand processes occuring in vicinity of strong gravitational fields. The importance of numerical investigations is supported by the fact that although gravitational wave GW150914 attributed to merger of black holes was discovered by a templateless algorithm, the later discovered fainter signals most probably would have gone unnoticed without accurate waveforms generated by numerical simulations. Despite their success the methods used are expensive in terms of time and computational power and not appropriate for every usecase hence even at this point it is clear that further improvements are needed if we are to keep up with the improvements of the detectors' sensitivity. For example the understanding of GW170817 is already limited by our lack of understanding on tidal forces occuring in binary neutron star systems.

Even in binary black hole simulations a nonphysical noise appears and contaminates the signal due to the tidal forces. Those parts of the Einstein equation which determine the initial data form an underdetermined system of constraint equations. The generally accepted method to solve them iterprets the constraints as a system of elliptic equations. As a result of this during the specification of initial data we have to apply boundary conditions in the near vicinity of the black holes and also at infinity. While the latter migh guarantee that the asymptotic behaviour of the initial data is in accordance with expectations due to physical considerations, imposing correct boundary conditions in the strong field regime is an open problem.

For this reason investigating more accurate inner boundary conditions or developing methods not relying on boundary conditions in the strong field regime are both contributing greatly to understanding the dynamics of compact binaries.

A good example for interaction between anylitic and numeric studies is the investigation of linear perturbations of the Kerr black hole which represents rotating black holes. Based on the studies of simpler Minkowski and Schwarzschild spacetimes we assume that understanding the behaviour of linear perturbations is an important step towards proving the full stability of Kerr spacetime. There are analytic results regarding the decay of perturbations but the coupling due to the rotation of the black hole makes the problem so complicated that independent numerical studies are needed to confirm the validity of the used assumptions and approximations.

We can describe the superradiant scattering of these perturbations in the same framework. In Kerr spacetime the causal character of the worldline associated with the asymptotically stationary observers changes outside the black hole. As a result of this the energy which is clearly positive definite outside the ergoregion might became negative. When the negative energy falls into the black hole due to the conservation theorems the waves propagating towards infinity might carry more energy than the incoming wavepacket. In frequency domain the equations decouple and it is possible to deduce important result by only using analytic methods. However carrying out the corresponding time domain studies require the use of numerical methods.

In my dissertation I present a numerical approach which is a generalization of a method developed in the Wigner Research Centre for Physics earlier. Then I apply this method to the issues presented in the previous paragraphs: First I present my results regarding the decay rates and scattering of linear electromagnetic and gravitational perturbations propagating over a Kerr black hole background then discuss the results related to nonlinear perturbations of Schwarschild initial data in the framework of gravitational constraints as evolutionary equations.

2 The objective of the research

My work is centered on developing a spectral numerical method based on spin-weighted spherical harmonics. The transformation properties of spin-weighted spherical harmonics makes them especially well suited to represent components of tensorial quantites as linear combinations of spin-weighted spherical harmonics.

The equations governing the propagation of radiative linear perturbations over Kerr spacetimes decouple when expressed in terms of Newman–Penrose spin coefficients. Further simplification is possible since Teukolsky discovered that by use of appropriate rescalings, these equations can be written as a single master equation where the phyisical character of the perturbations is represented by a single parameter, the spin-weight of the fields. The accuracy and robustness of the developed application made possible to investigate the extreme late-time evolution of the perturbations. Special attention has been given to systematic study of non axially symmetric configurations. We note that the first detailed study of non axially symmetric dynamical processes is made possible by the efficiency of this code. In this same framework I have also investigated the superradiant scattering of initial data with compact radial support.

As opposed to the Teukolsky equation the use of spin-weighted variables in context of the constraint equations is not a standard practice. We introduce the spin-weighted variables in the following manner. By foliating the initial data surface by 2-surfaces the tensorial quantities satisfying the constraints might be decomposed into components parallel and perpendicular to these surfaces. Chosing four of these new variables to solve the equations for, we either arrive at system consisting of a parabolic and two hyperbolic partial differential equations or a system of two symmetrizable hyperbolic partial differential equations and an agebraic one.

My numerical method is applicable if we chose the 2-surfaces to be topological 2-spheres and we also define a complex null basis on them. The components of the tensorial quantities expressed on this basis are functions with a given spin-weight. We consider perturbed Schwarzschild initial data such that all the freely specifiable variables are set to the values dictated by a time slice of Schwarzschild solution while we integrate the variables subject to the constraints from different initial values. One of the most interesting question to ask is which non spherically symmetric perturbations result in asymptotically flat initial data similar to the Schwarzschild one. The importance of these deformed black hole initial data lies in their gravitational wave content.

3 The applied methods

The developed numerical method is applicable to handle tensorial equations over manifolds which can be foliated by topological 2-spheres. In such cases the components of tensor fields over an appropriate basis are spin-weighted functions thus we can use the ansatz of series expansion in terms of spin-weighted spherical harmonics. In practice I truncate the series expansion at a sufficiently high mode determined by the expected accuracy.

The advantage of spin-weighted spherical harmonics expansion becomes evident when we express the derivatives tangential to the topological 2-spheres as combination of the operators $\eth$, $\overline{\eth}$, and ∂_ϕ. Accordingly the angular derivatives reduce to algebraic operations so I can evaluate them analytically in an exact manner. Taking advantage of this we can reduce the partial differential equations to a system of coupled equations for the expansion coefficients. In case of the constraints this way we arrive at a system of coupled ordinary differential equations while the Teukolsky equation becomes a system of partial differential equations in 1+1 variables.

A distinguished feature of my method is the fully spectral approach in evaluating the nonlinear expressions in the equations. It is widespread technique to transform between the spectral and coordinate representation of the fields depending on which way it is easier to evaluate the operation. In my approach we do not need such transformations. The product of functions can be simply expressed using Gaunt coefficients. Division by a nowhere vanishing 0 spin-weight function within the desired accuracy can be reduced to a series of multiplications by use of the Neumann series expansion for inverse operators. Therefore every operation can be evaluated on the expansion coefficients and there is no need to evaluate the spin-weighted spherical harmonics on a grid and then expand the results on spin-weighted spherical harmonics.

4 Results

1. During my PhD I have developed a new, fully spectral method based on spin-weighted spherical harmonics. The most important formulae related to this method were presented in a condensed way in the appendix of [1].

   The method has already found two applications which are quite different in their theoretic background. It turned out that this method yields accurate and effective applications.

2. I have thoroughly investigated the long-time evolution and the asymptotic behaviour of electromagnetic and gravitational perturbations over Kerr spacetime. The use of conformal compactification and hyperboloidal initial value problem enabled me to study the decay rate of perturbations directly at the horizon, at future null infinity, and inbetween near future timelike infinity. Among such studies a unique feature is the systematic study of non axially symmetric configurations. I have pointed out that in certain cases current analytic models do not provide the correct results. Note that in order to support my claims of accuracy and correct implementation of the equations I have used a novel conserved current for the first time. The results are published in a PRD paper [1].

3. Using the same theoretical framework I have investigated the superradiant scattering of bosonic perturbations. I have computed the differential operators wich map a solution of TME of a given spin-weight to a solution of opposite spin-weight. Supplementing the conserved current from the previous point with these operators it was able to characterise the superradiant amplification of a single solution. The solutions considered were grouped into three sets depending on the overlapping of the initial data's compact radial support with

the ergoregion. My observations suggest that significant superradiance occurs only when the initial data is considerably overlapping with the ergoregion. This study is published by PRD [3].

4. I have studied the perturbed Schwarzschild solutions of the gravitational constraint equations. When interpreting the constraints as evolution equations it is not evident how to control the asymptotic behaviour of solutions. I have pointed out that the notion of near-Schwarzschild solutions used in the literature is too strict to contain asymptotically flat solutions other than the Schwarzschild one. In any other case the higher modes turn on a coupling which causes the monopole of the trace of the exterior curvature's tensorial projection to decay slower then needed. Following this I have shown that extending the family of near-Schwarzschild solutions the decay rate of the critical mode can be improved. I have proven that either the parabolic-hyperbolic form either the algebraic-hyperbolic form of the constraints possess infinitely many asymptotically flat near-Schwarzschild solutions. The results are published in CQG [2].

List of publications the thesis is based on

[1] K. Z. Csukás, I. Rácz, G. Zs. Tóth: *Numerical investigation of the dynamics of linear spin s fields on Kerr background: Late time tails of spin* $s = \pm 1, \pm 2$ *fields*, Phys. Rev. D 100, 104025, arXiv:1905.09082, (2019)

[2] K. Z. Csukás, I. Rácz: *Numerical investigations of the asymptotics of solutions to the evolutionary form of the constraints*, Class. Quantum Grav. 37, 155006, arXiv:1911.02900, (2020)

[3] K. Z. Csukás, I. Rácz: *Numerical investigation of the dynamics of linear spin s fields on Kerr background II: Superradiant scattering*, Phys. Rev. D 103, 084035, arXiv:2101.05530, (2021)