NUCLEAR INSTRUMENTS AND METHODS 140 (1977) 533–536; © NORTH-HOLLAND PUBLISHING CO.

A SIMPLE METHOD FOR MEASURING THE MOMENTS OF THE GAMMA PARTICLE MULTIPLICITY DISTRIBUTION

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Received 2 August 1976

In this paper we propose an efficient statistical method for estimating the moments of the secondary γ multiplicity distribution in high energy bubble chamber processes. Our method requires relatively small statistics, even if the detection losses are considerable, as is demonstrated by calculating the dispersion of the secondary π^0 multiplicity distribution from 1200 events with a detection probability of about 25% for γ's.

1. Introduction

This work was stimulated by the methodical difficulties due to the low efficiency of the measurements of neutral particles in bubble chambers[1]).

The neutral secondary particles are mostly γ's emerging from the decays of neutral pions. In order to take into account the detection losses, the probability ω of the e^+e^- pair creation in the fiducial volume has to be defined for each secondary γ particle:

$$\omega = 1 - \exp(-L_{\max}/L), \tag{1}$$

where L is the radiation length in the liquid of the chamber, $L_{\max}$ is the distance between the interaction vertex and the boundary of the fiducial volume measured in the direction of the momentum of the γ. In the literature, usually the conversion weight

$$W = 1/\omega \tag{2}$$

is used instead of ω.

In this paper we will investigate, what kind of conclusions can be made concerning the original multiplicity distribution of γ secondaries based on the conversion weights of the detected γ's. There are several works dealing with this problem, e.g. ref. 2, but their methods have not come into general use because of the particular conditions they require. Our method works without any unnatural assumption.

In sect. 2 we consider a simplified model and show the possibility of estimating multiplicity moments even for considerable detection losses. In sect. 3 we present and prove a new method for estimating the binomial moments of the secondary γ multiplicity distribution from the conversion weights. In sect. 4 some results on π° multiplicity distribution are presented. It is the first time that these π° multiplicity moments have been obtained in a model-independent way, merely analysing a sample of data of 1200 bubble chamber π^-p events at 40 GeV/c incident π^- momentum.

2. A simplified treatment: ω = const

In order to find a well estimable set of quantities characterizing the multiplicity distribution, in this section we investigate the simple case when ω is constant.

Let p_n be the probability that a source of a certain kind emits n signals at a given instant. Let our detector system detect a single signal with probability ω, independently of the other $n-1$ signals. Then the measured multiplicity $\bar{n}$ has a distribution $\bar{p}_{\bar{n}}$ which is related to p_n as follows:

$$\bar{p}_{\bar{n}} = \sum_{n \geq \bar{n}} p_n \binom{n}{\bar{n}} \omega^{\bar{n}} (1-\omega)^{n-\bar{n}}. \tag{3}$$

We can measure the distribution of $\bar{n}$, and in this way estimate the $\bar{p}_{\bar{n}}$'s. If in N experiments the multiplicity $\bar{n}$ is observed $N_{\bar{n}}$ times then

$$\bar{p}_{\bar{n}} = \lim_{N\to\infty} N_{\bar{n}}/N. \tag{4}$$

Our aim is to determine the properties of the original distribution p_n knowing the efficiency ω of the detection. By a simple algebraic inversion of eq. (3) we get:

$$p_n = \sum_{\bar{n} \geq n} \bar{p}_{\bar{n}} \binom{\bar{n}}{n} \left(\frac{1}{\omega}\right)^n \left(1-\frac{1}{\omega}\right)^{\bar{n}-n}. \tag{5}$$

Using eq. (4) this can be written in terms of the directly measured quantities:

$$p_n = \lim_{N\to\infty} \frac{1}{N} \sum_{\bar{n} \geq n} N_{\bar{n}} \binom{\bar{n}}{n} \left(\frac{1}{\omega}\right) \left(1-\frac{1}{\omega}\right)^{\bar{n}-n}. \tag{6}$$

For $n = 0$ the above formula gives:

$$p_0 = \lim_{N\to\infty} \frac{1}{N} \sum_{\bar{n}=0}^{\infty} N_{\bar{n}} \left(1 - \frac{1}{\omega}\right)^{\bar{n}}. \tag{7}$$

The formula (6) gives the distribution p_n in the limit $N \to \infty$ but we have to calculate the necessary minimal value of N for a fixed confidence level.

As a well computable example let us consider the Poisson distribution:

$$p_n = e^{-\lambda} \lambda^n / n!. \tag{8}$$

Therefore the distribution of $\bar{n}$ will again be a Poisson one but with an average value decreased by ω:

$$\bar{p}_{\bar{n}} = e^{-\lambda\omega} (\lambda\omega)^{\bar{n}} / \bar{n}!. \tag{9}$$

In this case we can compute the error of the estimation of p_0 to be

$$\begin{aligned}(\Delta p_0)^2 &\approx \frac{1}{N^2} \sum_{\bar{n}=0}^{\infty} N_{\bar{n}} \left(1 - \frac{1}{\omega}\right)^{2\bar{n}} \\ &\approx \frac{1}{N} e^{-\omega\lambda} \sum_{\bar{n}=0} \frac{(\omega\lambda)^{\bar{n}}}{\bar{n}!} \left(1 - \frac{1}{\omega}\right)^{2\bar{n}} \\ &= \frac{1}{N} \exp\left(\frac{\lambda}{\omega} - 2\lambda\right).\end{aligned} \tag{10}$$

If we keep Δp_0 fixed, the required statistics N varies with the efficiency ω as follows:

$$N \approx \left(\frac{p_0}{\Delta p_0}\right)^2 \exp\left(\frac{\lambda}{\omega}\right). \tag{11}$$

It can be seen that eqs. (5)–(7) cannot be used for small values of ω because N depends exponentially on ω^{-1}.

However what could be said about the distribution of n if the statistics N is too small to derive the p_n probabilities?

Let B_k denote the kth binomial moment of the distribution p_n:

$$B_k = \sum_{n \geq k} p_n \binom{n}{k}. \tag{12}$$

Let $\bar{B}_k$ be the same for the distribution $\bar{p}_{\bar{n}}$:

$$\bar{B}_k = \sum_{\bar{n} \geq k} \bar{p}_{\bar{n}} \binom{\bar{n}}{k}. \tag{13}$$

Using eq. (3) and the properties of the binomial coefficients we obtain the following formulae:

$$\bar{B}_k = \omega^k B_k, \tag{14}$$

$$B_k = \frac{1}{\omega^k} \bar{B}_k = \lim_{N\to\infty} \frac{1}{\omega^k N} \sum_{\bar{n} \geq k} N_{\bar{n}} \binom{\bar{n}}{k}. \tag{15}$$

They relate the "measured" and the "true" moments $\bar{B}_k$, B_k in a quite simple way.

Let us discuss the error of the estimation of B_k. From eq. (15) the error of the estimation can be written as:

$$\begin{aligned}(\Delta B_k)^2 &\approx \frac{1}{\omega^{2k} N^2} \sum_{\bar{n} \geq k} N_{\bar{n}} \binom{\bar{n}}{k}^2 \approx \frac{1}{N} \frac{e^{-\omega\lambda}}{\omega^{2k}} \times \\ &\quad \times \sum_{\bar{n} \geq k} \frac{(\omega\lambda)^{\bar{n}}}{\bar{n}!} \binom{\bar{n}}{k}^2.\end{aligned} \tag{16}$$

Therefore we can conclude that the required statistics N has a power-like rise with ω^{-1} if ω goes to 0:

$$N \sim (1/\omega)^k. \tag{17}$$

This fact makes it possible to obtain good estimation for the B_k's (if k is not too large) even if ω is much less than unity and thus the p_n's cannot be estimated.

3. Description of the general method for estimating the binomial moments ($\omega \neq$ const)

Considering N high energy events of a given type, let $\bar{n}^{(\alpha)}$ be the number of the detected secondary γ's in the αth event ($\alpha = 1, 2, \ldots, N$), and $W_r^{(\alpha)}$ ($r = 1, 2, \ldots, \bar{n}^{(\alpha)}$) the conversion weight of the detected γ's, see eqs. (1) and (2).

The kth binomial moment of the true multiplicity distribution of the γ's can be estimated by the following generalization of the formula (15):

$$B_k = \lim_{N\to\infty} \frac{1}{N} \sum_{\alpha=1}^{N} \left(\sum_{\binom{\bar{n}^{(\alpha)}}{k}} W_{i_1}^{(\alpha)} W_{i_2}^{(\alpha)} \ldots W_{i_k}^{(\alpha)} \right), \tag{18}$$

where the symbol $\sum_{\binom{\bar{n}}{k}}$ stands for the summation for all the $\binom{\bar{n}}{k}$ different set of indices i_r ($r = 1, 2, \ldots, \bar{n}$). The meaning of the right-hand side of eq. (18) is the following: for the αth event with the measured values $\bar{n}^{(\alpha)}$, $W_1^{(\alpha)}$, $W_2^{(\alpha)}, \ldots, W_{\bar{n}^{(\alpha)}}^{(\alpha)}$; ($\alpha = 1, 2, \ldots, N$) in which $\bar{n}^{(\alpha)} \geq k$ sum up all the $\binom{\bar{n}^{(\alpha)}}{k}$ different products of the W_r's of number k and average over the N events. Eq. (18) can be written in a more compact form:

$$B_k = \left\langle \sum_{\binom{\bar{n}}{k}} W_{i_1} W_{i_2} \ldots W_{i_k} \right\rangle, \tag{19}$$

where the bracket denotes the sample mean.

Proof: Introduce a distribution function φ of the multiplicity n and weights $W_1, W_2, \ldots, W_n$ of all secondary γ's; these have either been actually detected or not. We choose the following normalization:

$$\int \varphi_n(W_1, \ldots, W_n)\, \mathrm{d}W_1 \ldots \mathrm{d}W_n = n!\, p_n. \tag{20}$$

Note that φ is a symmetric function.

Let us define the distribution function $\bar{\varphi}$ of the multiplicity and weights of the detected γ's and choose the normalization as follows:

$$\int \bar{\varphi}_{\bar{n}}(W_1, \ldots, W_n)\, \mathrm{d}W_1 \ldots \mathrm{d}W_n = \bar{n}!\, \bar{p}_{\bar{n}}. \tag{21}$$

The relation between the two distributions is given by

$$\bar{\varphi}_{\bar{n}}(W_1, \ldots, W_{\bar{n}}) = \sum_{n \geqslant \bar{n}} \frac{1}{(n-\bar{n})!} \int \varphi_n(W_1, \ldots, W_n) \times \times \omega_1 \ldots \omega_{\bar{n}} (1-\omega_{\bar{n}+1})\, \mathrm{d}W_{\bar{n}+1} \ldots \ldots (1-\omega_n)\, \mathrm{d}W_n. \tag{22}$$

Introduce two further distribution functions, the so-called inclusive distributions of order k ([3]):

$$f_k(W_1, \ldots, W_k) = \sum_{n \geqslant k} \frac{1}{(n-k)!} \int \varphi_n(W_1, \ldots, W_n) \times \times \mathrm{d}W_{k+1} \ldots \mathrm{d}W_n, \tag{23}$$

$$\bar{f}_k(W_1, \ldots, W_k) = \sum_{\bar{n} \geqslant k} \frac{1}{(\bar{n}-k)!} \int \bar{\varphi}_{\bar{n}}(W_1, \ldots, W_{\bar{n}}) \times \times \mathrm{d}W_{k+1} \ldots \mathrm{d}W_{\bar{n}}. \tag{24}$$

Using eq. (22) the following relation between f_k and $\bar{f}_k$ can be proved([4]):

$$f_k(W_1, \ldots, W_k) = W_1 W_2 \ldots W_k \cdot \bar{f}_k(W_1, \ldots, W_k). \tag{25}$$

Using the definition (12) of the binomial moment B_k and also eqs. (20), (23) and (25) we get for B_k:

$$B_k = \frac{1}{k!} \int f_k(W_1, \ldots, W_k)\, \mathrm{d}W_1 \ldots \mathrm{d}W_k = \frac{1}{k!} \int W_1 W_2 \ldots W_k \cdot \bar{f}_k(W_1, \ldots, W_k)\, \mathrm{d}W_1 \ldots \mathrm{d}W_k. \tag{26}$$

This latter expression is nothing else but the experimental mean eq. (19).

4. Application for deriving the π° multiplicity moments

The method presented above seems to be adequate to determine the moments of the neutral pion multiplicity distribution in bubble chamber experiments. Starting with the assumption that all γ's are coming from the decays $\pi^\circ \to \gamma\gamma$ the π° multiplicity moments can be related to those of the γ's. For example the dispersion D of the π° multiplicity distribution can be obtained as follows:

$$D = \tfrac{1}{2} B_2 + \tfrac{1}{4} B_1 - \tfrac{1}{4} B_1^2. \tag{27}$$

The method has been tested on the data of 1200 $\pi^- p$ events at 40 GeV/c incident π^- momentum. In this experiment the mean efficiency of the γ detection was 25% and so the average value of conversion weights is about 4 (ref. 1).

In ref. 5 we have derived the dispersion D of the π° multiplicity distribution for all events and for the events of fixed number of charged secondaries (n_{ch}):

n_{ch}	D
2	1.30 ± 0.25,
4	1.50 ± 0.20,
6	1.65 ± 0.25,
8	1.40 ± 0.35,
all events	1.51 ± 0.08.

In spite of the relatively poor statistics the accuracy achieved in calculating the second moments is satisfactory. This indicates that it would be worthwhile to apply our method on larger statistics of events in order to obtain model selective results.

5. Conclusions

An efficient statistical method for estimating the moments of the secondary γ multiplicty distribution in high energy processes has been proposed. Having tested the method on 1200 bubble chamber events it is the first time that the dispersion of the secondary π° multiplicity distribution has been derived from experimental data without any restrictions previously imposed. Using a larger statistics of events, model-selective results could be obtained.

I must express my sincere gratitude for the continuous support of S. Krasznovszky and the bubble chamber group. I would like to thank Drs. B. Lukàcs and A. Somogyi for the valuable remarks.

References

[1]) Budapest–Dubna–Hanoi, Dubna prep. P1-6928 (1973).
[2]) D. Fournier, J. F. Grivaz and J. J. Veillet, Orsay prep. 1269 (Sept. 1973).
[3]) R. P. Feynman, Phys. Rev. Lett. **23** (1969) 1415.
[4]) L. Diósi, KFKI-76-39.
[5]) L. Diósi, Thesis (Eötvös Loránd University Budapest, 1975).