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Comments on the non-Abelian Stokes theorem

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We prove the Stokes theorem for non-Abelian gauge fields using general surface coordinates. Our result contains both of the known versions of the non-Abelian Stokes theorem and allows us to get a new one which is explicitly invariant under rotations of coordinates.

I. INTRODUCTION

The partial differential equations of classical electrodynamics can be reformulated in the form of manifestly gauge-invariant integral rules. The well-known equivalence between differential and integral formulations is partially based on the Stokes theorem of differential geometry. In some recent papers,[1–3] Stokes's theorem has been formulated in non-Abelian gauge theories. Below we shall review the existing and the possible versions of the non-Abelian Stokes theorems (NAST's).

In Sec. II, we emphasize the restricted efficiency of NAST's compared to the Abelian one. Section III contains a proof of the NAST in a special coordinate system. Using this result we shall derive variants of the NAST in Sec. IV. Two of them will be equivalent to the results of Refs. 1 and 2. The third variant is the result of the present work.

II. NON-ABELIAN STOKES THEOREM

Consider a smooth simple connected surface σ and its boundary $\partial\sigma$ which forms a smooth closed contour. The Wilson loop operator[4] for a gauge field A is defined as

$$O_A(\partial\sigma) \equiv \left[P \exp\left(\oint_{\partial\sigma} A_\mu(x)\,dx^\mu\right)\right] , \tag{2.1}$$

where P represents path ordering in the integral which starts from a given point A of the contour $\partial\sigma$ (see Fig. 1).

In a given Abelian vector field A, the Stokes theorem states the following:

$$O(\partial\sigma) = \exp\left(\int_\sigma G_{\mu\nu}(x)\,d\sigma^{\mu\nu}\right) , \tag{2.2}$$

where $G_{\mu\nu}$ stands for the curl of the field A and $d\sigma^{\mu\nu}$ is the infinitesimal surface element. [In the Abelian case $O(\partial\sigma)$ does not depend on the point A.] The content of the Abelian Stokes theorem can be summarized as follows: The contour integral (2.1) can always be expressed in terms of the normal component of the field-strength tensor $G_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ on the surface σ spanned by the contour $\partial\sigma$.

In the non-Abelian case the situation will change. Usually, the non-Abelian field-strength tensor is defined as

$$G_{\mu\nu}(x) \equiv A_{\nu,\mu}(x) - A_{\mu,\nu}(x) - [A_\mu(x), A_\nu(x)] . \tag{2.3}$$

Here $G_{\mu\nu}$, taken on the surface σ, does not determine the loop integral $O_A(\partial\sigma)$. In the general case, we also need some surface component of the gauge field A itself for calculating O_A. The second anomaly of NAST's is that the integral in the right-hand side

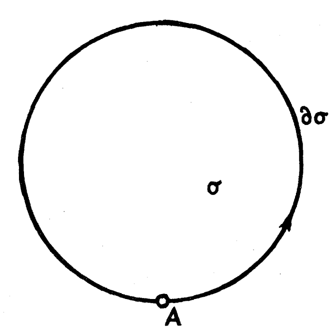

FIG. 1. Scheme of the simple connected smooth surface σ. The Wilson loop operator is defined on its boundary $\partial\sigma$. The loop integral starts from point A and goes in the arrowed direction.

(RHS) of Eq. (2.2) becomes a formal quadrature as it contains double ordering along certain surface coordinate lines.

III. NON-ABELIAN STOKES THEOREM FOR A SECTOR

Consider a smooth simple connected surface σ in the space of Cartesian coordinates x^μ and suppose it can be continuously mapped into a given sector ABC of the unit circle (see Fig. 2). Use polar coordinates r and φ for parametrizing the surface σ:

$$\sigma=\{(r,\varphi);0\leqslant r\leqslant 1;0\leqslant\varphi\leqslant\phi\} \ . \tag{3.1}$$

Here, ϕ is the central angle of the sector. We also require that the mapping $x(r,\varphi)$ be continuously differentiable.

Let us introduce new notation for gauge-field and field-strength variables according to our surface coordinates:

$$\mathfrak{A}_r(r,\varphi)=A_\mu(x)x^\mu{}_{,r} \ , \tag{3.2}$$

$$\mathfrak{A}_\varphi(r,\varphi)=A_\mu(x)x^\mu{}_{,\varphi} \ , \tag{3.3}$$

$$\mathcal{G}(r,\varphi)=G_{\mu\nu}(x)x^\mu{}_{,r}x^\nu{}_{,\varphi}=\mathfrak{A}_{\varphi,r}-\mathfrak{A}_{r,\varphi}-[\mathfrak{A}_r,\mathfrak{A}_\varphi] \ . \tag{3.4}$$

Let us define the Wilson loop operator (2.1) for the boundary $\partial\sigma$ of σ, with the starting point of this loop integral being at the vertex A of the sector ABC:

$$O_A(\partial\sigma)\equiv\left[P_r\exp\left(\int_0^1\mathfrak{A}_r(r,\phi)\,dr\right)\right]^{-1}\times\left[P_\varphi\exp\left(\int_0^\phi\mathfrak{A}_\varphi(1,\varphi)\,d\varphi\right)\right]\times\left[P_r\exp\left(\int_0^1\mathfrak{A}_r(r,0)\,dr\right)\right] \ . \tag{3.5}$$

Here, symbols P_r and P_φ stand for ordering to the left with increasing r and φ, respectively.

In our formulation the NAST is expressed as

$$O_A(\partial\sigma)=\left\{P_\varphi\exp\left[\int_0^\phi\left(\int_0^1\tilde{\mathcal{G}}(r,\varphi)\,dr\right)d\varphi\right]\right\} \tag{3.6a}$$

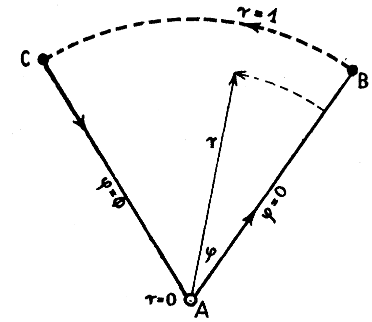

FIG. 2. Sector ABC and its parametrization by (r,φ) polar coordinates.

or, in more compact notation,

$$O_A(\partial\sigma)=\left[P_\varphi\exp\left(\int_\sigma\tilde{\mathcal{G}}\,dr\,d\varphi\right)\right] \ , \tag{3.6b}$$

where

$$\tilde{\mathcal{G}}(r,\varphi)=\left[P_r\exp\left(\int_0^r\mathfrak{A}_r(r',\varphi)\,dr'\right)\right]^{-1}\mathcal{G}(r,\varphi)\times\left[P_r\exp\left(\int_0^r\mathfrak{A}_r(r',\varphi)\,dr'\right)\right] \tag{3.7}$$

is the normal component $\mathcal{G}$ of the curl tensor G having been parallel transferred along a $\varphi=$const string to the point A.

Proof: Let us consider the RHS of (3.5) and (3.6) as functions of the central angle ϕ, and denote them by $R_5(\phi)$ and $R_6(\phi)$, respectively. Note the obvious fact

$$R_5(0)=R_6(0)=1 \ . \tag{3.8}$$

From (3.6), one gets

$$\frac{dR_6(\phi)}{d\phi}=\int_0^1\tilde{\mathcal{G}}(r,\phi)\,dr\,R_6(\phi) \ . \tag{3.9}$$

Now we calculate the same derivative of R_5. With the introduction of the brief notation

$$U_\varphi=\left[P_r\exp\left(\int_0^1\mathfrak{A}_r(r,\varphi)\,dr\right)\right] \ , \tag{3.10}$$

the derivative can be written in the following form:

$$\frac{dR_5(\phi)}{d\phi}=\frac{d}{d\phi}\left\{U_\phi^{-1}\left[P_\varphi\exp\left(\int_0^\phi\mathfrak{A}_\varphi(1,\varphi)\,d\varphi\right)\right]U_0\right\}=\left[\left(\frac{d}{d\phi}U_\phi^{-1}\right)U_\phi+U_\phi^{-1}A_\varphi(1,\phi)\,U_\phi\right]R_5(\phi) \ . \tag{3.11}$$

Let us perform identity transformations on the first term in square brackets:

$$\left(\frac{d}{d\phi}U_\phi^{-1}\right)U_\phi=-U_\phi^{-1}\left(\frac{d}{d\phi}U_\phi\right)=-U_\phi^{-1}\left(P_r\int_0^1\mathfrak{A}_{r,\varphi}(r,\phi)\,dr\,U_\phi\right)=-\int_0^1U_\phi^{-1}[P_r\mathfrak{A}_{r,\varphi}(r,\phi)\,U_\phi]\,dr$$

$$=-\int_0^1\left[P_r\exp\left(\int_0^r\mathfrak{A}_r(r',\phi)\,dr'\right)\right]^{-1}\mathfrak{A}_{r,\varphi}(r,\phi)\left[P_r\exp\left(\int_0^r\mathfrak{A}_r(r',\phi)\,dr'\right)\right]dr \ . \tag{3.12}$$

Note that the second term in square brackets can be rewritten as

$$U_\phi^{-1}\mathfrak{A}_\varphi(1,\phi)\,U_\phi=F(r)|_{r=1} \ , \tag{3.13}$$

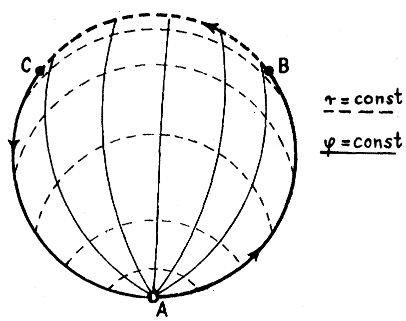

FIG. 3. Scheme of the coordinate choice of σ (see Fig. 1) induced by a continuous mapping of the surface σ onto the sector ABC of Fig. 2.

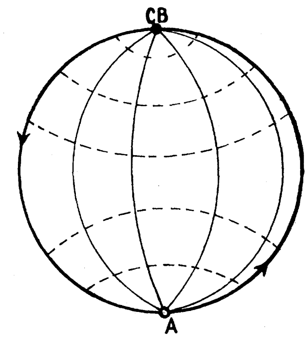

FIG. 4. Scheme of the degenerate case of Fig. 3-type coordinate choice of σ, when points C and B coincide.

where

$$F(r)=\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right]^{-1}\mathcal{A}_\varphi(r,\phi)\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right] . \tag{3.14}$$

Now we take the derivative with respect to r, and subsequently the integral over r, which results in

$$\begin{aligned}U_\phi^{-1}\mathcal{A}_\varphi(1,\phi)\,U_\phi&=\int_0^1\frac{dF(r)}{dr}\,dr\\&=\int_0^1\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right]^{-1}\{\mathcal{A}_{\varphi,r}(r,\phi)-[\mathcal{A}_r(r,\phi),\mathcal{A}_\varphi(r,\phi)]\}\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right]dr .\end{aligned} \tag{3.15}$$

In the first line of the above formula we made use of the fact that $\mathcal{A}_\varphi=0$ at the origin of the polar coordinate system since $x^\mu{}_{,\varphi}$ vanishes at $r=0$.

Substituting (3.12) and (3.15) into (3.11) we derive

$$\frac{dR_5(\phi)}{d\phi}=\int_0^1\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right]^{-1}\mathcal{G}(r,\phi)\left[P_r\exp\left(\int_0^r \mathcal{A}_r(r',\phi)\,dr'\right)\right]dr\,R_5(\phi) , \tag{3.16}$$

where $\mathcal{G}$ is taken from (3.4).

Considering (3.7), (3.9), and (3.16) one can see that both R_5 and R_6 satisfy the same first-order differential equation. Since the initial conditions (3.8) are the same for R_5 and R_6 we can conclude that $R_5(\phi)=R_6(\phi)$ for all possible values of ϕ. Thus statement (3.5) and (3.6) of the NAST is proved.

We can observe an explicit dependence of the RHS of (3.6) on the $\mathcal{A}_r$ component of the gauge field itself. However, any other components of the field A appear only indirectly through the field-strength tensor G. In some special cases, even this dependence on $\mathcal{A}_r$ can be eliminated if, for a proper choice of mapping and/or gauge, $\mathcal{A}_r$ would vanish.[1]

IV. ON SURFACE PARAMETRIZATION

In Ref. 3 an original proof of the NAST is given. The authors arrive at the heuristic analog of formulas (3.6) and (3.7), and they call attention to the broad ambiguity in the choice of string operators appearing in Eq. (3.7). We claim that this ambiguity corresponds to that of surface parametrization the ordered integrals in Eqs. (3.6) and (3.7) are depending on.

The RHS of Eq. (3.6) is, of course, invariant if we continuously change the mapping of the surface σ onto the sector ABC, i.e., change the parametrization of σ. This reparametrization is always possible. However, Eqs. (3.6) and (3.7) are not explicitly invariant for those changes.

In what follows, we shall point out that a special choice of (r,φ) coordinates on σ makes our NAST (3.6) and (3.7) equivalent with the result of Ref. 1

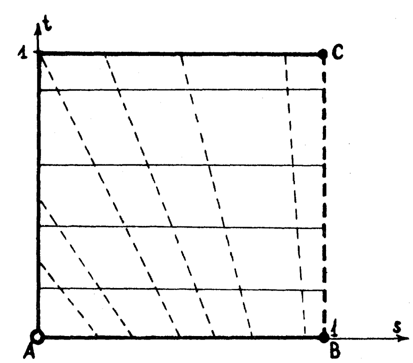

FIG. 5. Scheme of coordinate choice of the unit square induced by its continuous mapping (4.1) onto the sector ABC of Fig. 2. Coordinate system is degenerate along the $s=0$ side of the square.

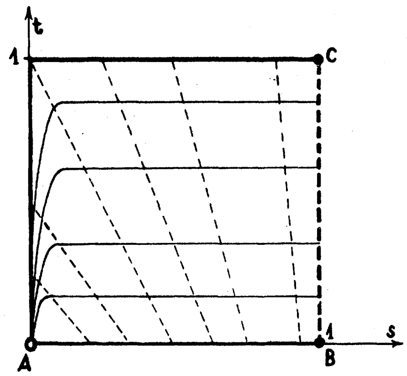

FIG. 6. Pattern of coordinate lines if one resolves the degeneration of the Fig. 5 case.

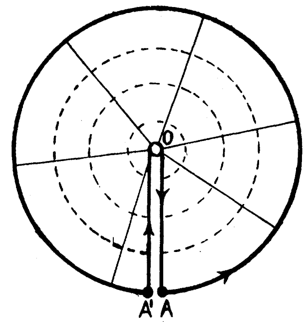

FIG. 7. Parametrization scheme of the surface σ induced by polar coordinates of the 2π angle sector $0AA'$, on which the surface σ is being continuously mapped.

and another coordinate choice gives the result of Ref. 2.

On Fig. 3 we have displayed our simple connected surface σ as an area enclosed by a circle and sketched the parametrization according to the case of Sec. III where σ was mapped on the sector ABC. As the most natural choice for the central angle ϕ of the sector we take π. If one starts to move points B and C closer and closer to each other, we obtain, finally, the parametrization of Fig. 4. The corresponding NAST is completely the same as that of Bralič in Ref. 1.

In Ref. 2 Aref'eva published a complete proof for a version of the NAST. Her paper contains results for σ's which can be mapped on the unit square $\{(s,t);0 \leq s \leq 1;\ 0 \leq t \leq 1\}$ of the (s,t) plane. Aref'eva's result belongs to extreme cases of Fig. 2-type parametrization where the coordinate system is degenerate along the $s=0$ side of the surface boundary $\partial\sigma$. A possible mapping of this kind from the unit square into a sector ABC with central angle $\phi=1$ can be done by the following formulas:

$$r=\begin{cases}\dfrac{t}{t+1}, & \text{if } s=0\ ,\\[2mm] \dfrac{t+s}{t+1}, & \text{if } 0<s\leq 1\ ,\end{cases} \qquad \varphi=s\ . \tag{4.1}$$

The corresponding parametrization is shown in Fig. 5 and a qualitative picture is given in Fig. 6 showing coordinates before degeneration has been entered.

With the use of (4.1), the equivalence of Eqs. (3.5)–(3.7) and the results of Ref. 2 can be verified.

We have to note that in the parametrization of Ref. 2 the surface integral of the NAST has an explicit dependence on the gauge-field components $\mathfrak{a}_t(t,0)$ and $\mathfrak{a}_s(t,s)$, where

$$\mathfrak{a}_s=A_\mu x^\mu{}_{,s},\quad \mathfrak{a}_t=A_\mu x^\mu{}_{,t}\ . \tag{4.2}$$

Finally, we construct a new form of NAST which fits, first of all, to rotation-invariant fields. Let us map the simple connected surface σ on the unit circle area which is parametrized by (r,φ) polar coordinates. Let $\varphi=0$ correspond to the point A on $\partial\sigma$ where the Wilson loop (2.1) starts. Then make a cut on σ at the corresponding $\varphi=0$ line; thus we get a sector $0AA'$ with the central angle 2π, see Fig. 7. For this new surface we can apply formulas (3.6) and (3.7):

$$O_A(\partial\sigma_{\text{with cut}})=\left[P_\varphi \exp\left(\int_\sigma \tilde{\mathfrak{G}}\, dr\, d\varphi\right)\right]\ . \tag{4.3}$$

The original Wilson loop operator (2.1) then will be the following:

$$O_A(\partial\sigma)=U_0^{-1}\left[P_\varphi \exp\left(\int_\sigma \tilde{\mathfrak{G}}\, dr\, d\varphi\right)\right]U_0\ , \tag{4.4}$$

where string operator U_0 is defined by Eq. (3.10). Note that the U's are canceled from the trace of the Wilson loop operator:

$$W(\partial\sigma)\equiv \mathrm{Tr}\,O_A(\partial\sigma)=\mathrm{Tr}\left[P_\varphi \exp\left(\int_\sigma \tilde{\mathfrak{G}}\, dr\, d\varphi\right)\right]\ . \tag{4.5}$$

The resulting formula is explicitly rotation invariant.

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