GRAVITATION AND QUANTUM-MECHANICAL LOCALIZATION OF MACRO-OBJECTS

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We propose a nonlinear Schrödinger equation with a gravitational self-interacting term. The separability conditions of Bialynicki-Birula are satisfied in an asymptotic sense. Soliton-like solutions were found.

There is an old-established common knowledge that when extending quantum-mechanical laws to macroscopic bodies one is confronted, among other things, with the following problem.

According to classical physics, in the absence of external forces the center of mass of a given macro-object either moves uniformly along a straight line or, in the particular case, rests at a certain point. Unfortunately, the Schrödinger equation of a free particle does not have localized stationary solutions. Wave-packet solutions which are possibly the best representation for the free motion of a macroscopic body are not stationary. On the contrary, the wave-packet corresponding to the CM continually widens with time, thus the position of the CM becomes more and more uncertain. At the same time, experience shows that a macroscopic object always has a well-defined position.

A possible way to circumvent this contradiction is to exclude the initial states which develop a measurable spread of the CM of the given macro-object. For instance, if the wavefunction of a given body with a weight of several grams is initially localized in a volume with a linear size of about 10^{-8} cm (or larger), the quantum-mechanical spreading of its CM will be extremely slow. The initial position displays no change even for thousands of years, and the wave-packet of the CM is apparently stationary with very good precision.

People often argue that it is meaningless to suppose that a macroscopic body can have a more accurate localization than a typical atomic size of 10^{-8} cm. Nevertheless, this is merely a bad guess. Let us accept that the position of a single atom is usually in a volume of atomic size. Then we must conclude that the CM position of a group of many atoms will be defined much more accurately than the position of the single atoms.

This reasoning shows that the atomic size of 10^{-8} cm does not give an absolute limitation for localizing macro-objects. Thus it would be conceivable to suppose a macroscopic object of 1 mg with a wave-packet of 10^{-12} cm width. However, this initial width becomes several times larger even in a few minutes. Hence, quantum mechanics would predict non-stationary behavior for the free motion of a macro-object and this anomaly could, in principle, be detected in certain extreme experiments [1].

However, Nature can single out another possible way for solving the problem of wavefunction localization: we cannot exclude the existence of a mechanism which modifies the laws of quantum mechanics for macroscopic objects. A modified Schrödinger equation will then have localized stationary solutions describing the state of macro-objects. Such arguments were put forward in ref. [2], where a nonlinear but local term was added to the Schrödinger equation and soliton-like solutions were found.

In the present work we show that the gravitational interaction possibly could prevent the unbounded quantum-mechanical spreading of the CM position of macro-objects, at least in certain quantum states. If this interaction is included, it destroys the linearity of quantum mechanics [3]. In the nonrelativistic

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case, newtonian gravitation can explicitly be built into the Schrödinger equation. We arrive then at a nonlinear integro-differential equation possessing soliton-like solutions, the ones we need to describe the well-localized macro-objects.

A theory, satisfactorily unifying quantum mechanics and gravitation in every respect, still has not been found. Here we are going to apply the approach of Møller and Rosenfeld [4,5]:

$$R_{ab} - \tfrac{1}{2} g_{ab} R = (8\pi G/c^4)\langle\psi|\hat{T}_{ab}|\psi\rangle, \tag{1}$$

where g is the metric, R_{ab} is the Ricci tensor, G stands for the constant of Newton and c denotes the velocity of light. We put the expectation value of the energy–momentum tensor operator $\hat{T}_{ab}$ in the actual quantum state ψ on the RHS of the Einstein equation.

This equation is certainly not correct if the fluctuation of T_{ab} is too large in the quantum state ψ, e.g. when macroscopically different densities of the energy and momentum are superposed [6]. But, if the given quantum state ψ can definitely be associated with only one macrostate, there can be no a priori objection against eq. (1). Actually, this equation is to be applied as long as we do not quantize gravity.

Henceforth we shall discuss nonrelativistic systems. Let us consider the Schrödinger equation for a system of N particles having masses $m_1, m_2, ..., m_N$:

$$\mathrm{i}\hbar\, \partial\psi(X,t)/\partial t = \left(-\sum_{r=1}^{N} \frac{\hbar^2}{2m_r} \frac{\partial^2}{\partial x_r^2} + \sum_{r,s=1}^{N} V_{rs}(x_r - x_s) + \sum_{r=1}^{N} m_r \phi(x_r,t)\right)\psi(X,t). \tag{2}$$

Here, $X \equiv (x_1, x_2, ..., x_N)$ stands for the space coordinates of the particles, V_{rs} is the interaction potential and ϕ denotes the newtonian gravitational potential given by the nonrelativistic equivalent of the Einstein equation (1):

$$\Delta\phi(x,t) = -4\pi G \int \mathrm{d}^{3N}X' |\psi(X',t)|^2 \sum_{r=1}^{N} \delta^{(3)}(x - x_r). \tag{3}$$

If we solve the Poisson equation (3) explicitly, we can eliminate the potential ϕ from eq. (2). Thus we are led to the following nonlinear integro-differential equation:

$$\mathrm{i}\hbar\, \partial\psi(X,t)/\partial t = \left(-\sum_{r=1}^{N} \frac{\hbar^2}{2m_r} \frac{\partial^2}{\partial x_r^2} + \sum_{r,s=1}^{N} V_{rs}(x_r - x_s) - G \sum_{r,s=1}^{N} \int \frac{m_r m_s}{|x'_s - x_r|} |\psi(X',t)|^2 \mathrm{d}^{3N}X'\right)\psi(X,t). \tag{4}$$

For one free point-like object of mass M, eq. (4) reduces to the following nonlinear Schrödinger equation with a non-local self-interacting term:

$$\mathrm{i}\hbar\, \partial\psi(x,t)/\partial t = -(\hbar^2/2M)\Delta\psi(x,t) - \left(GM^2 \int \frac{|\psi(x',t)|^2}{|x'-x|} \mathrm{d}^3x'\right)\psi(x,t). \tag{5}$$

An important feature of eq. (4) is that it asymptotically satisfies the separability condition of Bialynicki-Birula [2]: Let $\psi^{(A)}(x_A, t)$ and $\psi^{(B)}(x_B, t)$ be solutions to eq. (5) for single particles A and B, respectively. If the spatial separation of A and B is large enough to neglect both the potential V_{AB} and the gravitational interaction between A and B, then the wavefunction $\psi^{(AB)}(x_A, x_B, t) = \psi^{(A)}(x_A,t)\psi^{(B)}(x_B,t)$ is a solution to the two-particle equation (4) with $N = 2$.

Let us remind that in ref. [2] only mathematically local nonlinearities were discussed. Our nonlinear term is nonlocal.

We note that in eqs. (4), (5) the wavefunctions must be normalized to unity. The nonlinear Schrödinger equation (5) preserves this normalization:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |\psi(x,t)|^2 \,\mathrm{d}^3x = 0, \tag{6}$$

and the expectation value of the momentum operator $\hat{p}$ and that of the energy operator $\hat{E}$ are conserved as well:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\hat{p}|\psi\rangle \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int \psi^*(x,t)(-\mathrm{i}\hbar\nabla)\psi(x,t)\,\mathrm{d}^3x = 0, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|\hat{E}|\psi\rangle \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int \psi^*(x,t)\left(-\frac{\hbar^2}{2M}\Delta - \frac{GM^2}{2}\int \frac{|\psi(x',t)|^2}{|x'-x|}\mathrm{d}^3x'\right)\psi(x,t)\,\mathrm{d}^3x = 0. \tag{8}$$

Naturally, eq. (5) is covariant against galilean transformations. It can be shown that if $\psi(x,t)$ solves eq. (5) then the function

$$\psi(x-r-vt,t)\exp[-(\mathrm{i}/\hbar)\tfrac{1}{2}Mv^2t+(\mathrm{i}/\hbar)Mvx] \quad (9)$$

will also be a solution, where r and v are arbitrary constants.

Certain solutions of unit norm can conveniently describe the quantum-mechanical propagation of a given point-like macro-object of mass M. We are going to show that the solution of minimal energy is a soliton-like fixed wave-packet with static spatial density.

Let us consider the normalized function $\varphi(x)$ minimizing the energy functional (8):

$$E=\int\varphi^*(x)\left(-\frac{\hbar^2}{2M}\Delta-\frac{GM^2}{2}\int\frac{|\varphi(x')|^2}{|x'-x|}\,\mathrm{d}^3x'\right)\varphi(x)\,\mathrm{d}^3x$$
$$=\min., \quad (10)$$

$$\int|\varphi(x)|^2\,\mathrm{d}^3x=1. \quad (11)$$

One can easily verify that the phase of φ will not depend on the variable x, thus we can choose $\varphi(x)$ to be a real function. The resulting minimum problem is

$$\frac{\hbar^2}{2M}\int[\nabla\varphi(x)]^2\,\mathrm{d}^3x-\frac{GM^2}{2}\iint\frac{\varphi^2(x')\varphi^2(x)}{|x'-x|}\,\mathrm{d}^3x'\,\mathrm{d}^3x$$
$$-\epsilon\int\varphi^2(x)\,\mathrm{d}^3x=\min., \quad (12)$$

where ϵ is a Lagrange multiplier.

It can be proved that if $\varphi_0(x)$, ϵ_0 satisfy the minimum condition (12) and also the normalization (11) then the wavefunction

$$\varphi_0(x,t)=\varphi_0(x)\exp(-\mathrm{i}\epsilon_0 t) \quad (13)$$

is a solution to the nonlinear Schrödinger equation (5). Indeed, substituting the ansatz (13) into eq. (5) one arrives at the nonlinear time-independent Schrödinger equation for $\varphi_0(x)$. This latter equation is the same as the variational equation corresponding to the minimum problem (12). Thus, the function (13) proves to be the ground-state solution to eq. (5).

Finally, we have to find the function $\varphi_0(x)$. Let us suppose that $\varphi_0(x)$ is a smooth real function of unit norm, which has a peak with a characteristic width a at the origin and tends to zero outside this region. We can qualitatively evaluate the expression (10) of the energy E, which is now depending on the width a:

$$E\approx\hbar^2/Ma^2-GM^2/a. \quad (14)$$

By minimizing this expression we get the characteristic width a_0 of the ground-state wavefunction $\varphi_0(x)$:

$$a_0\approx\hbar^2/GM^3. \quad (15)$$

Hence, this value can be taken as the measure of the quantum-mechanical uncertainty in the position of a free point-like macroscopic object. The expression (13) is the stationary ground-state wavefunction of an object located at the origin. Applying the galilean transformation (9), one can construct the stationary wavefunction corresponding to an arbitrary uniform rectilinear motion of the object.

In addition to these one-soliton solutions, the nonlinear Schrödinger equation (5), unfortunately, possesses other solutions too. These latter are associated with quantum-mechanical states which generally cannot occur in the world of macro-objects. We do not know precisely how to exclude these paradoxical solutions from the theory. The most natural idea is to suppose that a certain physical mechanism destroys such states.

Let us demonstrate a typically unphysical two-soliton solution. The propagation of the given point-like macro-object is described by two wave-packets of width of about a_0. Both of them are normalized to 1/2. The two wave-packets are moving around each other as if they were two objects with mass $M/2$, gravitationally attracting each other.

Formula (15) yields the width of the wave-packet of a free point-like macro-object, i.e., the extension of the object is much less than the spread a_0 of its position. Now we estimate the value of a_0 for a homogeneous spherical object of radius R and mass M, supposing that $a_0 \ll R$. The only change appears in the interaction term in the functionals (10), (12). The simple newtonian kernel $-GM^2|x'-x|^{-1}$ has to be substituted by the effective interaction potential $V(x'-x)$ of two homogeneous spheres with radius R and mass M:

$$V(x'-x)$$

$$= -\frac{GM^2}{(4\pi R^3/3)^2}\int_{r<R} d^3r \int_{r'<R} d^3r' \frac{1}{|x'+r'-x-r|}$$

$$= (GM^2/R)[-\tfrac{6}{5} + \tfrac{1}{2}|(x'-x)/R|^2$$

$$+ O(|(x'-x)/R|^3)]\,. \qquad (16)$$

The characteristic a-dependence of the energy E is the following:

$$E \approx \hbar^2/Ma^2 - GM^2/R + (GM^2/R^3)a^2. \qquad (17)$$

The width $a_0^{(R)}$ of the ground-state wave-packet is given by the minimization of E:

$$a_0^{(R)} \approx (\hbar^2/GM^3)^{1/4} R^{3/4} = a_0^{1/4} R^{3/4}, \qquad (18)$$

where a_0 is the spread of the point-like object, see formula (15).

We consider formulae (15) and (18) as the main result of this work. We claim that these expressions define the natural width of the wave-packet of any macroscopic object. The similar problem of the natural uncertainty in the orientation of an extended macro-object can be discussed in this frame also.

It is interesting to note that, in ref. [1], the same result (15) was obtained from certain principles of the metrical smearing of space–time. For extended objects the relation $a_0^{(R)} \approx a_0^{1/3} R^{2/3}$ was derived, which is not identical with our result (18). However, if a critical size R_c is defined by the condition $a_0^{(R)} = R_c$, then ref. [1] and formula (18) yield the same value, $R_c \approx 10^{-5}$ cm, for objects of normal density. In ref. [2] special considerations are used to estimate the critical size and also a value of about 10^{-5} cm was predicted. Both papers [1,2] and the present one too, adopt the idea that a breakdown of the superposition principle is foreseen in the macroworld and R_c defines the line of demarcation between micro- and macro-objects.

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