ORTHOGONAL JUMPS OF THE WAVEFUNCTION IN WHITE-NOISE POTENTIALS

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The master equation of the quantum noise theory is derived by means of Feynman's path integral method. We propose an equivalent stochastic process where the wavefunction satisfies a nonlinear Schrödinger equation except for random moments at which it shows orthogonal jumps.

Since the early works [1–3] on the theory of quantum noise, great interest has been shown in quantum-mechanical systems affected by random external forces and also new aspects have appeared [4–6]. Usually, the quantum state of such systems is described by a density operator obeying a deterministic differential equation (master equation). If we attribute a state vector (instead of the density operator) to the given system, then the state vector should be considered as a stochastic variable governed by a certain stochastic process.

In the first part of our work we derive the master equation for the density operator by a seemingly new application of the path integral technique. Then, we construct a stochastic process for the wavefunction of the system, while retaining its equivalence with the density operator formalism. In the final part we shortly discuss favoured features of the above stochastic process proposed for the wavefunction.

This work is an anticipation of a more general and lengthy explanation of the stochastic properties of the wavefunction in the presence of random forces.

Now, we are going to introduce the notion of stochastic potential of white-noise type. Here we shall investigate the effect of gaussian white-noise potentials. For brevity, we restrict ourselves to the case of a single point-like particle moving in one dimension.

Let us assume that the potential $V(x, t)$ acting on the particle is a stochastic variable (x, t stand for the coordinate and time). We define the probability distribution of V by the following generator functional $G[h]$:

$$G[h] \equiv \left\langle \exp\left(\mathrm{i}\int V(r,t)h(r,t)\,\mathrm{d}r\,\mathrm{d}t\right)\right\rangle = \exp\left(-\tfrac{1}{2}\int h(r,t)h(r',t)f(r-r')\,\mathrm{d}r\,\mathrm{d}r'\,\mathrm{d}t\right), \tag{1}$$

where h is an arbitrary function. The symbol $\langle\ \rangle$ stands for expectation values evaluated by means of the probability distribution of V. Functional differentiation of $G[h]$ gives rise to the two typical relations of moments of white-noise type:

$$\langle V(r,t)\rangle = 0, \qquad \langle V(r,t)V(r',t')\rangle = \delta(t-t')f(r-r'). \tag{2, 3}$$

For further use, we introduce the following notation:

$$g(r) = f(0) - f(r). \tag{4}$$

Now we investigate the effect of the white-noise potential (1) on the quantum-mechanical motion of a given point-like particle of mass m.

If we single out a given potential V then the wavefunction $\psi_t(x)$ of the particle will satisfy the Schrödinger equation of motion

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$$\partial\psi_t(x)/\partial t = (i\hbar/2m)\partial^2\psi_t(x)/\partial x^2 - (i/\hbar)V(x,t)\psi_t(x), \tag{5}$$

where x is the spatial coordinate and t refers to the actual value of the time. Taking the initial wavefunction $\psi_0(x)$ at $t = 0$, one can express the solution for $\psi_t(x)$ by means of Feynman's path integral formula [7]:

$$\psi_t(x_t) = \int \exp\left(\frac{i}{\hbar}\int_0^t [\tfrac{1}{2}m\dot{x}_\tau^2 - V(x_\tau,\tau)]\,d\tau\right)\psi_0(x_0)\,Dx_\tau, \qquad t > \tau \geqslant 0. \tag{6}$$

In our case, $V(x,t)$ is a stochastic variable, thus $\psi_t(x)$ will evolve in time according to a given stochastic process. We shall not derive the rules of this stochastic process. Instead, we shall construct another stochastic process for $\psi_t(x)$ which leads to the correct statistical predictions and which has genuine features from the viewpoint of measurement theory.

In the generic case, the quantum state of the particle is uniquely characterized by the density matrix [8]

$$\rho_t(x,y) \equiv \langle \psi_t(x)\psi_t^*(y)\rangle. \tag{7}$$

Indeed, it can be shown [9] that ρ_t yields all the usual statistical predictions of quantum mechanics. Namely,

$$O_t = \int \hat{O}(y,x)\rho_t(x,y)\,dx\,dy, \tag{8}$$

where $\hat{O}$ is the hermitian operator of an arbitrarily given dynamical quantity and O_t stands for its predicted value at time t.

First, we shall prove that the density matrix (7) satisfies a parabolic differential equation of motion. Using eq. (6) along with eq. (7), one gets

$$\rho_t(x_t,y_t) = \left\langle \int \exp\left(\frac{i}{\hbar}\int_0^t \{\tfrac{1}{2}m(\dot{x}_\tau^2 - \dot{y}_\tau^2) - [V(x_\tau,\tau) - V(y_\tau,\tau)]\}\,d\tau\right)\rho_0(x_0,y_0)\,Dx_\tau\,Dy_\tau\right\rangle, \qquad t > \tau \geqslant 0, \tag{9}$$

where ρ_0 is the initially given density matrix: $\rho_0(x,y) = \psi_0(x)\psi_0^*(y)$. On the r.h.s. one can substitute

$$\left\langle \exp\left(-\frac{i}{\hbar}\int_0^t [V(x_\tau,\tau) - V(y_\tau,\tau)]\,d\tau\right)\right\rangle = \exp\left(-\frac{1}{\hbar^2}\int_0^t g(x_\tau - y_\tau)\,d\tau\right), \tag{10}$$

which obviously follows from eq. (1) if we insert there $h(r,\tau) = -(1/\hbar)(\delta(r - x_\tau) - \delta(r - y_\tau))$ and use eq. (4). Thus we have a path-integral representation for the density matrix at arbitrary time t:

$$\rho_t(x_t,y_t) = \int \exp\left(\int_0^t [i(m/2\hbar)(\dot{x}_\tau^2 - \dot{y}_\tau^2) - (1/\hbar^2)g(x_\tau - y_\tau)]\,d\tau\right)\rho_0(x_0,y_0)\,Dx_\tau\,Dy_\tau, \qquad t > \tau \geqslant 0. \tag{11}$$

Differentiating both sides of eq. (11) with respect to t, one gets the equation of motion for the density matrix in the gaussian white-noise potential (1):

$$\partial\rho_t(x,y)/\partial t = (i\hbar/2m)(\partial^2/\partial x^2 - \partial^2/\partial y^2)\rho_t(x,y) - (1/\hbar^2)g(x-y)\rho_t(x,y). \tag{12}$$

This equation is sometimes called the master equation of the quantum noise theory. Our path integral method seems to be very effective for deriving the master equation even in more general external noises.

The second term on the r.h.s. of eq. (12) is a typical damping term known from the quantum theory of reservoir effects (cf. coarse grained approximation in ref. [3]). This term cannot be reproduced by any given non-stochastic hamiltonian.

Besides damping, a very peculiar feature of eq. (12) is that it produces mixed quantum states from a pure one in a continuous manner. Exploiting the nature of this permanent quantum state mixing we shall construct the stochastic process for the evolution of the wavefunction itself.

In order to make the calculations as simple as possible we suppose that

$$g(r) = \tfrac{1}{2}A^2r^2 + \text{higher order terms in } r, \quad A = \text{const}, \tag{13}$$

and we shall neglect the "higher order terms" by assuming that the width of the wavefunction will always be small enough.

Thus, eq. (12) takes the form

$$\partial\rho_t(x,y)/\partial t = [(\mathrm{i}\hbar/2m)(\partial^2/\partial x^2 - \partial^2/\partial y^2) - (A^2/2\hbar^2)(x-y)^2]\,\rho_t(x,y). \tag{14}$$

If at time t the particle is in a pure quantum state with the given wavefunction ψ_t then [7]

$$\rho_t(x,y) = \psi_t(x)\psi_t^*(y), \tag{15}$$

and eq. (14) yields

$$\rho_{t+\epsilon}(x,y) = [1 - (\epsilon A^2/2\hbar^2)(x-y)^2]\,[1 + \mathrm{i}(\hbar\epsilon/2m)\partial^2/\partial x^2]\,\psi_t(x)[1 - \mathrm{i}(\hbar\epsilon/2m)\partial^2/\partial y^2]\,\psi_t^*(y), \tag{16}$$

for infinitesimal $\epsilon > 0$. Now, the r.h.s. is not a single product like it was in eq. (15). Nevertheless, it can be orthogonally decomposed into the sum of hermitian diadic terms. The diagonalization of the operator $\rho_{t+\epsilon}(x,y)$ in the lowest order of ϵ yields only two diads due to the approximation made by eq. (13):

$$\rho_{t+\epsilon}(x,y) = (1-\epsilon w)\psi_{t+\epsilon}(x)\psi_{t+\epsilon}^*(y) + \epsilon w\tilde{\psi}_{t+\epsilon}(x)\tilde{\psi}_{t+\epsilon}^*(y), \tag{17}$$

where

$$w = (A^2/\hbar^2)\sigma_\psi^2 \tag{18}$$

is the mixing rate, the dominant wavefunction $\psi_{t+\epsilon}$ is infinitesimally close to ψ_t:

$$\psi_{t+\epsilon}(x) = \{1 - (\epsilon A^2/2\hbar^2)[(x-x_\psi)^2 - \sigma_\psi^2] + \mathrm{i}(\epsilon\hbar/2m)\partial^2/\partial x^2\}\,\psi_t(x), \tag{19}$$

and the contaminating wavefunction $\tilde{\psi}_{t+\epsilon}$ is orthogonal to $\psi_{t+\epsilon}$,

$$\tilde{\psi}_{t+\epsilon}(x) = [(x-x_\psi)/\sigma_\psi + (\epsilon A^2/2\hbar^2)a_\psi^3/\sigma_\psi]\,[1 + \mathrm{i}(\hbar\epsilon/2m)\partial^2/\partial x^2]\,\psi_t(x). \tag{20}$$

We have introduced the following notations:

$$x_\psi = \int x|\psi_t(x)|^2\mathrm{d}x, \quad \sigma_\psi^2 = \int (x-x_\psi)^2|\psi_t(x)|^2\mathrm{d}x, \quad a^3 = \int (x-x_\psi)^3|\psi_t(x)|^2\mathrm{d}x. \tag{21}$$

It is easy to verify that $\psi_{t+\epsilon}$ and $\tilde{\psi}_{t+\epsilon}$ are normalized to unity and orthogonal to each other (in the lowest order of ϵ, of course). After straightforward calculations eqs. (18)–(21) lead to identity of the r.h.s.'s of eqs. (16) and (17).

Now, let us read out the statistical meaning of eq. (17): in an infinitesimally short time ϵ, the quantum state ψ_t of the particle either evolves continuously into the neighbouring state $\psi_{t+\epsilon}$, or, with the infinitesimal probability ϵw, jumps to the state $\tilde{\psi}_{t+\epsilon}$, which is orthogonal to $\psi_{t+\epsilon}$.

Thus, the stochastic process governing the evolution of the wavefunction is as follows. The wavefunction $\psi_t(x)$ of the given particle satisfies the following non-linear equation of motion [‡1]:

$$\partial\psi_t(x)/\partial t = \mathrm{i}(\hbar/2m)\partial^2\psi_t(x)/\partial x^2 - (A^2/2\hbar^2)[(x-x_\psi)^2 - \sigma_\psi^2]\,\psi_t(x),$$

$$x_\psi = \int x|\psi_t(x)|^2\mathrm{d}x, \quad \sigma_\psi^2 = \int (x-x_\psi)^2|\psi_t(x)|^2\mathrm{d}x, \tag{22}$$

‡1 Eq. (22) in itself, possesses soliton-like solutions. For a similar mechanism see, e.g., ref. [10].

apart from discrete orthogonal jumps

$$\psi_{t+0}(x) = [(x - x_\psi)/\sigma_\psi]\,\psi_t(x) \equiv \tilde{\psi}_t(x), \tag{23}$$

which occur at random in time. A jump is performed with probability according to the time-dependent transition rate

$$w = (A^2/\hbar^2)\sigma_\psi^2 . \tag{24}$$

By its construction, this stochastic process leads to the same physical predictions in average as eq. (14) did. Namely, given the initial density matrix $\rho_0(x,y)$, we can decompose it as

$$\rho_0(x,y) = \sum_r p_r \psi_0^{(r)}(x)[\psi_0^{(r)}(y)]^*, \tag{25}$$

where the $\psi_0^{(r)}$ form an orthonormal system. Let us regard equality (25) as if the particle were in the pure state $\psi_0^{(r)}$ with probability p_r. Starting the stochastic process (22)–(24) from these initial wavefunctions, each of them will give rise to the quantity

$$\sum_r p_r \psi_t^{(r)}(x)[\psi_t^{(r)}(y)]^* . \tag{26}$$

The stochastic average of this expression over the histories $\psi_t^{(r)}$ is equal to $\rho_t(x,y)$.

We have to note that many other stochastic processes for ψ_t can be constructed with the same $\rho_t(x,y)$. Nevertheless, we would like to underline that the orthogonality of the stochastic jumps (23) is very crucial from the viewpoint of measurement theory. We show that by means of continuous measurement [‡2] we can registrate all stochastic jumps, *without disturbing the measured particle.* A more sophisticated formalism of non-demolition continuous quantum measurements was recently constructed by Barchielli [12].

Let us suppose that we know the wavefunction ψ_t at a given moment t. Then we define the following hermitian operator

$$\hat{O}_{t+0}(x,y) = \psi_t(x)\psi_t^*(y). \tag{27}$$

If at time t we measure the dynamical quantity corresponding to the operator $\hat{O}$ then the result of the measurement is

$$O_{t+0} = \left| \int \psi_{t+0}^*(y)\hat{O}_{t+0}(y,x)\psi_{t+0}(x)\,\mathrm{d}x\,\mathrm{d}y \right|^2 = \left| \int \psi_t^*(x)\psi_{t+0}(x)\,\mathrm{d}x \right|^2 . \tag{28}$$

If at time t the wavefunction evolves continuously then $\psi_{t+0} = \psi_t$ and $O_{t+0} = 1$. Oppositely, if ψ_t jumps then $\psi_{t+0} = \tilde{\psi}_t$ (cf. eq. (23)) and O_{t+0} vanishes. Observe that in both cases ψ_{t+0} is an eigenvector of $\hat{O}_{t+0}$, therefore this measurement does not react on the measured system (i.e. it does not force the wavefunction ψ_{t+0} to collapse). In addition, the measured value (28) is definitely 0 if a jump has occurred and it is 1 in the opposite case.

Thus, keeping in mind the causal equation (22), we can follow the stochastic changes too, if we continuously measure the operator (27). Of course, it is not obvious at all how should we realize the proper measuring apparatus in practice.

Finally, we note that in the Langevin approach of quantum damping [2] also a stochastic process is constructed, but this process is related to the evolution of the density matrix, not to the pure quantum state of the system.

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‡2 Continuous measurement may totally suppress the own dynamics of the measured system ("Zeno paradox", see e.g. ref. [11].) In white-noise potentials, however, this paradox can easily be disproved.

References

[1] W.H. Louisell and L.R. Walker, Phys. Rev. 137 (1965) B204.
[2] M. Lax, Phys. Rev. 145 (1966) 110.
[3] W.H. Louisell, Quantum statistical properties of radiation (Wiley, New York, 1973).
[4] F. Károlyházy, A. Frenkel and B. Lukács, in: Physics as natural philosophy, eds. A. Shimony and H. Feschbach (MIT Press, Cambridge, 1982).
[5] J.M. Ziman, Models of disorder (Cambridge Univ. Press, London, 1979).
[6] N. Gisin, Phys. Rev. A28 (1983) 2981.
[7] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965).
[8] L.D. Landau and E.M. Lifshits, Quantum mechanics (Pergamon, Oxford, 1977).
[9] J. von Neumann, Matematische Grundlagen der Quantenmechanik (Springer, Berlin, 1932).
[10] L. Diósi, Phys. Lett. 105A (1984) 199;
I. Bialynicki-Birula and J. Mycielski, Ann. Phys. 100 (1976) 62.
[11] A. Peres, Am. J. Phys. 48 (1980) 931.
[12] A. Barchielli, Nuovo Cimento 74B (1983) 113.