THERMODYNAMICS OF THE VACUUM

L. DIÓSI and B. LUKÁCS
Central Research Institute for Physics, Budapest, Hungary

K. MARTINÁS
Department of Low Temperature Physics, Roland Eötvös University, Budapest, Hungary

and

G. PAÁL
Konkoly Observatory of the Hungarian Academy of Sciences, Budapest, Hungary

(Received 16 December, 1985)

Abstract. The fact that the energy-momentum tensor has nonzero vacuum expectation value in some space-times (the so-called back reaction) indicates a nontrivial thermodynamics of such vacua. A consequent thermodynamic analysis of the problem in Robertson–Walker space-times shows that, in the generic case, the number of the independent extensives is 2, in spite of the fact that the energy density is completely determined by the time-evolution of the geometry since the preparation of the vacuum state. The appearance of a second independent extensive seems to be in direct connection with the extra entropy term in the *Generalized Second Law* of Bekenstein and Hawking.

1. Introduction

It is well known that the formalisms of general relativity and quantum field theories are not fully mutually consistent, which fact is a consequence of the more or less independent development of these theories, some discrepancies may be demonstrated even on a more primitive level between Newtonian gravity and quantum mechanics (Diósi and Lukács, 1986). One can expect that gravity, relativity, and quantization can be unified into a single theory, and in such a unified formalism the mentioned discrepancies will vanish. Nevertheless, the final unification is obviously not yet at reach, and there are difficulties even in guessing the main features of the future theory. Today there is no common opinion even about the general direction in which the unified theory lies; it may be, e.g., geometrization of interactions (Kaluza, 1921), intermediate particles for gravity (as in supergravity (van Nieuwenhuizen, 1981)), the existence of a deeper quantum spacetime (Banai, 1985), and so on. This variety of propositions demonstrates that the problem is serious indeed.

The problem is suggestively illustrated by the Hawking radiation (Hawking, 1974). The formalism of the standard quantum field theory predicts some kind of radiation in the neighbourhood of a Schwarzschild black hole; then this radiation should be taken into account on the right-hand side of the Einstein equation, in order to preserve energy conservation. But this radiation has no classical (unquantized) limit, so it does not possess a c-number energy-momentum tensor. Therefore, the back reaction cannot be

Astrophysics and Space Science **122** (1986) 371–386.

calculated in a rigorous way in the present theory. A similar radiation is expected in expanding universes too.

In most cases this problem may be disturbing from the viewpoint of principles, but it is practically negligible: familiar black holes above 2 solar masses may have a radiation of several kilometer wavelength and ca. 10^{-20} erg s^{-1} total power; mini black holes are neither necessary nor indicated; the analogous radiation of the Universe is hopelessly dominated by the 3K blackbody radiation. So in the worst case ambiguous theoretical predictions are confronted with impossible observations. Nevertheless, there is at least one case when the existence and details of this radiation should be known. In some inflationary universe models (constructed for solving the monopole dominance problem) after the symmetry breaking GUT phase transition a substantial supercooling is needed, which may be stopped at the characteristic temperature of the Hawking radiation, thus the present day observables, as the monopole/proton ratio are influenced (Kibble, 1982).

So, Hawking radiation is an important effect both from theoretical and from practical viewpoints. As we have seen, its most direct problem is the incorporation of the back reaction. Now, the back reaction problem, when handled in usual general relativity, can be formulated as follows: the curvature causes a non-vanishing vacuum expectation value of T_{ik}, which is to be taken as a source term of curvature (Wada and Azuma, 1983). But this formulation demonstrates that Hawking radiation leads to a double thermodynamic peculiarity.

If the vacuum expectation value of T_{ik} does not vanish, then the vacuum carries, e.g., energy. One may think that the Hawking radiation is a manifestation of the (not yet fully understood) vacuum of the unified theory. In fact, Bekenstein (1973) and Hawking (1975) have shown that this radiation effects the direction of thermodynamic processes. A generalized second law has been proposed, with a correction to the entropy

$$S = S_{\text{matter}} + S_{\text{black hole}}, \tag{1.1}$$

a similar correction has been constructed for de Sitter universes. The extra term reflects some properties of the vacuum. Until now no situation violating this generalized second law is known.

On the other hand, T_{ik} carries direct thermodynamic meaning too. First, its certain projections yield the energy density and conductive energy current (Ehlers, 1973) standing for an extensive in the local description (de Groot and Mazur, 1962). Second, in the axiomatic treatments of thermodynamics the intensive p is taken from (or in accord with) continuum mechanics (cf., e.g., Tisza, 1961), and in continuum mechanics p is again a projection of T_{ik} (Ehlers, 1973). Thus, from thermodynamic viewpoint, the situation can be summarized in the following way. In a curved space-time the vacuum of quantum field theory possesses nonvanishing values of some extensives and intensives; and in the same time it seems to carry a nontrivial entropy too. So the Hawking radiation is not without interest from thermodynamic viewpoint, and, if the vacuum of quantum field theory can be identified with the vacuum of thermodynamics (a question which will be discussed in this paper), then the thermodynamics of the vacuum is not trivial. By investigating this thermodynamics one may get a deeper insight into the meaning of the generalized second law.

This paper is not intended to derive new phenomena of gravity or quantum field theory; its goal is to put the statements of these two theories into a consistent thermodynamic framework. This highly phenomenological approach was triggered by Press's work (Press, 1983), which demonstrates that a statistical treatment can reproduce the main features of a genuine 'quantum gravity' effect. We will restrict ourselves to Robertson–Walker geometries where the symmetries are very convenient. It would not be purposeful to complicate the discussion with onthologic problems about the existence of the radiation. A possible viewpont is that the description at reach is an inadequate and incomplete limit of the true (but still unknown) unified theory and then presumably some predictions of it (as possibly the existence of the Hawking radiation, not necessarily with its all details) survive in the true theory. If not, the whole problem vanishes, but there is no indication for such a kindness of nature.

2. The Vacuum

One should first define the vacuum. There are at least three possible definitions.

(1) General relativity: the Riemann tensor can be measured by the Szekeres detector (Pirani, 1964) (in principle). So the Ricci tensor on the left-hand side of the Einstein equation can be measured. The right-hand side vanishes for vacuum. A state is called vacuum, if the corresponding curvature is that of vacuum.

(2) Quantum field theory: the vacuum is the state on which the effect of the annihilation operators is zero.

(3) Thermodynamics: the only independent extensive of the vacuum is the volume V. This can be seen by a Gedanken-experiment: remove everything from a volume and surround it by ideal walls isolating all the extensives. Then, operating in this region, the only possible freedom in separating a subsystem is to choose its volume.

Nevertheless these familiar and obvious propositions for definitions are unsatisfactory, and partially inconsistent. Proposition (1) would need a unique choice of the cosmologic constant (Csernai and Lukács, 1984; Lukács and Martinás, 1984a). This illustrates a more fundamental problem: the true gravitational equation is not necessarily known, and the missing terms may imitate an energy-momentum tensor. This possibility is even more interesting if the missing terms come from quantum corrections to gravity, explicitely containing $\hbar$ and derivatives of g_{ik}.

In connection with proposition (2) observe that the Heisenberg state vectors are time-independent. So, in a nonstationary space-time, where the Hamiltonian is generally time-dependent, the diagonalizing creation and annihilation operators are also time-dependent, and they cannot give 0 on any state vector except for a specific moment (Veselov *et al.*, 1984). On the other hand, defining time-independent annihilation operators, this definition is not unique, and cannot guarantee the vanishing of T_{ik}.

Now, the problem with proposition (3) can be clearly seen too. Remove everything from a moderate volume at $t = t_0$, and surround it by ideal walls. Nevertheless, there are no walls against curvature and gravity is a long-range effect as it can be seen, e.g., from the linearized Einstein equation (Landau and Lifshitz, 1984). It has a structure similar to the Maxwell equation for the potential, and can be solved also by retarded potential integrals. Therefore, the actual value of the metric at a point is determined by the surrounding volume (Diósi *et al.*, 1985), so generally some nontrivial metric will appear even in the walled region, generating nontrivial values of some extensives.

Observe that all the listed difficulties lie in the same direction, and can be eliminated in the same way. Let us accept a series of vacua, labelled by a time parameter (Veselov *et al.*, 1984). Then $|0\rangle_{t_1}$ stands for a state

$$a_{t_1}|0\rangle_{t_1} = 0 \,, \tag{2.1}$$

where a_{t_1} is an annihilation operator, element of the set of creation and annihilation operators diagonalizing the time-dependent Hamiltonian at $t = t_1$. The the energy density ρ vanishes at t_1 (up to renormalization); for later times $\langle T_{ik}\rangle$ is given by the metric and its derivatives (together with prehistory terms negligible in the long run, and expandable in time derivatives too (Veselov *et al.*, 1984)). Such terms are just expected on the right-hand side of the Einstein equation, if some terms are missed in proposition (1), and within the ideal walls of proposition (3) too. Therefore, a definition of vacuum states according Equation (2.1) may be practical, and it may preserve as much of the naive notion of vacuum as possible. The set of different vacua is possibly surprising, but one cannot help it; each member of the set $|0\rangle_{t_i}$ belongs to a naive vacuum energy-momentum tensor at a specific moment, and if no preferred time moment exists, then one is as good as the other. (Of course, the prehistory terms may fade out for asymptotic times (Veselov *et al.*, 1984).) We are going to see that a consequent thermodynamic formulation of these states are possible indeed.

It is not a goal of this paper to develop any new treatment from the viewpoint of quantum field theory. Following Veselov *et al.* (1984), one can see that there is a formal coordinate dependence in the method, since the Hamiltonian is a component of T_{ik}; this is not necessarily a mere formal problem as demonstrated by the particle detection of accelerating observers in the empty Minkowski space-time (Unruh, 1976); this means that the expectation values calculated by quantizations in different systems generally cannot be connected by the transformation of general relativity. Since here we are confronted with the most fundamental problem of unifying general relativity and quantum field theory, it would not be too promising to discuss this now; a more pragmatic approach is to look for a preferred velocity field, then to quantize according to it, and finally to transform the so obtained T_{ik} appropriately. This pattern will be followed here; fortunately Robertson–Walker universes define a unique velocity field through their symmetries (Robertson and Noonan, 1969; Lukács and Mészáros, 1985).

3. The Thermodynamic Quantities of the Vacuum

We have seen that, due to quantum field effects, generally the vacuum expectation value of the energy-momentum tensor does not vanish; in usual situations the components of T_{ik} carry thermodynamic meaning. So it seems that in a general space-time the thermodynamics of the vacuum is not trivial. Now we are going to investigate this thermodynamics.

Consider a small spatial volume in a space-time, from which all the matter was removed at $t = t_1$. Then the state of that small volume is the ground state, or vacuum $|0_{t_1}\rangle$. If the volume considered is small enough, then it is not self-determining for the geometry due to the long-range nature of gravitation (Landau and Lifshitz, 1984; Diósi *et al.*, 1985); therefore, there the metric is prescribed by the neighbourhood: i.e., for the volume considered the geometry is externally given. Here, for the sake of argument, we assume that this metric is not stationary. First, let us fix the time to t_0; we ignore the spatial dependence of g_{ik}, being the volume small.

Now separate a subvolume V. Since the vacuum expectation value of T_{ik} is completely determined by the given metric, the only free parameter of the separated part is V, and, from the same reason, the extensives are additive in a unification of the (fictitiously) separated subvolumes (Kirschner, 1969, 1970, 1971). Thus, e.g.,

$$E(t_0) = E(t_0; V) = \rho(t_0)V; \tag{3.1}$$

E being the energy. Similarly, for the entropy S, which is the proper thermodynamic potential when using extensives,

$$S(t_0) = S(t_0; V). \tag{3.2}$$

Since S is extensive too, from the homogeneous linearity required for extensives

$$S(t_0; V) = s(t_0)V. \tag{3.3}$$

In these formulae ρ and s are the densities of energy and entropy, respectively.

Now, let the time go by, Equations (3.1)–(3.3) obviously remain valid with a general t, but the Gibbs–Duhem relation (Glansdorff and Prigogine, 1971) of thermodynamics will not hold, because of the explicit time dependence of S. In order to see this, consider a system of n independent extensives X^i with a time-dependent potential $S(t; X^i)$. Then the change of S can be evaluated in two different ways. First, through its variables, one obtains

$$dS = Y_r\, dX^r + \dot{S}\, dt, \tag{3.4}$$

where the Einstein convention is used for summation, and the intensives Y_i are defined as

$$Y_i = \frac{\partial S}{\partial X^i}. \tag{3.5}$$

On the other hand, S is a homogeneous linear function of the extensives; therefore, the Euler identity

$$S = Y_r X^r \tag{3.6}$$

holds; and, by differentiation,

$$\mathrm{d}S = X^r\,\mathrm{d}Y_r + Y_r\,\mathrm{d}X^r\,. \tag{3.7}$$

By comparing Equations (3.4) and (3.7) one obtains

$$X^r\,\mathrm{d}Y_r = \dot{S}\,\mathrm{d}t\,, \tag{3.8}$$

which is definitely not the Gibbs–Duhem relation $X^r\,\mathrm{d}Y_r = 0$.

Now, remember that we are just manufacturing the proper thermodynamic description for our vacuum. Generally, the number of thermodynamically independent degrees of freedom is not a trivial, a priori settled question (Landsberg, 1961; Lukács and Martinás, 1984b); if the Gibbs–Duhem relation does not hold, then it is a signal that some necessary degrees of freedom, either obvious or not, are missed. In our case the nonvanishing right-hand side of Equation (3.8) is caused by the fact that one variable t of S is not an extensive, so it is not a subject of the Euler identity. Therefore, one can conclude that the necessary condition for a canonical thermodynamic formalism is to express the variable t by means of extensives.

The energy density T_{00} is expected to depend on t in the generic case. If, however, it turns out to be independent of t, then

$$E = E(V) = \rho_0 V\,. \tag{3.9}$$

Since $E(X^i)$ is also a legal thermodynamic potential (Gibbs, 1931), now one can use the energy convention, when the only energetic intensive is the pressure p,

$$-p = \frac{\partial E}{\partial V} = \rho_0\,; \tag{3.10}$$

so that $\rho + p = 0$. This is conform with the balance equation $T^{ir}{}_{;r} = 0$, the dynamic and thermodynamic pressure coincide, and the potential (3.9) yields the full thermodynamic description of the (almost trivial) situation. On the other hand, if $\dot{\rho} \neq 0$, then one can invert the function $\rho(t)$ as

$$t = t(\rho) = t(E/V)\,; \tag{3.11}$$

and then, substituting into the potential $S(t;\,V)$ one gets

$$S = S(V, E)\,. \tag{3.12}$$

Here both V and E are extensives occurring in the Euler identity (3.6), so the obstacle of satisfying the Gibbs–Duhem relation has been removed.

Equation (3.12) is just the usual form of the potential for a continuum without any particle degree of freedom, as, e.g., for black-body radiation; however, note that a formal thermal degree of freedom (the variable E) has appeared, which had not been assumed at the beginning. If the vacuum state is fixed, one cannot prepare states of different energy in a given moment, but the system evolves through them.

Now, one can introduce the entropic intensives Y_i with the usual notations

$$\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{\mathrm{d}s}{\mathrm{d}\rho}\,, \qquad \frac{p}{T} = \frac{\partial S}{\partial V} = s - \frac{\rho}{T}\,; \tag{3.13}$$

where T and p stand for the thermodynamic temperature and pressure, respectively.

Obviously, if this thermodynamic formalism is not fully formal, then the dynamic pressure P occurring in the energy-momentum tensor and thermodynamic one p must be closely and intimately related to each other. Of course, one can always achieve $P = p$: namely,

$$P = P(t) = P(t(\rho)) = P(\rho)\,, \tag{3.14}$$

via Equation (3.11), and then the equation

$$p(\rho) = P(\rho)\,, \tag{3.15}$$

is an ordinary differential equation for $s(\rho)$, by using Equations (3.13). So, for any function $P(\rho)$ there exist entropy functions leading to relation (3.15): namely,

$$E(\rho) = s_0 \exp\left\{\int \frac{\mathrm{d}\rho}{\rho + P(\rho)}\right\}; \tag{3.16}$$

where s_0 is a free constant unaffecting the dynamics. Nevertheless, irreversible processes may lead to pressure correction (Ehlers, 1973; Heller *et al.*, 1973; Diósi *et al.*, 1984), thus it is still necessary to discuss the validity of fundamental laws of thermodynamics. Nevertheless, for simplicity's sake now we are going to restrict ourselves to universe solutions with the vacuum expectation value of T_{ik}.

4. Vacuum Universes

The simplest example for the back reaction problem is a pure vacuum universe, as discussed, e.g., by Wada and Azuma (1983) and Veselov *et al.* (1984). Now we have obtained a thermodynamic formalism for the vacuum state, this formalism can be applied to the problem.

It is the widespread opinion that the Universe possesses the maximal symmetry. This opinion is sometimes formulated as cosmologic principle (Anderson, 1967). However, it is not quite clear, which symmetries are possible. The mathematically possible maximal symmetry involving ten Killing vectors is inconsistent with observations (Robertson and Noonan, 1969), therefore, it is usual to assume maximal spatial symmetry with six spatial Killing vectors (Robertson and Noonan, 1969). This leads to the Robertson–Walker line elements, which, in coordinates adapted to the symmetries, possess the common form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\,\{\mathrm{d}x^2 + f^2(x)\,(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2)\}\,, \tag{4.1}$$

where

Symmetry	SO(4)	$E(3)$	SO(3, 1)
$k =$	$+1$	0	-1
$f(x) =$	$\sin x$	x	$\mathrm{sh}\, x$

The existence of the six spatial Killing vectors uniquely defines a cosmological velocity field – i.e., a vector field of unit length

$$u^i = \delta^i_0 \,. \tag{4.2}$$

In the real universe observations, in fact, verify that the peculiar velocities are moderate, so this unique vector field is a physical reality. Now, the energy-momentum tensor can be decomposed according to any time-like unit vector field (Ehlers, 1973), using this u^i one gets

$$T^{ik} = \rho u^i u^k + q^i u^k + q^k u^i + p^{ik} \,, \qquad u^r q_r = u^r p_{ir} = 0 \,, \tag{4.3}$$

where q^i is the (thermal) energy flux, ρ is the energy density, and p^{ik} is the spatial stress. These individual projections possess as much physical meaning as the cosmological velocity field. The consequence of the spatial symmetries is that

$$q^i = 0 \,, \qquad p^{ik} = -P(g^{ik} - u^i u^k) \,. \tag{4.4}$$

The energy-momentum tensor is then represented by two scalar functions: the energy density ρ, and the (dynamic) pressure P. These quantities are to be calculated from the detailed theory of the matter filling the Universe, and then the Einstein equations assume the form

$$\dot{R}^2 = -k + \frac{8\pi G}{3} \rho R^2 \,, \qquad \ddot{R} = -\frac{4\pi G}{3} (\rho + 3P) R \,. \tag{4.5}$$

The second one of Equations (4.5) can be substituted by the integrability condition of the system

$$\rho + 3 \frac{\dot{R}}{R} (\rho + P) = 0 \,. \tag{4.6}$$

Thus there are two equations for three quantities: namely, R, ρ, and P; nevertheless, the third relation is generally provided by the theory of the matter.

Now, as a simplification, let us complete ignore the matter. Then T_{ik} is purely the vacuum expectation value, which should be evaluated by field theoretical methods. The problem has been extensively studied for de Sitter cases

$$k = 0 \,, \qquad R = R_0 \, e^{t/t_0} \,. \tag{4.7}$$

In particular, Wada and Azuma (1983) have shown that the back reaction then gives a correction to the original cosmological constant. Observe that here we choose the specific form of the Einstein equations (4.5) in which there is no cosmological constant in the geometric terms; so such a constant, if exists, should be attributed to the energy-momentum tensor, which can always be done (Csernai and Lukács, 1984; Lukács and Martinás, 1984a). Naturally, the complete vacuum expectation value of T_{ik} automatically contains any type of Hawking radiation, if such a radiation occurs indeed in the energy-momentum tensor. On the other hand, Gibbons and Hawking (1977) have found a characteristic temperature for de Sitter universes to be

$$T_{\mathrm{H}} = \frac{1}{2\pi}\,\frac{\ddot{R}}{R}\,. \tag{4.8}$$

This temperature defines the equilibrium states of freely moving detectors in de Sitter space-times. One possible interpretation of this result is the effect of a radiation on the detectors, nevertheless, as Gibbons and Hawking (1977) ephasize, since the temperature is the same for any freely moving detector, it may not be described by an energy-momentum tensor. Let us demonstrate the problem from another viewpoint. Assume a black-body radiation of temperature T_{H} as the only real matter field, then

$$\rho = 3p = \frac{\pi^2}{30}\,T_{\mathrm{H}}^4\,; \tag{4.9}$$

and the energy-momentum tensor is expressed by $R(t)$, so the two Einstein equations (4.5) overdetermine $R(t)$: in fact, the only solution is then R = const., $k = 0$ (the trivial Minkowski space-time).

Then the problem is not simply to look for an appropriate radiation. Instead of this, now we turn to the Einstein equations (4.5) with vacuum expectation values (fulfilling Equation (4.6)) on the right-hand side, and look for the proper thermodynamic quantities s, p, *and* T (according to Equations (3.13) and (3.16)). By checking the thermodynamic one can decide if the above formalism is self-consistent or not.

Consider first the function $\rho(t)$. Obviously one cannot expect to get the explicit forms for $\rho(t)$ and $P(t)$ by means of a pure field theoretical calculation without recourse to the Einstein equation. In the best case, when evaluating the expectation values with a given metric (which is represented here by a given scale factor $R(t)$ and by k), the result is of form (Wada and Azuma, 1983; Veselov *et al.*, 1984)

$$\rho = \rho\left(\frac{k}{R^2}, \frac{\dot{R}}{R}, \frac{\ddot{R}}{R}\right), \qquad P = P\left(\frac{k}{R^2}, \frac{\dot{R}}{R}, \frac{\ddot{R}}{R}\right), \tag{4.10}$$

where the arguments on the right-hand side are the quantities building up the Riemann tensor characterizing the curvature. Now, k/R^2 and $\ddot{R}/R$ can be expressed by ρ and P via the Einstein equations, and then, finally

$$\rho = \rho(\dot{R}/R)\,, \qquad P = P(\dot{R}/R)\,. \tag{4.11}$$

The remaining variable (apart from a constant factor) is just the expression on the right-hand side of Equation (4.8) which, henceforth, will be referred to as the Hawking temperature T_H independently of the geometry of the space-time. Here the actual form of $P(T_H)$ is not needed, because Equation (4.6) contains P algebraically, and the results of a self-consistent field theoretical calculation have to fulfil Equation (4.6) (Veselov *et al.*, 1984). In this section we do not specialize the form of the function $\rho(\dot{R}/R)$.

Consider first the case $k = 0$. Then, via Equations (4.5) and (4.11)

$$x^2 - a^2\rho(x) = 0 , \tag{4.12}$$

where

$$a^2 = \frac{8\pi}{3} G = \frac{8\pi}{3} \frac{1}{M_{Pl}^2} \quad \text{and} \quad x = \frac{\dot{R}}{R} \tag{4.13}$$

and $\hbar = c = 1$. An alternative arises: in the generic case Equation (4.12) leads to a constant x, whence, via Equation (4.11), $\rho = 0$, and, from Equation (4.6), $\rho + P = 0$. Then we have arrived at the almost trivial subcase $E = E(V)$ of Section 3, without any thermal degree of freedom. The other possibility is that the quantum field theory yields $\rho = x^2/a^2$, when the evolution of x remains free. In what follows we do not discuss this exceptional case.

If $k \neq 0$, Equation (4.6) expresses P in terms of $\rho(x)$, $\rho'(x)$, x, and $\dot{x}$ (henceforth the prime stands for x derivative). On the other hand, $\dot{x}$ can be obtained by taking the time derivative of the first of Equations (4.5). Substituting this expression into Equation (3.16) one obtains

$$s = s_0 \left| x^2 - a^2\rho(x) \right|^{3/2} . \tag{4.14}$$

Since this formula contains a square root, the sign convention has still to be fixed.

There is also the question whether the first and second laws of thermodynamics hold or not with such pressure and entropy functions. (The third law obviously cannot be checked without specifying the function $\rho(x)$.) Let us first tentatively identify p with P (which equality, as it has been seen, holds if the entropy is given by Equation (4.14)). Then Equation (4.6) is just the differential form of the first law. Now the evolution of ρ is known, and by calculating the entropy production via Equations (4.6), (3.13), and (3.15)–(3.16) one gets

$$\dot{s} + 3 \frac{\dot{R}}{R} s = 0 , \tag{4.15}$$

which can be seen also by substituting the first of Equation (4.5) into Equation (4.14) when the result $s = s_0 R^{-3}$ is equivalent with Equation (4.15). Thus the second law is also fulfilled.

Observe now that the Gibbs–Duhem relation holds after introducing E instead of t, in the same time, the entropy matrix (Kirschner, 1969, 1970, 1971)

$$g_{ik} = \frac{\partial^2 s}{\partial x^i \, \partial x^k} \tag{4.16}$$

(where x^i stands for extensive densities), which in our case consists of a single element $\mathrm{d}^2s/\mathrm{d}\rho^2$, does not degenerate (except for the pathologic case P = const.). Thus, we have arrived at the necessary and sufficient set of extensives (cf. Landsberg, 1961). Then one can conclude that the presented thermodynamic formalism is consistent indeed with the evolution of the model universe. Therefore, one may also consider T defined in Equations (3.13) as the temperature of the (quantum field) vacuum.

Equation (4.14) suggests a convenient interpretation. Assume that there exists a limit where $a^2\rho \ll x^2$ (which is, of course, impossible for $k \neq -1$). Then there $s = s_0(2\pi)^3 T_H^3$, which, with an appropriate choice of s_0, is the same form as for a black-body radiation of Hawking temperature. Nevertheless, the function $s(T)$ is not a thermodynamic potential (as $s(\rho)$ is); therefore, it does not carry the complete information about the system; and $s \sim T_{\mathrm{H}}^3$ is of a limited validity only. Thus, a more systematic discussion is needed, which we are going to perform on a simplified model system.

5. An Explicit Example

In order to proceed further one needs a specific form for $\rho(x)$. Now, Wada and Azuma (1983) give the vacuum expectation value of T_{ik} for N independent massive scalar fields in a de Sitter universe. If one is contented with massless free limits then

$$\rho = \frac{N}{64\pi^2}\, h(\tilde{R})\,, \tag{5.1}$$

where $\tilde{R}$ stands for the Ricci scalar, which in the particular case is of the form

$$\tilde{R} = 6\left(x^2 + \frac{\ddot{R}}{R}\right); \tag{5.2}$$

and the function h is composed of $\tilde{R}^2$ and $\tilde{R}\ln\tilde{R}$ terms. Ignoring the latter (first from technical reasons, second, because $\ln\tilde{R}$ terms would need a dimensional constant whose unknown value would influence the final result) one gets

$$\rho \simeq \frac{29N}{3840\pi^2}\left(x^2 + \frac{\ddot{R}}{R}\right)^2, \tag{5.3}$$

where $\ddot{R}/R$ can be substituted from the second of Equations (4.5). Since $\rho + P = 0$, in the particular case one obtains

$$\rho \simeq C^2(x^2 + a^2\rho)^2 \quad \text{and} \quad C^2 = \frac{29N}{3840\pi^2}\,. \tag{5.4}$$

In what follows we shall regard this formula as a simplified prediction of quantum field theory, although it has been obtained only for massless free scalar fields in a de Sitter universe.

Now, one can see that the third law (Callen, 1960) holds good for such a energy density: namely,

$$\lim_{x \to X} \left(\frac{ds}{d\rho}\right)^{-1} = 0\,, \tag{5.5}$$

where X is a real root of the equation

$$s(X) = 0\,, \tag{5.6}$$

as it can be seen by direct calculation.

Well below M_{Pl} in T_{H} Equation (5.4) can be solved as

$$\rho \simeq C^2 x^4\,. \tag{5.7}$$

This formula remains valid as an order of magnitude estimation at $x \sim 1/a \sim M_{\mathrm{Pl}}$, therefore, one may use Equation (5.7) as a simple approximation. We are going to solve the Einstein equation (4.5) with a right-hand side according to Equation (5.7), and to investigate the behaviour of the solutions.

The evolution of the scale factor R is governed by the equation

$$x^2 - b^2 x^4 = -kR^{-2}\,; \qquad b \equiv Ca\,, \tag{5.8}$$

which, together with Equation (4.13), is a first-order differential equation. It can be solved by quadrature which, however, cannot be performed analytically. The thermal history is described by the thermodynamic temperature T defined in Equation (3.13). This is the moment when one has to choose some sign convention for the square-root in Equation (4.14); this convention is, of course, arbitrary, because it does not affect the dynamics. Nevertheless, we proceed as follows. For $k = -1$ the first term in the square root in Equation (4.14) dominates the second because of the Einstein equation (5.8); there are asymptotic states with $\rho/x^2 \to 0$, and there s/T_{H}^3 becomes constant. Therefore, we arrive at

$$s = s_0 x^3 \sqrt{(1 - b^2 x^2)^3} \tag{5.9}$$

(square-roots are taken with positive signs). Therefore, $s(x)$ is an odd function. On the other hand, for $k = +1$ the second term dominates the first one; the case $s \sim T_{\mathrm{H}}^3$ never recovers. Therefore, then we choose

$$s = s_0 x^6 \sqrt{\left(1 - \frac{1}{b^2 x^2}\right)^3 b^3}\,, \tag{5.10}$$

which is an even function. Thus, for the temperature one gets

$$\begin{aligned} T &= \frac{4C^2}{3s_0} \frac{|x|}{\sqrt{b^2x^2 - 1}\,(2b^2x^2 - 1)}\,, \quad k = +1\,, \\ T &= \frac{4C^2}{3s_0} \frac{x}{\sqrt{1 - b^2x^2}\,(1 - 2b^2x^2)}\,, \quad k = -1\,. \end{aligned} \tag{5.11}$$

Equation (5.8) is quadratic in x^2; it has only one positive root if $k = +1$. For $k = -1$ there are two positive roots, but one of them leads to $Ts < 0$, hence, $\rho + p < 0$, which violates the energy positivity condition (Hawking and Ellis, 1973). Thus, we regard it as a physically impossible solution; and we are left with

$$x^2 = \frac{1}{2b^2}\left(1 + k\sqrt{1 + 4kb^2R^{-2}}\right), \tag{5.12}$$

which represents a contracting and expanding solution for both k. For the temperature one obtains

$$R = \frac{4C^2}{3s_0 b}\,\frac{\sqrt{k\sqrt{(1 + 4kb^2R^{-2})} + 1}}{\sqrt{\sqrt{(1 + 4kb^2R^{-2})} - k}}\,\frac{1}{\sqrt{1 + 4kb^2R^{-2}}}\,(\operatorname{sgn}(x))^{(1-k)/2}\,. \tag{5.13}$$

Since the evolution equation cannot be analytically solved, we are looking for asymptotic solutions for large and small values of R. Let us start with $k = +1$. Consider a contracting solution at $R \gg b \sim \lambda_{\mathrm{Pl}}$. Then x is approximately constant according to Equation (5.12), so that

$$R \simeq R_0\, e^{-(t - t_0)/b} \quad \text{and} \quad T/R \simeq \text{const} = 4C^2/3s_0b^2\,. \tag{5.14}$$

On the other hand, well below b one gets a diverging Hawking temperature, together with

$$R \simeq \frac{1}{4b}(t - t_1)^2 \quad \text{and} \quad T/R \simeq 2C^2/3s_0b^2\,. \tag{5.15}$$

Therefore, the solution formally turns back at $R = 0$; the other branch is just the expanding one. The temperature is almost proportional to R, therefore, it moves inversely with the entropy density, so the specific heat is negative, and the solution is unstable thermodynamically (Kirschner, 1969, 1970, 1971). This seems to indicate unstability against inhomogeneities (Lukács and Csernai, 1984). Another strange feature is the opposite motion of T and T_{H}.

Nevertheless, this solution is not too significant physically: namely, ignoring such number constants as, e.g., C, which cannot be too far from unity in any state either R is below Planck length or T is above Planck temperature.

For $k = -1$ the global behaviour is completely different. Consider an expanding solution. There cannot be states below $R = 2b$ because of the square-roots. Then $\dot{R}/R = x = 1/\sqrt{2}\,b$; so the Universe, thereafter, expands exponentially with characteristic time in the order of t_{Pl}. The temperature starts from positive infinity, and decreases as

$$T \simeq \frac{4C^2b}{3s_0}\sqrt{\left(\frac{R}{2b} - 1\right)^{-1}}\,. \tag{5.16}$$

When $R \gg b$, one obtains a uniform expansion

$$R = R_0 + t, \tag{5.17}$$

with a temperature

$$T = \frac{4C^2}{3s_0}\frac{1}{R} = \frac{4C^2}{3s_0}x = \frac{8\pi C^2}{3s_0}T_{\mathrm{H}}. \tag{5.18}$$

Hence, one can fix the free constant s_0 by requiring that

$$\lim_{R \to \infty} (T/T_{\mathrm{H}}) = 1. \tag{5.19}$$

Nevertheless, this choice leads to an 'effective' equation of state

$$p = \tfrac{29}{720}\pi^2 NT^4, \tag{5.20}$$

while for a true black-body radiation of massless spin 0 particles the number factor would be $\frac{1}{90}$ (Barrow, 1983). This difference again indicates that ρ is not the energy density of a real radiation.

The contracting solution is just the time-reflection of the expanding one; it ceases to exist at $R = 2b$. Observe that, choosing again a positive s_0, T is negative for contraction. This indicates a possibility for an intimate connection between the direction of the evolution of the Universe and the statistical laws, which kind of connection was discussed by Penrose (1977). For $k = +1$, T does not change its sign at the turning back.

Obviously the $k = -1$ universes are geodetically incomplete; they appear or vanish at finite size and expansion rate. However, note that this radius is in the order of the Planck length; it is not evident that the known 'classical' general relativity would be valid below λ_{Pl}. One may guess, for example, that without the full theory a point and a cell of Planck size cannot be clearly distinguished, in which case the starting point of the Universe is simply the moment when we can recognize it first time by means of classical tools. Note that $T = \infty$ in this initial state; this suggests that from thermodynamic viewpoint it is, in fact, a natural initial state, representing a 'singularity' or realization of all degrees of freedom as fully as possible without the unified theory.

6. Conclusions

In this paper we have defined the notion of vacuum in such a way that it keep the maximal possible content of independent definitions in general relativity, quantum field theory, and thermodynamics. The so-defined vacuum states depend on the time of their preparation, at a generic time moment the vacuum expectation value of the energy-momentum tensor does not vanish (e.g., the Hawking radiation is incorporated into it) and then nontrivial values of thermodynamic quantities belong to this state. On the other hand, pioneering works by Bekenstein and Hawking about generalization of the second law of thermodynamics of gravitating systems suggest special features of the thermodynamics when both general relativity and quantum field theory are taken into account. Since Press's statistical treatment has demonstrated the possibility of a phenomenological treatment of such systems, it is worthwhile to look for the proper thermodynamic description of this problem. (By this term we mean a formalism for the quantities of canonical thermodynamics itself, not the analogous treatment of black-hole characteristics and de Sitter horizon areas, sometimes referred to also as thermodynamics. Note that, e.g., the potential function of black-hole 'thermodynamics' is not even homogeneous linear in its variables.) Here we wanted to give this proper formalism, from technical reasons only for a simplified model, which was a vacuum universe.

First we have determined the number of independent thermodynamic extensives according to Landsberg's criteria that the Gibbs–Duhem relation already hold but the entropy matrix still be nondegenerate; the result is that such a set, in fact, exists in the generic case, when it consists of two extensives, e.g., of the volume and energy. The second degree of freedom was not expected from any 'naive' definition of the vacuum; it corresponds to the nontrivial evolution of a quantum field in a time-dependent geometry.

Then we have constructed a thermodynamic potential for this vacuum, if the independent variables are V and E, it must be the entropy S. The so constructed entropy function obeys the first and second laws of thermodynamics, while the validity of the third law depends on the details of quantum field theory, however, an explicit example has been shown when it holds in the usual form too. Thus, one may conclude that a consequent thermodynamic description of the vacuum of quantum field theories in curved space-times is, in fact, possible. Then one obtains a proper thermodynamic temperature of the vacuum, which may or may not coincide with the Hawking temperature felt by detectors immersed into the investigated space-time. From thermodynamic viewpoint the detector does not belong to the vacuum; the direct meaning of our thermodynamic temperature should be clarified in models containing material degree of freedom too.

By use of a simplified quantum field treatment, closed ($k = +1$) and an open ($k = -1$) model was discussed. The closed model did not possess a Friedmannian regime. As in the results of Veselov *et al.* (1984), the open subcase seemed to be more realistic, tending to an expanding Minkowski solution. In this latter case the temperature scale could be chosen in such a way that Hawking and thermodynamic temperatures asymptotically approach each other. Nevertheless, even then the explicit form of the thermodynamic potential differs from that of a free radiation field.

In this vacuum model the evolution of the Universe is still reversible, so the 'geometric' entropy does not offer an 'arrow of time'. This property may change in the presence of material degrees of freedom.

References

Anderson, J. L.: 1967, *Principles of Relativity Physics*, Academic Press, New York.
Banai, M.: 1985, *Found. Phys.* **15**, 1203.

Barrow, J. D.: 1983, in V. M. Canuto and B. G. Elmegreen (eds.), *Fundamentals of Cosmic Physics*, Gordon and Breach, New York, p. 83.

Bekenstein, J. D.: 1973, *Phys. Rev.* **D7**, 2333.

Callen, H. B.: 1960, *Thermodynamics*, John Wiley, New York.

Csernai, L. P. and Lukács, B.: 1984, *Acta Phys. Pol.* **B15**, 149.

Diósi, L. and Lukács, B.: 1986, *Ann. Phys.* (in press).

Diósi, L., Keszthelyi, Bettina, Lukács, B., and Paál, G.: 1984, *Acta Phys. Pol.* **B15**, 909.

Diósi, L., Keszthelyi, Bettina, Lukács, B., and Paál, G.: 1985, *Astron. Nachr.* **306**, 213.

Ehlers, J.: 1973, in W. Israel (ed.), *Relativity, Astrophysics, and Cosmology*, D. Reidel Publ. Co., Dordrecht, Holland, p. 1.

Gibbons, G. W. and Hawking, S.: 1977, *Phys. Rev.* **D15**, 2738.

Gibbs, J. W.: 1931, *Collected Works*, Longmans-Green, Vol. 1.

Glansdorff, P. and Prigogine, I.: 1971, *Thermodynamic Theory of Structure, Stability and Fluctuations*, Wiley-Interscience, New York.

De Groot, S. R. and Mazur, P.: 1962, *Non-Equilibrium Thermodynamics*, North-Holland, Amsterdam.

Hawking, S. and Ellis, G. F. R.: 1973, *The Large-Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge.

Hawking, S.: 1974, *Nature* **248**, 30.

Hawking, S.: 1975, *Commun. Math. Phys.* **43**, 199.

Heller, M., Klimek, Z., and Suszycki, L.: 1973, *Astrophys. Space Sci.* **20**, 205.

Kaluza, Th.: 1921, *Sitzungsb. Preuss. Akad. Wiss. zu Berlin*, p. 966.

Kibble, T. W. B.: 1982, in M. S. Craigie *et al.* (ed.), *Monopoles in Quantum Field Theories*, World Scientific, Singapore, p. 341.

Kirschner, I.: 1969, *Mag. Fiz. Foly.* **18**, 71.

Kirschner, I.: 1970, *Acta Phys. Hung.* **29**, 209.

Kirschner, I.: 1971, *Acta Phys. Hung.* **30**, 61.

Landau, L. D. and Lifshitz, E. M.: 1984, *Theoreticheskaya Fizika II*, Theoriya polya, Nauka, Moscow.

Landsberg, P. T.: 1961, *Thermodynamics*, Interscience Publ., New York.

Lukács, B. and Csernai, L. P.: 1984, *Acta Phys. Slov.* **34**, 161.

Lukács, B. and Martinás, K.: 1984a, KFKI-33.

Lukács, B. and Martinás, K.: 1984b, KFKI-110.

Lukács, B. and Mészáros, A.: 1985, *Astrophys. Space Sci.* **114**, 211.

Penrose, R.: 1977, *Proc. 1st Marcel Grossmann Meeting*, Trieste, North-Holland, p. 173.

Pirani, F. A. E.: 1964, *Lectures on General Relativity*, Vol. 1, Prentice Hall, p. 249.

Press, W. H.: 1983, in Joan Centrella (ed.), *J. Wilson 60th Birthday Volume*, Center for Astrophysics, Cambridge, Massachusetts.

Robertson, H. P. and Noonan, T. W.: 1969, *Relativity and Cosmology*, Saunders.

Tisza, L.: 1961, *Ann. Phys.* **13**, 1.

Unruh, W. G.: 1976, *Phys. Rev.* **D14**, 870.

Van Nieuwenhuizen, P.: 1981, *Phys. Reports* **68**, 189.

Veselov, A. I. *et al.*: 1984, in K. P. Stanyukovich (ed.), *Problemy teorii gravitacii i elementarnyh chastic*, Vol. 14, Energoatomizdat, Moscow.

Wada, S. and Azuma, T.: 1983, *Phys. Letters* **132B**, 313.