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Comments on continuous observation in quantum mechanics

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It is shown that in open quantum systems the so-called Zeno paradox is not valid. The equations of ideal continuous measurement for Markovian open systems are elaborated and applied to Pauli's simple open system, the actual energy level of which is shown to be monitorable by a continuous nondemolition measurement.

I. INTRODUCTION

In a series of original papers quoted in Ref. 1, Barchielli and co-workers proposed a theory of continuous measurement for quantum-mechanical systems (see especially Refs. 2 and 3). The theory reviewed in Ref. 1 was applied to the problem of gravity-wave detection. The author pays a great deal of attention to the continuous observation of the so-called open quantum systems[4] and it is on this that we wish to comment.

Let us recall the main stages of creating the theory of continuous measurement, especially of that for open systems.

In 1977 Misra and Sudarshan[5] pointed out that for conventional Hamiltonians the exact transition rates between orthogonal states vanish. This fact, called the quantum-mechanical Zeno paradox, makes nonsense of any possible concept of continuous observation based on the standard theory[6] of ideal measurements. It was subsequently shown that Zeno's paradox is an extreme case of the "watchdog effect":[7] The back reaction of the measuring apparatus on the measured system increases with the strength of the coupling between the apparatus and the system, and this back reaction will overcome the original dynamics of the system when the measurement becomes ideal. Generally, the Zeno paradox does not arise if the precision of the continuous measurement is below the ideal quantum-mechanical limit dictated by the Heisenberg uncertainty.

Fortunately, such a generalization of the ideal measurement[6] had already been developed by, for example, the measurement phenomenology based on operations.[8] If this formalism is applied to usual Hamiltonian quantum systems, a reasonable model of continuous observation can be built.[2] During such observation, a given Hamiltonian quantum system becomes a type of Markovian open system. It was obviously this fact that suggested an *ab initio* open quantum system be chosen and that its continuous observation be considered.[3] In Ref. 1 (originally in Ref. 3) the operation-based continuous measurement formalism[2] was indeed successfully extended to open quantum systems.

Nevertheless, we would like to return to the problem of the Zeno paradox. In Refs. 1, 2, and 3 we meet the statement that one needs operation-based measurements to avoid the Zeno paradox since this would veto the standard theory of ideal measurements. Though this statement is true for Hamiltonian systems it is no longer true for open quantum systems. It is well known[4] that an open quantum system possesses genuinely nonvanishing exact transition rates and this feature disproves Zeno's paradox.

Consequently, it is possible to consider the ideal continuous observation of a given open quantum system; i.e., to obtain a reasonable model there is no need to generalize the standard measurement phenomenology of von Neumann.[6]

Section II will be devoted to the general equations of the ideal continuous measurement in Markovian open quantum systems. In Sec. III we apply these results to Pauli's open system and show an example for ideal continuous observation of the nondemolition type.

II. IDEAL CONTINUOUS MEASUREMENT IN OPEN QUANTUM SYSTEMS

As it is generally used,[4] the density operator $\rho(t)$ of a given Markovian open quantum system satisfies the master equation

$$\dot{\rho}(t)=L[\rho(t)] \tag{2.1}$$

for all time t. The right-hand side (RHS) is linear in $\rho(t)$ and, to preserve the Hermiticity and the normalization of $\rho(t)$, the evolution operator L must be Hermitian and traceless:

$$L^{\dagger}[\rho]=L[\rho] \ , \tag{2.2}$$

$$\mathrm{tr}L[\rho]=0 \tag{2.3}$$

for all density operators ρ. From the positivity of the density operator $\rho(t)$ some further conditions for L follow.

Given a complete orthogonal system of pure states $\{P_n; n=1,2,\ldots,N\}$, $N\leq\infty$,

$$P_n^{\dagger}=P_n \ , \tag{2.4a}$$

$$\mathrm{tr}P_n=1 \ , \tag{2.4b}$$

$$P_nP_m=\delta_{nm}P_n; \quad n,m=1,2,\ldots,N \ ; \tag{2.4c}$$

$$\sum_{n=1}^{N}P_n=\mathbf{1} \ ; \tag{2.4d}$$

the transition rates

$$w_{n\to m}=\mathrm{tr}(P_mL[P_n])\quad (n\neq m) \tag{2.5}$$

must be non-negative.[4] We introduce the full decay rate for the state P_n by

$$w_{n\to}=\sum_{\substack{m=1\\ m\neq n}} w_{n\to m}=-\mathrm{tr}(P_nL[P_n]) \ . \tag{2.6}$$

Let us now define the ideal continuous measurement of a given time-dependent observable $A(t)$ on a certain Markovian open system.

In this paper we shall assume that the measurements are complete; i.e., the Hermitian operator $A(t)$ has a nondegenerate discrete spectrum for all t. Without restricting the generality of our work we can suppose that

$$A(t)=\sum_{n=1}^{N} nP_n(t) \ , \tag{2.7}$$

where $\{P_n(t);n=1,2,\ldots,N\}$ forms a complete orthogonal system of pure states. It is also assumed that the operator $A(t)$ and, consequently, the projectors $\{P_n(t)\}$ are smooth functions of time t. Let us recall that the $P_n(t)$'s are uniquely given by the diagonalization (2.7) of $A(t)$, up to the freedom in the labeling of the terms.

Consider now the series of moments $t=t_\alpha\equiv\alpha\epsilon$ with $\epsilon>0$ and α running over all integers; then perform an ideal measurement[4] of the quantity $A(t_\alpha)$, for each α in turn. In the usual way,[5,7] the continuous measurement of $A(t)$ is defined as the $\epsilon\to+0$ limit of the above procedure with repeated observations. We show that this definition yields a reasonable result for open quantum systems.

First we derive the stochastic process which characterizes the measured value $\bar n(t)$ of $A(t)$ (2.7) and also the measured state $\bar\rho(t)$ of the system.

Let ϵ still be finite and denote the measured value $\bar n(t_\alpha)$ of $A(t_\alpha)$ by $\bar n_\alpha$ for each α in turn. According to the standard measurement theory[6] the measured value $\bar n_\alpha$ may be some positive integer, cf. Eq. (2.7). The measured state $\bar\rho(t)$ of the system is then collapsed[6] onto the corresponding pure eigenstate of the observable:

$$\bar\rho(t_\alpha)=P_{\bar n_\alpha}(t) \tag{2.8}$$

after each measurement. In between the measurements, the measured state $\bar\rho(t)$ obeys the continuous master equation (2.1) of the system:

$$\dot{\bar\rho}(t)=L[\bar\rho(t)];\quad t_\alpha\leq t<t_{\alpha+1}\ \text{ for all } \alpha \ ; \tag{2.9}$$

therefore the collapsed pure states (2.8) become mixed when the next measurement is due to take place:

$$\begin{aligned}\bar\rho(t_{\alpha+1}-0)&=\bar\rho(t_\alpha)+\epsilon L[\bar\rho(t_\alpha)]+O(\epsilon^2)\\ &=P_{\bar n_\alpha}(t_\alpha)+\epsilon L[P_{\bar n_\alpha}(t_\alpha)]+O(\epsilon^2) \ .\end{aligned} \tag{2.10}$$

It is well known that the outcome of a quantum measurement[6] is always of a statistical nature. In our case the probability of a given outcome $\bar n_\alpha$ at $t=t_\alpha$ is

$$\mathrm{tr}[P_{\bar n_\alpha}(t_\alpha)\bar\rho(t_\alpha-0)] \tag{2.11}$$

for arbitrary α.

Then, applying Eq. (2.10) we can express the probability distribution of $\bar n_{\alpha+1}$ as the function of the previously observed $\bar n_\alpha$, viz.,

$$\omega_{\bar n_{\alpha+1}}(\bar n_\alpha)=\mathrm{tr}[P_{\bar n_{\alpha+1}}(t_{\alpha+1})\bar\rho(t_{\alpha+1}-0)]=\mathrm{tr}[P_{\bar n_{\alpha+1}}(t_{\alpha+1})P_{\bar n_\alpha}(t_\alpha)]+\epsilon\,\mathrm{tr}\{P_{\bar n_{\alpha+1}}(t_{\alpha+1})L[P_{\bar n_\alpha}(t_\alpha)]\}+O(\epsilon^2) \ . \tag{2.12}$$

At this point, because of the smooth time dependence of $A(t)$, we can replace $P_{\bar n_{\alpha+1}}(t_{\alpha+1})$ by its expansion $P_{\bar n_{\alpha+1}}(t_\alpha)+\epsilon\dot P_{\bar n_{\alpha+1}}(t_\alpha)+O(\epsilon^2)$. Thus, Eq. (2.12) has the form

$$\omega_{\bar n_{\alpha+1}}(\bar n_\alpha)=\mathrm{tr}[P_{\bar n_{\alpha+1}}(t_\alpha)P_{\bar n_\alpha}(t_\alpha)]+\epsilon\,\mathrm{tr}\{\dot P_{\bar n_{\alpha+1}}(t_\alpha)P_{\bar n_\alpha}(t_\alpha)+P_{\bar n_{\alpha+1}}(t_\alpha)L[P_{\bar n_\alpha}(t_\alpha)]\}+O(\epsilon^2) \ . \tag{2.13}$$

The first term on the RHS yields $\delta_{\bar n_{\alpha+1}\bar n_\alpha}$. The term containing $\dot P$ will vanish since the identity $\mathrm{tr}(\dot P_nP_m)=0$. In view of this the probability distribution of the measured value $\bar n_{\alpha+1}$ of $A(t_{\alpha+1})$ depends on $\bar n_\alpha$ explicitly, viz.,

$$\omega_{\bar n_{\alpha+1}}(\bar n_\alpha)=\begin{cases}\epsilon w_{\bar n_\alpha\to\bar n_{\alpha+1}}(t_\alpha)+O(\epsilon^2) & \text{if } \bar n_{\alpha+1}\neq\bar n_\alpha \ ,\\ 1-\epsilon w_{\bar n_\alpha\to}(t_\alpha)+O(\epsilon^2) & \text{if } \bar n_{\alpha+1}=\bar n_\alpha \ ,\end{cases} \tag{2.14}$$

where the transition rates w were introduced according to the notations (2.5) and (2.6). One can see that the series $(\bar n_\alpha)$ of the measured values is governed by a certain Markovian stochastic process with the transition matrix given by the formula (2.14) up to ϵ^2 terms

We now take the $\epsilon\to+0$ limit of the above measurement procedure.

From Eqs. (2.8) and (2.9) we see that when ϵ goes to zero, the measured state $\bar\rho(t)$ tends—at least weakly—to a certain pure state of the form

$$\bar\rho(t)=P_{\bar n(t)}(t) \tag{2.15}$$

for all t. At the same time, the measured label function $\bar n(t)$ will obey the continuous analogue of the discrete stochastic process (2.14). Namely, the transition rate from a given integer value, say n of $\bar n(t)$, to another m, will be

$$w_{n\to m}(t)=\mathrm{tr}\{P_m(t)L[P_n(t)]\} \ . \tag{2.16}$$

Equations (2.15) and (2.16) are the main goal of this paper. It follows from these that if at a given moment t_1 we observed the open system in the nth eigenstate of the observable $A(t_1)$ then for a period $t_1\leq t\leq t_2$ the measured state will be equal to the (time-dependent) nth eigenstate, i.e.,

$$\bar\rho(t)=P_n(t),\ \ t_1\leq t\leq t_2 \ , \tag{2.17}$$

with the probability

$$\exp\left[-\int_{t_1}^{t_2} w_{n\to}(t)\,dt\right]\equiv\exp\left[\int_{t_1}^{t_2}\mathrm{tr}\{P_n(t)L[P_n(t)]\}\,dt\right] \ ; \tag{2.18}$$

but $\bar\rho(t)$ can decay into some other eigenstate $P_m(t)$ with the transition rate (2.16).

Qualitatively speaking, the measuring apparatus captures the state $\bar{\rho}(t)$ of the system in a certain time-dependent eigenstate of the observed quantity $A(t)$. The dynamics of the open system may, however, overcome the constraining effect of the measurement and the measured state $\bar{\rho}(t)$ will jump stochastically to another eigenstate with given transition rates.

Conversely, if the system is Hamiltonian, then $L[.] = -i[H,.]$, and thus the transition rates (2.16) vanish. The state $\bar{\rho}(t)$ will then be captured forever in a given eigenstate of the observable operator. This is just the Zeno paradox which, as we can see now, is connected with Hamiltonian quantum systems.

And finally, let us derive the evolution equation for the density operator $\rho(t)$ of the continually observed open system.

We have shown that the measured quantum state $\bar{\rho}(t)$ of the system is always a pure state (2.15) governed by the continuous stochastic process with transition rates (2.16). Obviously, the density operator $\rho(t)$ must be expressed by the stochastic mean of the measured state $\bar{\rho}(t)$:

$$\rho(t) = \langle \bar{\rho}(t) \rangle_{\text{mean}} = \langle P_{\bar{n}(t)}(t) \rangle_{\text{mean}} \quad . \tag{2.19}$$

The RHS takes diagonal form according to the distribution $\omega_n(t)$ of the probabilities of $\bar{n}(t) = n$ at time t, $n = 1, 2, \ldots, N$:

$$\rho(t) = \sum_{n=1}^{N} p_n(t) P_n(t) \quad , \tag{2.20}$$

where $p_n(t) \geq 0$ for all n and $\sum_{n=1}^{N} p_n(t) = 1$.

We present the evolution equation for $\rho(t)$ (2.20) in terms of a master equation for the probabilities $p_n(t)$. It is trivial to show that they obey a Pauli-type master equation with transition rates (2.16):

$$\dot{p}_n(t) = \sum_{\substack{m=1 \\ m \neq n}} p_m(t) w_{m \to n}(t) - p_n(t) w_{n \to}(t) \quad . \tag{2.21}$$

Thus, Eqs. (2.20) and (2.21) yield the deterministic evolution equation for the expected state $\rho(t)$ of the open system during the continuous observation of the quantity $A(t)$, while the pure state $\bar{\rho}(t)$, actually measured in a given continuous observation process, satisfies the stochastic evolution rule (2.15) and (2.16). For better understanding let us consider an analogy with classical mechanics: The density distribution of a Brownian particle obeys the diffusion equation which is deterministic; the individual path of the particle is governed, however, by the Wiener stochastic process.

It is worthwhile to stress that the ideal continuous measurement process explained in our paper represents a different point of view with respect to the theory of Barchielli and co-workers[1-3] as well as to other works (see, e.g., Ref. 9). The ideal measurement process contains instantaneous collapses of the quantum state while the quoted theories work with a kind of "continuous collapse."

III. NONDEMOLITION CONTINUOUS MEASUREMENT IN PAULI'S OPEN SYSTEM

Let us apply the results of Sec. II to the simple open quantum system of Pauli.[10]

Consider the constant Hamiltonian H with a discrete nondegenerate spectrum:

$$H = \sum_{n=1}^{\infty} E_n P_n = \sum_{n=1}^{\infty} E_n |n\rangle\langle n| \quad , \tag{3.1}$$

where $\{P_n \equiv |n\rangle\langle n|;\ n = 1, 2, \ldots\}$ forms a complete orthogonal system of pure states. It will be useful to introduce the matrix notation $\rho_{nm} = \langle n|\rho|m\rangle$ for the density operator ρ.

The following master equation[10] may then be used for the diagonal elements of ρ:

$$\dot{\rho}_{nn}(t) = \sum_{\substack{m=1 \\ m \neq n}} \rho_{mm}(t) w_{m \to n} - \rho_{nn}(t) w_{n \to} \quad , \tag{3.2}$$

where $w_{n \to} = \sum_{m=1}^{\infty} w_{n \to m}$ and $\{w_{n \to m};\ n, m = 1, 2, \ldots;\ n \neq m\}$ are given constant positive transition rates calculated from the detailed dynamics of the system and its environment.

In the simplest cases, the off-diagonal elements of ρ are neglected. (In particular, they are killed by equations of type $\dot{\rho}_{nm} = -\rho_{nm}/\tau_{nm}$ with $\tau_{nm} > 0$, for all pairs of $n \neq m$.) Therefore, it is usual to suppose that ρ is always diagonal in the stationary basis:

$$\rho(t) = \sum_{n=1}^{\infty} \rho_{nn}(t) |n\rangle\langle n| \quad . \tag{3.3}$$

We now propose to measure continuously the value of Hamiltonian H (3.1) on the open system defined by Eqs. (3.2) and (3.3). Observe that $[H, \rho(t)] \equiv 0$; therefore, such measurement does not affect the density operator. A second observer would not be able to discover any change in the given open system when the apparatus measuring H is switched on or off. In view of this, surely such a measurement is of nondemolition type.[11]

In order to be consistent with Sec. II we redefine the observable $A(t)$, similarly to Eq. (2.7):

$$A(t) \equiv A \equiv \sum_{n=1}^{\infty} n |n\rangle\langle n| \quad . \tag{3.4}$$

Here the observable A is, of course, time independent, since $|n\rangle$'s are the stationary states of Hamiltonian H.

According to Eq. (2.15) the measured pure state of Pauli's open system will have the form

$$\bar{\rho}(t) = |\bar{n}(t)\rangle\langle \bar{n}(t)| \quad , \tag{3.5}$$

where the measured value $\bar{n}(t)$ of the observable A obeys the stationary continuous stochastic process with Markovian transition rates $w_{n \to m}$, cf. Eq. (2.16). In other words, the observed state of the system behaves as follows: The system remains in a given stationary state $|n\rangle$ for an average lifetime $1/w_{n \to}$ and jumps into another stationary state $|m\rangle$ ($m \neq n$) with constant transition rate $w_{n \to m}$, the coefficient of the master equation (3.2) of the Pauli system.

Invoking now Eq. (2.19) we can define the density operator $\rho(t)$ of the observed Pauli system by the stochastic mean of the pure state (3.5). We can already see at first glance that the evolution equations (2.20) and (2.21) for the observed system are identical with the master equation (3.2) for the density operator (3.3) of the free system. This fact is in line with previously stated nondemolitioness of the measurement.

Therefore, since continuous observation of operator A (3.4) [or, equivalently, that of the Hamiltonian (3.1)] causes no change to the system (except for information gained on it), we are led to a delicate point: Namely, we can claim that the pure state $|\bar{n}(t)\rangle$, evolving stochastically as dictated by the Markovian process above, yields an alternative representation for the Pauli open system. This pure state representation[12] is exactly equivalent with the density operator formalisms (3.2) and (3.3) commonly used. The equivalence is a genuine mathematical fact independent of whether the pure state $|\bar{n}(t)\rangle$ was really measured or not.

IV. CONCLUSION AND OUTLOOK

This paper was devoted to the problem of continuous measurement in open quantum systems. References 1–3 exploit the operation theory of quantum measurements. Operations may be the most suitable tool, especially when we measure quantities with a continuous spectrum as in Refs. 1–3. Nevertheless, we wish to emphasize that the quantum-mechanical Zeno paradox does not veto the concept of ideal continuous measurement for open quantum systems.

We derived the appropriate equations in Sec. II and applied them to the simple open system of Pauli. Moreover, we showed an example of ideal nondemolition continuous observation leading to the pure state representation of the Pauli open system. Another example was given earlier for a particle affected by white-noise potential.[13] Here we guess that nondemolition ideal continuous measurement should exist for any given open quantum system.

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