EXACT SOLUTION FOR PARTICLE TRAJECTORIES IN MODIFIED QUANTUM MECHANICS

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Considering a simple quantum master equation for a free particle we prove that after a given period of relaxation the solutions can be interpreted exactly in terms of uniform localized wave packets whose centers imitate classical trajectories.

1. Introduction

According to ordinary quantum mechanics, a generic wave packet of a free particle will permanently expand so the particle trajectory becomes undefined. While this phenomenon is deeply understood for the case of elementary objects we would nevertheless expect the trajectories to be restored for macroscopic particles. However, unitary evolution of the wave function must evidently be violated in order to have localized stationary wave packets for a free particle.

For example, by appending a non-linear term to the ordinary Schrödinger equation, it acquires soliton solutions [1,2]. However there are several reasons [3,4] to preserve the linearity of the equation of motion. We shall concentrate on the simplest quantum master equation [5] for the density operator $\hat{\rho}(t)$ of a given free particle of mass m:

$$\dot{\hat{\rho}}(t)=L[\hat{\rho}(t)]\equiv-\mathrm{i}[\hat{p}^2/2m,\hat{\rho}(t)]-\tfrac{1}{2}D[\hat{x},[\hat{x},\hat{\rho}(t)]]\quad(\hbar=1)\,, \tag{1}$$

where double commutator violates the unitary evolution. ($\hat{x}$ and $\hat{p}$ are operators for the cartesian canonical coordinates and momenta, respectively; the constant D couples the non-unitary term.) Such a master equation has frequently been suggested [6–10] to eliminate unwanted long distance coherence from ordinary quantum theory. Note that the same equation appears in fact if the particle is thought to interact with a reservoir whose dynamics we integrate out, see e.g. refs. [5,8].

In this paper, in section 2, we shall present a theorem showing an exact solution to the master equation (1) in terms of uniform wave packets which move along random trajectories. Our results represent a development of statements and proofs which have appeared recently in the works of Joos and Zeh [8], Ghirardi, Rimini and Weber [10] as well as of the present author [9].

2. Preparation

If $\hat{Q}$ stands for a certain observable quantity then for the time derivative of its expectation value, eq. (1) yields

$$(\mathrm{d}/\mathrm{d}t)\,\mathrm{Tr}\{\hat{\rho}(t)\hat{Q}\}=\mathrm{Tr}\{\hat{\rho}(t)L^*[\hat{Q}]\}\equiv\mathrm{Tr}\{\hat{\rho}(t)\ (\mathrm{i}[\hat{p}^2/2m,\hat{Q}]-\tfrac{1}{2}D[\hat{x},[\hat{x},\hat{Q}]])\}\,, \tag{2}$$

where L^* is the dual [5] evolution operator.

In what follows, we shall often use the characteristic length

$$\sigma=(2Dm)^{-1/4} \tag{3}$$

and time

$$\tau=(2m/D)^{1/2} \tag{4}$$

instead of the original parameters m and D.

In previous papers [11,12] we suggested that each markovian master equation $\dot{\hat{\rho}}=L[\hat{\rho}]$ is *associated* with a given non-linear Schrödinger equation

$$\dot{\psi}=-\mathrm{i}\hat{H}_{\mathrm{fr}}\psi\equiv L[\psi\psi^+]\psi-(\psi^+L[\psi\psi^+]\psi)\psi\,,$$

where ψ is the state vector and $\hat{H}_{\mathrm{fr}}$ denotes the so-called frictional hamiltonian [13]. For the case of eq. (1) this yields [11]

$$\dot{\psi}=-\mathrm{i}\hat{H}_{\mathrm{fr}}\psi\equiv[-\mathrm{i}(\hat{p}^2/2m)-\tfrac{1}{2}D(\hat{x}-x_\psi)^2+\tfrac{1}{2}D\sigma_\psi^2]\psi;\quad x_\psi\equiv\psi^+x\psi\,,\quad \sigma_\psi^2\equiv\psi^+\hat{x}^2\psi-x_\psi^2\,. \tag{5}$$

For short enough periods, this frictional Schrödinger equation is physically equivalent [13] to the master equation (1). This equivalence breaks down for longer periods, of course. Nevertheless, it is a surprising fact that certain solutions of the frictional Schrödinger equation play a central role in constructing exact solutions of eq. (1).

So, observe that eq. (5) possesses stationary solutions with localized gaussian shaped wavefunctions. Let ψ_0 denote the state whose wave packet rests at the origin:

$$\psi_0(x)=(2\pi\sigma^2)^{-3/4}\exp[(\mathrm{i}-1)\,(x/2\sigma)^2]\,. \tag{6}$$

It can be shown that $\psi_0\exp(-\mathrm{i}t/2\tau)$ satisfies eq. (5). By applying translation and boost to this solution, one can generate new localized solutions of eq. (5) in the form

$$\psi_\Gamma(x-\bar{p}t/m)\exp[-\mathrm{i}t(\tfrac{1}{2}\tau^{-1}+\tfrac{1}{2}\bar{p}^2/m)]\,,$$

where

$$\psi_\Gamma(x)=\psi_0(x-\bar{x})\exp[\mathrm{i}\bar{p}(x-\bar{x})]=(2\pi\sigma^2)^{-3/4}\exp[(\mathrm{i}-1)\,[(x-\bar{x})/2\sigma]^2+\mathrm{i}\bar{p}(x-\bar{x})] \tag{7}$$

and the compact notation

$$\Gamma=(\bar{x},\bar{p}) \tag{8}$$

stands for the expectation values of the canonical variables in the state $\psi_\Gamma(x)$. When Γ runs over the whole phase space the vectors $\{\psi_\Gamma\}$ form an overcomplete system in the Hilbert space of the particle states.

Let us introduce the density operators $\hat{Q}(\Gamma)$ belonging to the localized solutions (7) of the frictional Schrödinger equation:

$$\hat{Q}(\Gamma)=\psi_\Gamma\psi_\Gamma^+\,. \tag{9}$$

Using now the wavefunctions (7) along with the definitions (1) and (2) of the evolution operators, one can prove the following relations:

$$\mathrm{Tr}[\hat{Q}(\Gamma')\hat{Q}(\Gamma)]=\exp[-\tfrac{1}{2}(\Gamma'-\Gamma)M(\Gamma'-\Gamma)]\,, \tag{10}$$

$$L[\hat{Q}(\Gamma)]=\left(\frac{\bar{p}}{m}\frac{\partial}{\partial\bar{x}}+\frac{1}{2\tau}\frac{\partial}{\partial\Gamma}M^{-1}\frac{\partial}{\partial\Gamma}\right)\hat{Q}(\Gamma)\,, \tag{11}$$

$$L^*[\hat{Q}(\Gamma)]=\left(-\frac{\bar{p}}{m}\frac{\partial}{\partial\bar{x}}-\frac{1}{2\tau}\frac{\partial}{\partial\Gamma}M^{-1}\frac{\partial}{\partial\Gamma}+D\frac{\partial^2}{\partial\bar{p}^2}\right)\hat{Q}(\Gamma)\,. \tag{12}$$

The constant positive matrices M and M^{-1} are defined by

$$M=\begin{bmatrix}\sigma^{-2} & -1\\ -1 & 2\sigma^2\end{bmatrix},\quad M^{-1}=\begin{bmatrix}2\sigma^2 & 1\\ 1 & \sigma^{-2}\end{bmatrix}. \tag{13}$$

Theorem. Assume that the density operator $\hat{\rho}(t)$ satisfies the master equation (1) and let $\hat{\rho}(0)$ be an arbitrarily given initial state at $t=0$. Then for $t>\text{const}\times\tau$ the density operator takes the following form:

$$\hat{\rho}(t)=\int f(\Gamma,t)\hat{Q}(\Gamma)\,\mathrm{d}\Gamma\,,\quad \mathrm{d}\Gamma\equiv\frac{\mathrm{d}\bar{x}\mathrm{d}\bar{p}}{2\pi}\,, \tag{14}$$

where f is a time dependent distribution over the phase space, i.e.

$$f(\Gamma,t)\geqslant 0\,,\quad \int f(\Gamma,t)\,\mathrm{d}\Gamma\equiv 1\,, \tag{15}$$

whose evolution is governed by the following Fokker–Planck diffusion equation:

$$\dot{f}(\Gamma,t)=\left(-\frac{\bar{p}}{m}\frac{\partial}{\partial\bar{x}}+\frac{1}{2\tau}\frac{\partial}{\partial\Gamma}M^{-1}\frac{\partial}{\partial\Gamma}\right)f(\Gamma,t)\,. \tag{16}$$

Proof. Let us postpone the proof of the relaxation property and *assume temporarily* that the expansion (14) exists for a certain moment. Then we are going to show that eq. (16) ensures that the expansion (14) remains valid in the future too.

Thus, let us assume the expansion (14) exists for a certain t and express the time derivative via eq. (1):

$$\dot{\hat{\rho}}(t)=\int f(\Gamma,t)L[\hat{Q}(\Gamma)]\,. \tag{17}$$

Observe now that, due to relation (11), the time derivative $\dot{\hat{\rho}}(t)$ *remains in the linear space spanned by the* $\hat{Q}$. By substituting expression (11) into the r.h.s. of eq. (17) and by partial integration we get

$$\dot{\hat{\rho}}(t)=\int\left[\left(-\frac{\bar{p}}{m}\frac{\partial}{\partial\bar{x}}+\frac{1}{2\tau}\frac{\partial}{\partial\Gamma}M^{-1}\frac{\partial}{\partial\Gamma}\right)f(\Gamma,t)\right]\hat{Q}(\Gamma)\,\mathrm{d}\Gamma\,. \tag{18}$$

So it is easy to see that the expansion (14) will hold also at time $t+\mathrm{d}t$ (and so later as well) if the coefficient $f(\Gamma,t)$ satifies the Fokker–Planck equation (16).

We are still owing the proof that for an arbitrarily chosen initial state $\hat{\rho}(0)$, eq. (14) will get a solution $f(\Gamma, t)$ after a certain relaxation period of several times τ (4). First, let us find the *formal* solution of eq. (14).

Let us multiply eq. (14) by $\hat{Q}(\Gamma)$, take the trace of both sides and substitute relation (10) in it:

$$\mathrm{Tr}[\hat{\rho}(t)\hat{Q}(\Gamma)]=\int\exp[-\tfrac{1}{2}(\Gamma-\Gamma')M(\Gamma-\Gamma')]f(\Gamma',t)\,\mathrm{d}\Gamma'\,. \tag{19}$$

In order to invert this equation it is worthwhile to introduce the following Fourier transformations:

$$\tilde{f}(\Omega,t)=\int\exp(\mathrm{i}\Omega\Gamma)f(\Gamma,t)\,\mathrm{d}\Gamma\,, \tag{20}$$

$$\tilde{\hat{Q}}(\Omega)=\int\exp(\mathrm{i}\Omega\Gamma)\hat{Q}(\Gamma)\,\mathrm{d}\Gamma\,, \tag{21}$$

where

$$\Omega \equiv (P, -X) . \tag{22}$$

In the Fourier transformed representation the convolution equation (19) can formally be solved as

$$\tilde{f}(\Omega, t) = \exp(\tfrac{1}{2}\Omega M^{-1}\Omega)\ \mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)] . \tag{23}$$

However, since the matrix M (13) is positive definite, the r.h.s. of eq. (23) may have no Fourier inverse at $t=0$. Nevertheless, we are going to show that, as time goes by, the factor $\mathrm{Tr}[Q(\Omega)\rho(t)]$ will compensate the blow up of the exponential for $\Omega\to\infty$ and so the *formal solution (23) becomes valid for large enough t.* So we need to discuss the time evolution of the term $\mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)]$.

Let us evaluate the time derivative of $\mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)]$. Using formula (12) and applying Fourier transformation to it, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)] = \left[\frac{P}{m}\frac{\partial}{\partial X} + \frac{1}{2\tau}\Omega M^{-1}\Omega - DX^2\right]\mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)] . \tag{24}$$

This equation yields the following solution:

$$\mathrm{Tr}[\tilde{\hat{Q}}(\Omega)\hat{\rho}(t)] = \exp[(t/2\tau)(\Omega M^{-1}\Omega - 2\sigma^{-2}X^2)]\ \mathrm{Tr}[\tilde{\hat{Q}}(\Omega_t)\hat{\rho}(0)] \tag{25}$$

with the notation

$$\Omega_t = (P, -X_t) \equiv (P, -X - Pt/m) . \tag{26}$$

Consider now the definition (7), (9) of the states $\hat{Q}$ and perform the Fourier transformation (21); in coordinate representation one obtains

$$\langle x|\tilde{\hat{Q}}(\Omega)|y\rangle = \delta(x-y-X)\ \exp\{-(1/8\sigma^2)[(x-y)^2 + (x-y-2\sigma^2 P)^2] + \tfrac{1}{2}\mathrm{i}P(x+y)\} . \tag{27}$$

Substituting this result into eq. (25), the r.h.s. of eq. (23) can be rewritten as follows:

$$\tilde{f}(\Omega, t) = \exp[\tfrac{1}{2}\Omega M^{-1}\Omega + (t/2\tau)(\Omega M^{-1}\Omega - 2\sigma^{-2}X^2)]\int \mathrm{d}x\,\mathrm{d}y\ \langle y|\hat{\rho}(0)|x\rangle\,\delta(x-y-X-Pt/m)$$

$$\times\exp\{-(1/8\sigma^2)[(x-y)^2 + (x-y-2\sigma^2 P)^2] + \tfrac{1}{2}\mathrm{i}P(x+y)\} . \tag{28}$$

As was mentioned above (cf. paragraph after eq. (23)) we expect that $\tilde{f}(\Omega, t)$ will not diverge for $\Omega\to\infty$ if t exceeds a certain threshold. Indeed, by collecting the leading terms on the r.h.s. of eq. (28) we can evaluate its asymptotic Ω dependence as follows :

$$\tilde{f}(\Omega, t) = \int \mathrm{d}x\,\mathrm{d}y\,\delta(x-y-X-Pt/m)\ \exp\{-\tfrac{1}{2}\sigma^2 P^2[4(t/\tau)^3 - 8(t/\tau)^2 - 6t/\tau - 1] + \mathrm{O}(P)\} . \tag{29}$$

Thus, $\tilde{f}(\Omega, t)$ will allow inverse Fourier transformation if

$$4(t/\tau)^3 > 8(t/\tau)^2 + 6(t/\tau) + 1 , \tag{30}$$

so the existence of $f(\Gamma, t)$ and of the expansion (14) has been proved for $t > \mathrm{const}\times\tau \approx 2.611\tau$.

3. Interpretation

Let us expand the r.h.s. of the Fokker–Planck diffusion equation (16). Invoking the definition (13) of M^{-1} we get

$$\dot{f}(\Gamma, t) = \left[-\frac{\bar{p}}{m}\frac{\partial}{\partial x} + \frac{1}{2\tau}\left(2\sigma^2\frac{\partial^2}{\partial\bar{x}^2} + 2\frac{\partial^2}{\partial\bar{x}\partial\bar{p}} + \sigma^{-2}\frac{\partial^2}{\partial\bar{p}^2}\right)\right]f(\Gamma, t) . \tag{31}$$

Almost the same equation (i.e. with $(\partial^2/\partial\bar{x}\partial\bar{p})f$ term $2^{1/2}$ times larger) was derived first in ref. [10]. If one assumes that eq. (1) governs the dynamics of the center of mass motion for a given macroscopic object then an appealing interpretation follows: After a finite transient period of order τ (4), the wave packet of the object has a constant gaussian shape of width σ (3) and its center moves along random trajectory characterized by the diffusion equation (31). So classical behaviour has been restored if the parameter D in eq. (1) was properly chosen. Let us emphasize that our eq. (31) is exact if the master equation (1) is valid.

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