ON THE MINIMUM UNCERTAINTY OF SPACE–TIME GEODESICS

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Inspired by a construction of Wigner we calculate the quantum relativistic limitations of measuring the metric tensor of a certain space–time. Our suggestion is that the result is an estimate for fluctuations of g_{ab} whose rigorous determination will be a subject of a future relativistic quantum gravity.

Although various attempts for systematic quantization of the space–time geometry ("gravitation") have appeared, none of them is considered fully consistent or final. Still, even without a consistent quantization method one can estimate the obligatory quantum unsharpness. The method goes back to the early trick of Landau and Peierls [1], originally applied to the electromagnetic field. In their gedankenexperiment a quantum particle was placed into a classical electromagnetic field, in order to test the field strength, and, due to quantum uncertainties of the probe coordinates and momenta, the classical field strength turned out to be tested with a certain uncertainty. Later Bohr and Rosenfeld [2] showed that, by optimal preparation of the probe, this uncertainty coincides with the quantum unsharpness of the field strength in the standard quantization of the electromagnetic field.

This success has encouraged transplantation of the above method to test quantum fluctuations (whatever they are) of the space–time geometry. For some 50 years, various gedankenexperiments [3–10] were elaborated where quantum particles may test the local curvature, the Christoffels or, alternatively, the geodesics. It seems that different works suggest different unsharpness for the space–time. Remind that in our case, unlike quantum electrodynamics, still no final criteria are within reach to verify a specific estimation. Consequently, our work is not intended to suggest an exclusive solution to the problem. We confine ourselves to find the *minimal uncertainties of geodesics*. Here we essentially append Wigner's original thoughts [3,4] by a gedankenexperiment [6–8] proposed first in 1966 and do not aim to resolve discrepancies among the great number of other works mentioned above.

Obviously, the ultimate object of measurement is the space–time geometry. It is better to avoid constructions exploiting rigid rods, etc. Fortunately, the measurement can be performed purely via time measurements on time-like geodesics and Wigner elaborated such a construction [3,4]. There we need a *net* of time-like geodesics as tight as possible. However, these geodesics are realized by real bodies (clocks) subject to quantum physics. Therefore, for a length $s=cT$ a geodesic will develop a space-like uncertainty at least [4,8,10]

$$\Delta x \approx (\hbar s/Mc)^{1/2}\,, \tag{1}$$

where M is the mass of the body. Then geodesics become time-like world tubes rather than sharp world lines. Next, Wigner imposed a plausible condition on the expected accuracy of measuring s:

$$\Delta s \approx \Delta x\,, \tag{2}$$

i.e. space-like and time-like accuracies are identical. (Another possible argument by ref. [8]: cutting a world tube is regarded to be a causal process.) Even here we mention that this restriction of cutting seems too strict; we will return to this condition later.

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Now the net is to be tightened as far as possible, but the clocks travelling along the geodesics have some size R which is a limit for the tightening. For $R > \Delta x$ the net is unnecessarily thin. So a suitable condition is

$$R \approx \Delta x . \tag{3}$$

A smaller R is not convenient, either, as we shall see later.

Here the original construction stops but we must continue as far as our goal is considered. Namely, by increasing the mass M of the clock the accuracy of measuring the metric increases without limit. However, the masses distort the space–time just under measurement. This distortion leads to a further deviation in measuring the length s, giving the estimation [8]

$$\Delta s' \approx (GM/Rc^2) s . \tag{4}$$

Now one sees that a small distortion $\Delta s'$ needs large R, cf. our remark after the condition (3). One intends to measure the structure of the space–time so the quantum uncertainty (2) and the distortion (4) are both limitations. The first one decreases by taking a larger mass M while the second increases. There is an optimum at [8]

$$\Delta s \approx \Delta s' . \tag{5}$$

Eqs. (1)–(5) build up a closed system of conditions to derive minimum uncertainties. We are not yet ready, however. First, ref. [8] did not choose this way. It adopted conditions (1), (2) of Wigner, appended them by the new conditions (4), (5) but condition (3) was replaced by the less strong one $R < s$. (The extension of the clock could be much larger than the width of the world tube and it is limited only by causality.) Hence ref. [8] concluded in an *absolute* limitation on defining the length of an *individual* geodesic as

$$(\Delta s)^2 \approx \Lambda^{4/3} s^{2/3} , \tag{6}$$

where Λ is the Planck length. (See also refs. [6,7].)

Now, note that by taking literally this construction the optimal mass of the clock would be about $\hbar c^3 s/G^2$ which is enormously large. This follows from the large size ($R \approx s$) of the clock. Such a clock may measure quite well the length of an *individual* geodesic, it is hardly optimal for constructing a tight net. Therefore for our goal we suggest to retain the original condition (3).

The system (1)–(5) is closed and would lead to an absolute limitation (viz. $(\Delta s)^2 \approx \Lambda s$) in measuring the distances in a net so also for space–time metric. However, we have reasons to revise condition (2) as promised. No doubt, condition (2) is clearly obligatory for a simple "cutting" process. Still it does not seem to be an absolute limitation for more sophisticated ones. Ad absurdum, condition (2) would only allow the preparation of light-like hypersurfaces and the realization of space-like surfaces (believed possible in measurements) would be prohibited. Having made this general remark, we propose to pass this "causality" limitation. Even then there remains a quantum one: the localization of a free body along its world line can never be determined better than its Compton wavelength. Hence in the optimal case

$$\Delta s \approx \hbar / Mc . \tag{7}$$

Our proposed system of conditions is thus (1)–(5) but with (7) instead of (2). Eqs. (3)–(5) and (7) lead to the absolute limit

$$(\Delta s)^2 \approx \Lambda^2 s / R \tag{8}$$

via the optimal mass $(\hbar R/GT)^{1/2}$. Recalling that we aim at a tight net of geodesics, R is to be regarded rather as the cell size of the net.

Formally condition (1) restricts the validity of eq. (8) by limiting the length s. However, the squared accuracy $(\Delta s)^2$ of measuring s is *proportional* to s hence successive independent measurements possess uncorrelated inaccuracies. Consequently, eq. (8) remains valid for a world line of any (large) length s since one can measure its length by successive measurements on shorter periods (cf. also ref. [10]).

The content of eq. (8) is an inevitable (and, if the present conditions are correct, ultimate) uncertainty of measured lengths of time-like geodesics. Via Wigner's construction, this uncertainty propagates into the space–time geometry which therefore cannot be sharply measured, so there is no possibility to consider it sharp.

Now we are in the position to calculate the uncertainty of the metric tensor g_{ab}. Here it is done in the simplest case when the gravitation is weak and the geometry is nearly static. Then the metric tensor is Minkowskian except for $g_{00}=1-2\Phi/c^2$ where Φ is the Newton potential ($|\Phi|\ll c^2$). For simplicity we assume the background to be just the Minkowski metric and thus $\Phi\equiv 0$ in average. The uncertainty Δg_{ab} of the metric is contained in Φ:

$$\Delta g_{00}\approx\Phi/c^2\,. \tag{9}$$

Since it seems hardly possible that large relative velocities in the net of geodesics could improve the accuracy, we have chosen a net with vanishing relative velocities. For a period T, a clock measures a proper time about

$$\int\sqrt{1-2(\Phi)_R/c^2}\;c\,\mathrm{d}t\approx[1-(\Phi)_{R,T}/c^2]s\,, \tag{10}$$

where $(\Phi)_R$, $(\Phi)_{R,T}$ denote the Newton potential averaged over the volume of the clock and, respectively, over the volume and the time T as well. From eq. (10) the uncertainty of the length s is $\approx(\Phi)_{R,T}s/c^2$ and this has to coincide with Δs of eq. (8). As a result, one obtains the unsharpness (9) of the metric by

$$(\Phi)_{R,T}\approx\sqrt{\hbar G/RT}\,. \tag{11}$$

It is true that the uncertainty decreases with increasing R or T; but then the measured quantity is less and less the local metric.

Following e.g. ref. [8] or ref. [10], one may consider Φ as a stochastic variable whose fluctuations yield just the measurement accuracy (11). In ref. [10] it has been proved that the proper correlation function of Φ is of the form

$$\langle\Phi(\boldsymbol{r},t)\Phi(\boldsymbol{0},0)\rangle=\mathrm{const}\times\hbar G r^{-1}\delta(t)\,. \tag{12}$$

Two appealing features are to be noticed here. First, c has been cancelled in eq. (12) (as well as in eq. (11)) thus the non-relativistic Newton potential possesses even a non-relativistic quantum fluctuation. The existence of a genuine *non-relativistic* unsharpness has been predicted for the Newtonian gravity [10]. (This would not happen with the alternative conditions.) Second, the correlation function is of white noise type because of $(\Delta s)^2\approx s$ implying statistical independence of fluctuations of Φ at different times.

Alternatively, had we chosen eq. (6) instead of eq. (8) $(\Delta s)^2$ would be *not* proportional to s therefore uncertainties along a given world line would have to be statistically correlated. Then, obviously, Φ would not be a white noise. In refs. [7] and [8] the following correlation function has been derived:

$$\langle\Phi(\boldsymbol{r},t)\Phi(\boldsymbol{0},0)\rangle=\mathrm{const}\times(\hbar G)^{2/3}c^2r^{-1}$$
$$\times[\mathrm{sgn}(r_+)|r_+|^{-1/3}+\mathrm{sgn}(r_-)|r_-|^{-1/3}]\,, \tag{13}$$

where $r_+=r+ct$ and $r_-=r-ct$. (Note that refs. [7,8] give the correlation (13) for the Fourier coefficients of $\gamma\equiv-2\Phi/c^2$.)

As demonstrated e.g. in refs. [6–8,10,11], stochastic fluctuations of g_{ab} offer some natural mechanism for *spontaneous reduction of wavefunctions*. Comparing the correlation functions (12) and (13) one can expect technical and quantitative differences between the corresponding reduction mechanisms as well.

If one cannot find a very sophisticated way to circumvent our accepted "quantum relativistic" limitations of measuring lengths of geodesic world tubes (1), (3)–(5), (7), then eq. (8) gives the final unsharpness in determining the distances in a space–time net drawn to measure the space–time geometry. (Note that even then one individual geodesic can be measured with higher accuracy, but that is not enough to determine g_{ab}.) If one represents the corresponding unsharpness of the geometry by properly adjusted stochastic fluctuations of the metric tensor, then, in the weak field approximation, it turns out that the fluctuations of the non-relativistic Newton potential are of white noise type and remain non-relativistic. Both characteristics are rather attractive.

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