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Unique quantum paths by continuous diagonalization of the density operator

Lajos Diósi [1]

KFKI Research Institute for Particle and Nuclear Physics, P.O. Box 49, H-1525 Budapest 114, Hungary

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Abstract

In this short note we show that for a Markovian open quantum system it is always possible to construct a unique set of perfectly consistent Schmidt paths, supporting quasi-classicality. Our Schmidt process, elaborated several years ago, is the $\Delta t \to 0$ limit of the Schmidt chain constructed very recently by Paz and Zurek.

1. Introduction

In a very recent work, Paz and Zurek [1] discussed Markovian open quantum systems and constructed Schmidt paths showing exact decoherence. The authors, however, noticed their Schmidt paths were quite unstable under, e.g., varying the number of subsequent projections. They thought to eliminate the problem by tuning time intervals between subsequent projections larger than the typical decoherence time.

In the present short note I propose the opposite. Let time intervals be infinitely short! It means that the frequency of the subsequent projections is so high that the Schmidt path will be defined for any instant t in the period considered. This infinite frequency limit exists and provides a unique consistent set of Schmidt paths. The issue was discussed [2] and completely solved [3] several years ago. Of course, the above limit is only valid up to Markovian approximation. In fact, the "infinitesimal" repetition interval is still longer than the response time of the reservoir.

For the sake of better distinction between ordinary [1] and hereinafter advocated [3] Schmidt paths, let us call them Schmidt *chain* and Schmidt *process*, respectively.

In the next section, Schmidt chains are briefly reviewed. Sect. 3 will recall the earlier results available now for Schmidt processes. Subsequently, in Sect. 4, we propose an application of Schmidt processes to the quantum Brownian motion where classicality might be demonstrated.

2. Schmidt path–Markov chain

Consider the reduced dynamics of a given subsystem,

$$\rho(t') = J(t'-t)\rho(t) \quad (t' > t), \tag{1}$$

where ρ is the reduced density operator, J is the Markovian evolution superoperator. For a given sequence $t_0 < t_1 < \ldots < t_n$ of selection times, let us have the corresponding sequence of pure state (Hermitian) projectors: $\{P^0, P^1, \ldots, P^n\}$ is a *Schmidt chain* if

$$[P^{k+1}, J(t_{k+1}-t_k)P^k] = 0, \quad k = 0, 1, 2, \ldots, n-1, \tag{2}$$

[1] E-mail: diosi@rmki.kfki.hu.

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cf. Sect. 4 of Ref. [1]. For a fixed initial state $P^0 = \rho(t_0)$, let the probabilities

$$p(P^1, P^2, \ldots, P^n) = \mathrm{tr}[P^n J(t_n - t_{n-1}) P^{n-1} \ldots P^1 J(t_1 - t_0)\rho(t_0)] \tag{3}$$

be assigned to Schmidt chains. Schmidt chains satisfy the following sum rule,

$$\sum_{\text{Schmidt paths}} p(P^1, P^2, \ldots, P^n) P^1 \otimes P^2 \otimes \ldots \otimes P^n = \rho(t_1) \otimes \rho(t_2) \otimes \ldots \otimes \rho(t_n) \tag{4}$$

assuring the *consistency* [4] of the probability assignments (3).

Let us construct concrete Schmidt chains. To satisfy Eq. (2) for $k = 0$, let us first diagonalize the positive definite operator $J(t_1 - t_0)P^0$,

$$J(t_1 - t_0)P^0 = \sum_\alpha p^1_\alpha P^1_\alpha. \tag{5}$$

If our choice is $P^1 = P^1_{\alpha_1}$, whose probability is p_{α_1}, consider Eq. (2) for $k = 1$ and diagonalize $J(t_2 - t_1)P^1_{\alpha_1}$,

$$J(t_2 - t_1)P^1_{\alpha_1} = \sum_\alpha p^2_\alpha P^2_\alpha. \tag{6}$$

Single out $P^2 = P^2_{\alpha_2}$ at random, with probability $p^2_{\alpha_2}$, etc.

For the Schmidt chain $\{P^0, P^1_{\alpha_1}, P^2_{\alpha_2}, \ldots, P^n_{\alpha_n}\}$ one generates from the fixed initial state P^0, probability (3) takes the following factorized form,

$$p(P^1_{\alpha_1}, P^2_{\alpha_2}, \ldots, P^n_{\alpha_n}) \equiv p(\alpha_1, \alpha_2, \ldots, \alpha_n) = p^1_{\alpha_1} p^2_{\alpha_2} \ldots p^n_{\alpha_n}. \tag{7}$$

The Schmidt chain is the Markov chain. Given $P^0 = \rho(t_0)$, it will branch at t_1 into $P^1_{\alpha_1}$, i.e., into one of the eigenstate projectors of $J(t_1 - t_0)P^0$; the branching probability $p^1_{\alpha_1}$ is the corresponding eigenvalue. In the general case, $P^k_{\alpha_k}$ will branch into $P^{k+1}_{\alpha_{k+1}}$, i.e., into a certain eigenstate of $J(t_{k+1} - t_k)P^k_{\alpha_k}$, with branching probability $p^{k+1}_{\alpha_{k+1}}$ given by the corresponding eigenvalue.

3. Schmidt path–Markov process

In this section, we consider the limiting case of the Schmidt chains when the separations $t_{k+1} - t_k$ go to zero. The pure state path $\{P(t);\ t > t_0\}$, starting from the fixed initial state $P(t_0) = \rho(t_0)$, is a *Schmidt process* if (for $t > t_0$ and $\epsilon \equiv \mathrm{d}t > 0$) $P(t + \epsilon)$ branches into an eigenstate projector $P_\alpha(t)$ of $J(\epsilon)P(t)$ while the branching probability is the corresponding eigenvalue $p_\alpha(t)$. Branching rates $w_\alpha(t)$ are worthwhile to introduce by $p_\alpha(t) = \epsilon w_\alpha(t)$.

We follow the general results obtained in Ref. [3]. Let us introduce the Liouville superoperator L generating the Markovian evolution (1),

$$J(\epsilon) = 1 + \epsilon L. \tag{8}$$

Assume the Lindblad form [5],

$$L\rho = -\mathrm{i}[H, \rho] - \tfrac{1}{2}\sum_\lambda (F^\dagger_\lambda F_\lambda \rho + \rho F^\dagger_\lambda F_\lambda - 2F_\lambda \rho F^\dagger_\lambda), \tag{9}$$

where H is the Hamiltonian and $\{F_\lambda\}$ are the Lindblad generators. Following the method of Ref. [3], introduce the *frictional* (i.e. nonlinear-non-Hermitian) *Hamiltonian*

$$H_{\mathrm{P}} = H - \frac{1}{2\mathrm{i}}\sum_\lambda (\langle F^\dagger_\lambda\rangle F_\lambda - \mathrm{h.c.}) - \tfrac{1}{2}\mathrm{i}\sum_\lambda (F^\dagger_\lambda - \langle F^\dagger_\lambda\rangle)(F_\lambda - \langle F_\lambda\rangle) + \tfrac{1}{2}\mathrm{i}w \tag{10}$$

and the nonlinear positive definite *transition rate operator*

$$W_{\mathrm{P}} = \sum_\lambda (F_\lambda - \langle F_\lambda\rangle) P (F^\dagger_\lambda - \langle F^\dagger_\lambda\rangle), \tag{11}$$

where, e.g., $\langle F_\lambda\rangle \equiv \mathrm{tr}(F_\lambda P)$. We need the unit expansion of the transition rate operator,

$$W_{\mathrm{P}} = \sum_{\alpha=1}^{\infty} w_\alpha P_\alpha. \tag{12}$$

Observe that, due to the identity $W_{\mathrm{P}} P \equiv 0$, each P_α is orthogonal to P. The w_α are called *transition (branching) rates*. The total transition (branching) rate then follows from Eqs. (11) and (12),

$$w \equiv \sum_\alpha w_\alpha = \sum_\lambda (\langle F^\dagger_\lambda F_\lambda\rangle - \langle F^\dagger_\lambda\rangle\langle F_\lambda\rangle). \tag{13}$$

How to generate Schmidt processes? Given the initial pure state $\rho(t_0) = P(t_0)$, the pure state $P(t)$ evolves according to the deterministic frictional Schrödinger–von Neumann equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}P = -\mathrm{i}(H_\mathrm{P}P - PH_\mathrm{P}^\dagger) \tag{14}$$

except for discrete *orthogonal jumps* (branches)

$$P(t+0) = P_\alpha(t) \tag{15}$$

occurring from time to time at random with $P(t)$-dependent partial transition rates $w_\alpha(t)$. It is worthwhile to note that neither H_P nor W_P depend on the concrete Lindblad representation (9) of L, as it is clear in Ref. [3].

Mathematically, the above Schmidt path is the pure-state-valued Markov process of generalized Poissonian type. During a given infinitesimal period $(t,\ t+\mathrm{d}t)$, the probability of the branch-free (i.e., jump-free, continuous) evolution is $1-w(t)\,\mathrm{d}t$. Consequently, one obtains [6] the a priori probability of continuous evolution for an arbitrarily given period $(t_1,\ t_2)$ as

$$\exp\left(-\int_{t_1}^{t_2} w(t)\,\mathrm{d}t\right). \tag{16}$$

4. Classicality

Schmidt processes assure maximum classicality in "measurement situations". It has been shown in Ref. [6] that for large enough t, the Schmidt process converges to one of the pointer states while the overall probability of further branches tends to zero. Convergence is then dominated by the asymptotic solutions of the deterministic frictional Schrödinger–von Neumann equation (14).

To test classicality of Schmidt processes in less artificial situations, let us start with the (modified [7]) Caldeira–Leggett [8] master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = L\rho = -\mathrm{i}\frac{1}{2M}[p^2,\rho] - \mathrm{i}\gamma[q,\{p,\rho\}] - \tfrac{1}{2}\gamma\lambda_\mathrm{dB}^{-2}[q,[q,\rho]] - \tfrac{1}{2}\kappa\gamma\lambda_\mathrm{dB}^2[p,[p,\rho]], \tag{17}$$

where γ is (two times) the friction constant, λ_dB stands for the thermal de Broglie length of the Brownian particle of mass M. In Ref. [7] the value $\kappa = \frac{4}{3}$ has been suggested. For simplicity, we have omitted the usual renormalized potential term in the Hamiltonian, assuming it is zero or small enough. Hence we can model the quantum counterpart of pure frictional motion.

To cast Eq. (17) into the Lindblad form (9), two Lindblad generators F_1, F_2 have to be introduced in the general case. Then, applying mechanically Eqs. (10) and (11) calculate both the frictional Hamiltonian and the transition rate operator. In Ref. [3], the following expressions are directly derived from the Liouville superoperator L,

$$H_\mathrm{P} = \frac{1}{2M}p^2 + \tfrac{1}{2}\gamma\{q-\langle q\rangle, p-\langle p\rangle\} - \tfrac{1}{2}\mathrm{i}\gamma\{\lambda_\mathrm{dB}^{-2}[(q-\langle q\rangle)^2 - \sigma_{qq}^2] + \kappa\lambda_\mathrm{dB}^2[(p-\langle p\rangle)^2 - \sigma_{pp}^2]\}, \tag{18}$$

$$W_\mathrm{P} = \gamma\lambda_\mathrm{dB}^{-2}(q-\langle q\rangle)P(q-\langle q\rangle) + \kappa\gamma\lambda_\mathrm{dB}^2(p-\langle p\rangle)P(p-\langle p\rangle) - \mathrm{i}\gamma[(q-\langle q\rangle)P(p-\langle p\rangle) - (p-\langle p\rangle)P(q-\langle q\rangle)], \tag{19}$$

where $\sigma_{qq}^2 = \langle q^2\rangle - \langle q\rangle^2$ and $\sigma_{pp}^2 = \langle p^2\rangle - \langle p\rangle^2$. The total transition (branching) rate (13) takes the simple form

$$w = \gamma(\lambda_\mathrm{dB}^{-2}\sigma_{qq}^2 + \kappa\lambda_\mathrm{dB}^2\sigma_{pp}^2 - 1) \tag{20}$$

as can be easily verified by observing $w = \mathrm{tr}\,W_\mathrm{P}$.

For most of the time the Schmidt process is governed by the frictional Hamiltonian (18), via the nonlinear equation (14). This equation *itself* possesses a stationary solution $P(\infty)$ with a simple Gaussian wave function representing a standing particle. In the high temperature limit $kT >> \hbar\gamma$ the stationary wave function is $\sim \exp[-\frac{1}{2}(1-\mathrm{i})\sqrt{2\gamma kTM^2\hbar^{-3}}\,q^2]$ where the $\hbar = 1$ convention has been released in the notations. One can heuristically guess that the non-Hermitian terms of the frictional Hamiltonian establish quasi-classicality for arbitrarily given initial states. Obviously, the random jumps (15) will interrupt the deterministic evolution of the Schmidt process. Their overall rate (20), e.g. in the high temperature limit, is $w = \sqrt{2\gamma kT/\hbar}$. It is, nevertheless, not excluded that the jumps would only cause slight random walk and breathing to the otherwise quasi-classical wave function. This issue needs further investigations.

5. Conclusion

Schmidt processes offer a certain solution to the preferred basis problem of quantum mechanics, at least when the subsystem's reduced dynamics can be considered Markovian. It will be interesting to carry on analytic or numeric calculations for the Schmidt process of the Brownian motion, not at all exhausted in Sect. 4.

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