Coupling Classical and Quantum Variables using Continuous Quantum Measurement Theory

Lajos Diósi*
KFKI Research Institute for Particle and Nuclear Physics, H-1525, Budapest 114, POB 49, Hungary

Jonathan J. Halliwell†
Theory Group, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 13 May 1997; revised manuscript received 9 June 1998)

Continuous quantum measurement theory is used to construct a phenomenological description of the interaction of a quasiclassical variable X with a quantum variable x, where the quasiclassical nature of X is assumed to have come about as a result of decoherence. The state of the quantum subsystem evolves according to the stochastic nonlinear Schrödinger equation of a continuously measured system, and the classical system couples to a stochastic c number $\overline{x}(t)$ representing the imprecisely measured value of x. The theory gives intuitively sensible results even when the quantum system starts out in a superposition of well-separated localized states. [S0031-9007(98)07285-8]

PACS numbers: 03.65.Bz, 04.62.+v, 05.40.+j, 42.50.Ar

A variety of problems in a number of different fields involve coupling quantum variables to variables that are effectively classical. A case of particular interest is quantum field theory in curved space-time, where one would often like to understand how a quantized matter field affects a classical gravitational field. The most commonly postulated way of modeling this situation is the semiclassical Einstein equations [1]:

$$G_{\mu\nu} = 8\pi G\langle T_{\mu\nu}\rangle. \tag{1}$$

Here, the left-hand side is the Einstein tensor of the classical metric field $g_{\mu\nu}$ and the right-hand side is the expectation value of the energy momentum tensor of a quantum field.

Yet, one cannot realistically expect that an equation such as (1) could be valid in more than a very limited set of circumstances. One would expect it to be valid, for example, only when the fluctuations in energy density are small [2,3], and it is not difficult to produce situations in which its predictions are not physically reasonable [4,5]. In particular, when the quantum state of the matter field consists of a superposition of two well-separated localized states, Eq. (1) suggests that the gravitational field couples to the average energy density of the two states, while physical intuition suggests that the gravitational field feels the energy of one or the other of the localized matter states, with some probability. It therefore becomes of interest to ask, Is there a way of going beyond the naive mean field equations which sensibly accommodates a wide class of nontrivial matter states, but without having to tackle the considerably more difficult question of quantizing the gravitational field?

In this Letter we will present a simple scheme for coupling classical and quantum variables which goes far beyond the naive mean field equations, and produces intuitively sensible results in the key case of superposition states. We will not address the full problem of the semiclassical Einstein equations (1), but rather we will concentrate on a simple model in which the scheme is easily presented and perhaps verified. Our attempt to describe the coupling of classical and quantum variables is of course one of many [6–9].

We consider a classical particle with position X in a potential $V(X)$ coupled to a harmonic oscillator with position x, which will later be quantized. The action is

$$S = \int dt\left(\frac{1}{2}M\dot{X}^2 - V(X) + \frac{1}{2}M\dot{x}^2 - \frac{1}{2}m\omega^2x^2 - \lambda Xx\right). \tag{2}$$

Hence the classical equations of motion are

$$M\ddot{X} + V'(X) + \lambda x = 0, \tag{3}$$

$$m\ddot{x} + m\omega^2x + \lambda X = 0. \tag{4}$$

The naive mean field approach involves replacing (3) with the equation

$$M\ddot{X} + V'(X) + \lambda\langle\psi|\hat{x}|\psi\rangle = 0, \tag{5}$$

and replacing (4) with the Schrödinger equation

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}(\hat{H}_0 + \lambda X\hat{x})|\psi\rangle \tag{6}$$

for the quantum particle. $\hat{H}_0$ is the Hamiltonian of the quantum particle (in this case, a harmonic oscillator) and $-X(t)$ is regarded as an external classical force. As stated above, the scheme [(5) and (6)] is unlikely to have a very wide range of validity.

Generally, for a quantum system with wave function $\psi(x)$, there will be a nonzero probability for x to take any one of a range of values, and the expectation value $\langle\hat{x}\rangle$ [as in Eq. (5)] will not be representative of the distribution of x (unless the distribution just happens to be peaked about its expectation value). One would therefore expect the classical system to be stochastically influenced by the quantum system, and follow one of an

 0031-9007/98/81(14)/2846(4)$15.00

ensemble trajectories, with the probability distribution on the ensemble determined by the dynamics and quantum state of the quantum particle.

The question of coupling classical variables to quantum variables is, however, intimately connected to the question of how certain variables become classical in the first place. In this Letter, we adopt the point of view that there are no *fundamentally* classical systems in the world, only quantum systems that are effectively classical under certain conditions. The most comprehensive approach to obtaining generalizations of the semiclassical scheme [(5) and (6)] therefore consists of starting from the underlying quantum theory of the whole composite system and then *deriving* the effective form of that theory under the conditions in which one of the subsystems is effectively classical. The most important condition that needs to be satisfied for a subsystem to be effectively classical is *decoherence*—interference between histories of certain types of variables (in this case, position) must be destroyed (see, for example, Ref. [10]). Decoherence is typically brought about by some kind of coarse-graining procedure, of which perhaps the most commonly used procedure is to couple to a large environment (typically a heat bath) and then trace it out. The resulting decoherent variables are often referred to as quasiclassical, a nomenclature we shall adopt. Quasiclassical variables follow classical trajectories, but modified by fluctuations induced by the environment that decohered them. For sufficiently massive particles, these fluctuations have a negligible effect.

A derivation of an effective theory of coupled quasiclassical and quantum variables therefore involves a three-component quantum system consisting of a ("to be quasiclassical") particle with position X, coupled to an environment which is traced out to render X quasiclassical, and also coupled to the position x of another ("quantum") particle (not necessarily coupled to the environment). A particular class of models of this type was considered in Ref. [11]. Although the details are somewhat involved, the final form of the coupled quasiclassical-quantum theory is reasonably simple and intuitively appealing: The quasiclassical variable X couples to a stochastic c number $\bar{x}$ (instead of the deterministic $\langle\hat{x}\rangle$) whose probability distribution closely resembles the formula for continuous quantum measurement of the quantum system's position $\hat{x}$.

Emergent classicality is a widespread and generic phenomenon. It has been demonstrated in a wide variety of different circumstances using a variety of different approaches to decoherence. This, together with the simplicity of the above result, suggests that it should be possible to abstract the essential features of the model of Ref. [11] and write down directly a phenomenological model describing the coupling of the quasiclassical variable X to the quantum variable x, but without having to appeal to the full details of a specific decoherence calculation. Such a scheme would also have the advantage that it may be valid when the underlying quantum theory is not particularly manageable or even not known (as may be the case for gravity).

Our approach is to use the continuous quantum measurement theory together with a heuristic appreciation of decoherence to write down the desired phenomenological scheme. The basic idea is to think of the quasiclassical particle as in some sense "measuring" the quantum particle's position and responding to the measured c number result $\bar{x}$. (A precursor to this idea may be found in Ref. [12].) Quantum measurement theory is already a partial description of the interaction between quasiclassical and quantum systems, so its appearance in this context should be no surprise (it is also strongly suggested by the results of Ref. [11]). We do not model the decoherence of the quasiclassical particle explicitly, but appeal to general known features of the decoherence process where necessary. In particular, the assumed decoherence ensures that the quasiclassical particle remains quasiclassical (although it may be stochastically influenced) even when it interacts with the quantum particle in a nontrivial superposition.

Consider, therefore, the consequences of standard quantum measurement theory for the evolution of the coupled quasiclassical and quantum systems over a small interval of time δt. The state $|\psi\rangle$ of the quantum system will evolve, as a result of the measurement, into the (unnormalized) state

$$|\Psi_{\bar{x}}\rangle = \hat{P}_{\bar{x}} e^{-i\hat{H}\delta t}|\psi\rangle\,, \tag{7}$$

where $\hat{H} = \hat{H}_0 + \lambda X \hat{x}$ and $\hat{P}_{\bar{x}}$ is a projection operator which asks whether the position of the quantum particle is $\bar{x}$, to within some precision. [If the classical system couples to some operator of the quantum system other than position, e.g., momentum, then the projection operator in (7) is changed accordingly, e.g., to a momentum projector.] The probability that the measurement yields the result $\bar{x}$ is given by $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$. It is then natural to suppose that the classical particle, in responding to the measured result, will evolve during this small time interval according to the equation of motion

$$M\ddot{X} + V'(X) + \lambda\bar{x} = 0\,, \tag{8}$$

with probability $\langle\Psi_{\bar{x}}|\Psi_{\bar{x}}\rangle$.

Now we would like to repeat the process for an arbitrary number of time steps and then take the continuum limit. If $\hat{P}_{\bar{x}}$ is an exact projection operator, i.e., one for which $\hat{P}_{\bar{x}}^2 = \hat{P}_{\bar{x}}$, the continuum limit is trivial and of no interest (this is the watchdog effect). However, standard quantum measurement theory has been generalized to a well-defined and nontrivial process that acts continuously in time by replacing $\hat{P}_{\bar{x}}$ with a positive operator-valued measure (POVM) [12–15]. The simplest example, which we use here, is a Gaussian,

$$\hat{P}_{\bar{x}} = \frac{1}{(2\pi\Delta^2)^{1/2}} \exp\!\left(-\frac{(\hat{x}-\bar{x})^2}{2\Delta^2}\right), \tag{9}$$

and the continuum limit involves taking $\Delta \to \infty$ as $\delta t \to 0$ in such a way that $\Delta^2 \delta t$ is held constant. The evolution of the wave function of the quantum system is then conveniently expressed in terms of a path-integral expression for the unnormalized wave function:

$$\Psi_{[\bar{x}(t)]}(x', t') = \int \mathcal{D}x \exp\left[\frac{i}{\hbar}\int_0^{t'} dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \lambda x X\right)\right] \exp\left(-\int_0^{t'} dt \frac{(x - \bar{x})^2}{4\sigma^2}\right)\Psi(x_0, 0). \quad (10)$$

Here, the integral is over paths $x(t)$ satisfying $x(0) = x_0$ and $x(t') = x'$. The classical particle at each moment of time evolves according to Eq. (8), where the functional probability distribution of the entire measured path $\bar{x}(t)$ takes the form

$$p[\bar{x}(t)] = \langle \Psi_{[\bar{x}(t)]} | \Psi_{[\bar{x}(t)]} \rangle. \quad (11)$$

(The parameter σ in Eq. (10), representing the width of the effective "measurement" of the particle by the classical system, will be discussed below.)

The scheme is therefore as follows. We solve Eqs. (8) and (10), where $\bar{x}(t)$ is regarded as a stochastic variable whose probability distribution is given by (11). The final result is an ensemble of $\bar{x}$-dependent classical and quantum trajectories, respectively, for the two particles, with an interdependent probability distribution.

It turns out that this system [(8), (10), and (11)] can be rewritten in such a way that brings it closer to the form of the naive mean field Eqs. (5) and (6). The basic issue is that Eq. (11) gives the probability for an entire history of measured alternatives, $\bar{x}(t)$. Yet, the naive mean field Eqs. (5) and (6) are evolution equations defined at each moment of time. Fortunately, the system [(8), (10), and (11)] may be rewritten as follows. Consider the basic process (7) with the Gaussian projector (9), but, in addition, let the state vector be normalized at each time step. Then denoting the normalized state at each time by $|\psi\rangle$, and taking the continuum limit in the manner indicated above, it is readily shown [15] that $|\psi\rangle$ obeys a stochastic nonlinear equation describing a system undergoing continuous measurement:

$$\frac{d}{dt}|\psi\rangle = \left(-\frac{i}{\hbar}(\hat{H}_0 + \lambda X\hat{x}) - \frac{1}{4\sigma^2}(\hat{x} - \langle\hat{x}\rangle)^2\right)|\psi\rangle + \frac{1}{2\sigma}(\hat{x} - \langle\hat{x}\rangle)|\psi\rangle\eta(t). \quad (12)$$

Here, $\eta(t)$ is the standard Gaussian white noise, with linear and quadratic means,

$$M[\eta(t)] = 0, \qquad M[\eta(t)\eta(t')] = \delta(t - t'). \quad (13)$$

where $M(\cdots)$ denotes stochastic averaging. The noise terms are to be interpreted in the sense of Ito. The measured value $\bar{x}$ is then related to η by

$$\bar{x} = \langle\psi|\hat{x}|\psi\rangle + \sigma\eta(t). \quad (14)$$

Hence the final form of Eq. (8) [replacing Eq. (5)] is

$$M\ddot{X} + V'(X) + \lambda\langle\psi|\hat{x}|\psi\rangle + \lambda\sigma\eta(t) = 0, \quad (15)$$

and (6) is replaced by the stochastic nonlinear equation (12).

We now turn to the question of the value of the parameter σ. As discussed above, the quasiclassical particle suffers fluctuations as a result of interacting with the environment that decohered it. This must still be true even when it is not coupled to the quantum particle. We can therefore fix σ by demanding that, in Eq. (5), the term $\lambda\sigma\eta(t)$, in the limit $\lambda \to 0$, describe the environmentally induced fluctuations suffered by the classical particle. This forces us to choose σ to be proportional to λ^{-1}. Further information on the form of σ requires more specific details about the environment. In the particular but frequently studied case of a thermal environment, the random force should be $\sqrt{2M\gamma k_B T}\,\eta(t)$, in order to coincide with the standard Langevin equation of classical Brownian motion. From this we deduce that $\sigma^2 = 2M\gamma k_B T/\lambda^2$ (in agreement with Ref. [11]). The result is not hard to understand. Because of the environmentally induced fluctuations it suffers, the quasiclassical particle is necessarily limited in the precision with which it can "measure" the quantum particle, hence the width σ of the measurement is related to the fluctuations of the quasiclassical particle.

The formal solution to (12) describes a family of pure states, $|\psi\rangle = |\psi_{[\eta(t)]}\rangle$, one for each choice of function, $\eta(t)$. Correspondingly, in Eq. (15), with $|\psi\rangle = |\psi_{[\eta(t)]}\rangle$ inserted in the pure state expectation value, there is one evolution equation for each $\eta(t)$. For fixed initial data, $|\psi_0\rangle$, X_0, and $\dot{X}_0$, Eqs. (12) and (15) therefore describe an ensemble of quantum and classical trajectories $[|\psi_{[\eta(t)]}\rangle, X_{[\eta(t)]}(t)]$, with members labeled by $\eta(t)$. The probability for each member of the ensemble is that implied by the probability distribution of $\eta(t)$ [implicit in Eq. (13)].

There are two differences between the system [(12)–(15)] and the naive mean field Eqs. (5) and (6). One is the noise term η. In Eq. (15) [as compared to Eq. (5)] the noise clearly describes an additional (completely uncorrelated) random force. This type of modification to the semiclassical Einstein equations has been considered previously [3,16].

More important is the novelty that the state $|\psi\rangle$ evolves according to the stochastic nonlinear equation (12), and hence its evolution is very different to that under the usual Schrödinger equation [Eq. (6)]. In particular, it may be shown that all solutions to (12) undergo *localization* [17,18] on a time scale which might be extremely short compared to the oscillator's frequency ω. That is, every initial state rapidly evolves to a generalized coherent state centered around values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$ undergoing classical Brownian motion. [The results cited above are readily extended to the case here in which the Hamiltonian contains a linear coupling to an external force $-X(t)$.] Which particular solution the state becomes centered around depends statistically on the initial state of the system. For an initial state consisting of a superposition of well-separated coherent states,

$$|\psi\rangle = \alpha_1|x_1p_1\rangle + \alpha_2|x_2p_2\rangle, \qquad (16)$$

the state after localization time will, with probability $|\alpha_1|^2$, be as if the initial state were just $|x_1p_1\rangle$ and, with probability $|\alpha_2|^2$, will be as if the initial state were just $|x_2p_2\rangle$ [18]. The localization time $\sim 1/\sigma^2(x_1 - x_2)^2$ becomes, with our previous choice $\sigma^2 \sim M\gamma k_B T/\lambda^2$, very short indeed if the classical particle has a large mass M.

Hence, in the new semiclassical Eqs. (12)–(15), what happens is that effectively we solve separately for the two initial states $|x_1p_1\rangle$ and $|x_2p_2\rangle$, and the classical particle then follows the first solution with probability $|\alpha_1|^2$ and the second with probability $|\alpha_2|^2$. In simple terms, therefore, an almost classical system interacting through position with a quantum system in a superposition state (16) "sees" one or another of the superposition states, with some probability, and not the mean position of the entire state. This is the key case for which the naive mean field equations fail to give intuitively sensible results [5,19].

As noted earlier, decoherence is essential in our approach to preserve the quasiclassical behavior of one of the subsystems. (The possible significance of decoherence here was also noted in Ref. [6].) Weaker notions of classicality are sometimes used in this context. For example, it is sometimes argued that a massive particle starting out in a coherent state and evolved unitarily will behave "classically." Aside from the fact that a special initial state is required, the "classical" system is really still quantum, and its quantum nature may be seen if it interacts with another subsystem in a nontrivial superposition state, for then the entire composite system would go into a "nonclassical" superposition. The notion of classicality used here, which follows the standard decoherence literature [10], is more comprehensive, and is the appropriate one for the real physical systems that we observe to be effectively classical.

We have presented a scheme {Eqs. (12)–(15) [or Eqs. (8), (10), and (11)]} for coupling classical and quantum variables which appears to be reasonable on physical grounds and gives intuitively sensible results. It is based on the premise that the interaction between the classical and quantum variables may be regarded as a quantum measurement. The mathematics of continuous quantum measurement theory then fixes the overall structure of the scheme, but an additional physical argument is required to fix the parameter σ describing the precision of the measurement. Our proposed scheme, including the value of σ, is in broad agreement with the detailed derivation of such a particular scheme starting from an underlying quantum theory [11]. The theory of continuous quantum measurements is also closely related to the so-called hybrid representation of composite quantum systems [8,20], and this provides another possible framework for examining the emergence of the scheme. We do not claim, however, that our scheme eliminates all known controversies of the naive mean field method or of nonlinear quantum theories generally [21]. See Ref. [22] for a more detailed discussion of this work.

L. D. was supported by EPSRC and by Grant No. OTKA T016047. We are grateful to Chris Isham for useful comments.

*Email address: diosi@rmki.kfki.hu
†Email address: j.halliwell@ic.ac.uk

[1] C. Moller, in *Les Theories Relativistes de la Gravitation*, edited by A. Lichnerowicz and M. A. Tonnelat (CNRS, Paris, 1962); L. Rosenfeld, Nucl. Phys. **40**, 353 (1963).
[2] L. H. Ford, Ann. Phys. (N.Y.) **144**, 238 (1982).
[3] C-I. Kuo and L. H. Ford, Phys. Rev. D **47**, 4510 (1993).
[4] T. W. B. Kibble, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, New York, 1981).
[5] D. N. Page and C. D. Geilker, Phys. Rev. Lett. **47**, 979 (1981).
[6] A. Anderson, Phys. Rev. Lett. **74**, 621 (1995); **76**, 4090 (1996); in *Proceedings of the Fourth Drexel Symposium on Quantum Nonintegrability* edited by D. H. Feng (International Press, Cambridge, MA, 1996).
[7] I. V. Aleksandrov, Z. Naturforsch. **36A**, 902 (1981); W. Boucher and J. Traschen, Phys. Rev. D **37**, 3522 (1988); Ph. Blanchard and A. Jadcyk, Phys. Lett. A **175**, 157 (1993); K. R. W. Jones, Phys. Rev. Lett. **76**, 4087 (1996); L. Diósi, Phys. Rev. Lett. **76**, 4088 (1996); I. R. Senitzky, Phys. Rev. Lett. **76**, 4089 (1996).
[8] L. Diósi, quant-ph/9510028.
[9] A. Zoupas, "Coupling of Quantum to Classical in the Presence of a Decohering Environment," Imperial College, 1997 (to be published).
[10] M. Gell-Mann and J. B. Hartle, Phys. Rev. D **47**, 3345 (1993); E. Joos and H. D. Zeh, Z. Phys. B **59**, 223 (1985); W. Zurek, Prog. Theor. Phys. **89**, 281 (1993); Phys. Today **40**, 36 (1991).
[11] J. J. Halliwell, Phys. Rev. D (to be published).
[12] L. Diósi, Phys. Rev. A **42**, 5086 (1990).
[13] A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento Soc. Ital. Fis. **72B**, 79 (1982); V. P. Belavkin and P. Staszewski, Phys. Rev. A **45**, 1347 (1992).
[14] C. M. Caves and G. J. Milburn, Phys. Rev. A **36**, 5543 (1987).
[15] L. Diósi, Phys. Lett. A **129**, 419 (1988).
[16] B. L. Hu and A. Matacz, Phys. Rev. D **51**, 1577 (1995); A. Campos and E. Verdaguer, Phys. Rev. D **53**, 1927 (1996).
[17] L. Diósi, Phys. Lett. A **132**, 233 (1988); N. Gisin and I. C. Percival, J. Phys. A **26**, 2233 (1993); **26**, 2245 (1993); I. C. Percival, J. Phys. A **27**, 1003 (1994); Y. Salama and N. Gisin, Phys. Lett. A **181**, 269 (1993).
[18] J. J. Halliwell and A. Zoupas, Phys. Rev. D **52**, 7294 (1995); **55**, 4697 (1997).
[19] L. Diósi, Phys. Lett. A **105**, 199 (1984).
[20] L. Diósi, Quantum Semiclass. Opt. **8**, 309 (1996); in *New Developments on Fundamental Problems in Quantum Physics*, edited by M. Ferrero and A. van der Merwe (Kluwer, Dordrecht, 1997), p. 115.
[21] N. Gisin, Helv. Phys. Acta **62**, 363 (1989).
[22] J. J. Halliwell, gr-qc/9808071.