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Single qubit estimation from repeated unsharp measurements

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When estimating an unknown single pure qubit state, the optimum fidelity is $2/3$. As it is well known, the value $2/3$ can be achieved in one step, by a single ideal measurement of the polarization along a random direction. I analyze the opposite strategy which is the long sequence of unsharp polarization measurements. The evolution of the qubit under the influence of repeated measurements is quite complicated in the general case. Fortunately, in a certain limit of very unsharp measurements the qubit will obey simple stochastic evolution equations known for long under the name of time-continuous measurement theory. I discuss how the outcomes of the very unsharp measurements will asymptotically contribute to our knowledge of the original qubit. It is reassuring that the fidelity will achieve the optimum $2/3$ for long enough sequences of the unsharp measurements.

1 Introduction

Quantum measurement $\mathbf{M}$ means the procedure to obtain the value σ of some hermitian observable $\hat{\sigma}$ of the given quantum system. The *apriori* state $\hat{\rho}$ of the system transforms into the *aposteriori* state $\hat{\rho}(\sigma)$ conditioned on the measurement outcome σ. The theory of quantum measurement is well-known for projective (sharp) as well as for non-projective (unsharp) measurements. There is, however, a further task beyond quantum measurement. One can consider the *apriori* quantum state $\hat{\rho}$ of the given system as an additional object of inference [1, 2]. The estimation $\mathbf{E}$ is based on the measurement outcome σ. Hence the *estimate* state $\hat{\rho}'(\sigma)$ becomes, similarly to the aposteriori state $\hat{\rho}(\sigma)$, the function of the measurement outcome. This function depends on the estimation strategy $\mathbf{E}$ [3]. The flowchart of quantum inference, consisting of the quantum measurement $\mathbf{M}$ and of the estimation $\mathbf{E}$, is displayed on Fig. 1. Unlike the theory of measurement, the theory of estimation has not so far achieved a complete understanding. Most results are restricted for pure apriori states. A completely unknown state $\hat{\rho}$ can not be inferred from a *single* system: the *fidelity* of the estimate $\hat{\rho}'$ will be poor. If the apriori state $\hat{\rho}$ is pure then the estimate $\hat{\rho}'$ must

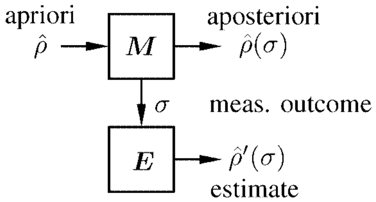

Fig. 1 Quantum inference: measurement (M) and estimation (E).

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also be pure, and the simple bilinear expression $F = \mathrm{tr}[\hat\rho'\hat\rho]$ defines its fidelity. If we assume that the apriori pure $\hat\rho$ is completely random then lower and upper limits become analytically calculable for the average fidelity $\bar F$ [5]. For a single two-state system (qubit) one obtains:

$$\tfrac{1}{2} \le \bar F \le \tfrac{2}{3}\,. \tag{1}$$

Any deliberate trial $\hat\rho'$, when completely unrelated to $\hat\rho$, will yield the same worst value $1/2$. The best value can be attained in many ways. Let us, for instance, measure the Pauli-polarization matrix $\hat\sigma$ along a single randomly chosen spatial direction. Let $\sigma = \pm 1$ be the results of the *projective* measurement. It is then natural to identify the estimate pure state $\hat\rho'(\sigma)$ with the standard *aposteriori* pure state $\hat\rho(\sigma)$ taught in textbooks:

$$\hat\rho'(\sigma) = \hat\rho(\sigma) \equiv \frac{\hat I + \sigma\hat{\vec\sigma}}{2}\,. \tag{2}$$

The average fidelity over random apriori pure states $\hat\rho$ is $2/3$. No quantum measurement however involved could improve on $\bar F = 2/3$. It would make no sense to perform a second projective measurement on the given single qubit. We can, however, consider *non-projective* (unsharp) measurements [7, 8] from the beginning. It makes sense to combine successive non-projective measurements on a single system [9] in order to improve fidelity. This I call sequential inference. Its flowchart is shown on Fig. 2. The question, discussed first in [10], is this. *For an unknown qubit $\hat\rho = \hat\rho^2$, do many ($n \gg 1$) unsharp measurements (of precision $\Delta \gg 1$) of random polarizations $\hat\sigma_1, \hat\sigma_2 \ldots, \hat\sigma_n$ allow an optimum estimate $\hat\rho'$ (of fidelity $2/3$)?* We shall see that they do!

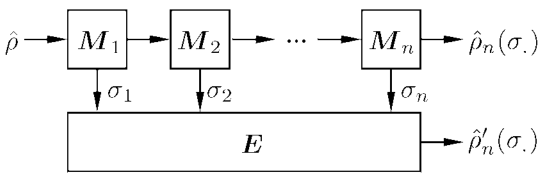

Fig. 2 Sequential quantum measurement and estimation.

There is a particular limit of sequential inference which is tractable by stochastic differential equations. This may be called continuous inference (Fig. 3). It is valid for very long sequences of very unsharp measurements. The elegant differential equations of the continuous measurement have been known for long [11]. The more complex differential equations of the continuous estimation give little assistance, at least in their first derived form [10]. To overcome their difficulty, I applied a trick. One can consider a hypothetical apriori state whose hypothetical aposteriori states are identical to the true estimate states. Hence the differential equations of continuous measurement can be used to derive certain properties like, in particular, the fidelity of the estimate state.

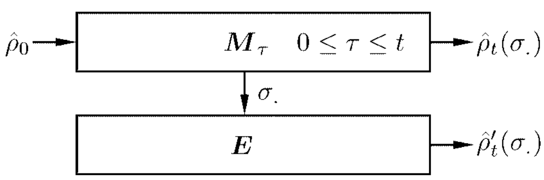

Fig. 3 Continuous quantum measurement and estimation.

2 Estimation from POVM

We approximate the exact eigenstates of a given hermitian observable $\hat{\sigma}$ by approximate Gaussian projectors of precision Δ:

$$\hat{\Pi}(\sigma) = \frac{1}{\sqrt{2\pi\Delta^2}} \exp\left[-\frac{(\hat{\sigma}-\sigma)^2}{2\Delta^2}\right] . \tag{3}$$

They satisfy the completeness condition

$$\int \hat{\Pi}(\sigma) d\sigma = \hat{I} \ , \tag{4}$$

and form a POVM [7, 8]. The corresponding (non-projective) measurement of $\hat{\sigma}$ will transform the apriori state $\hat{\rho}$ into the following aposteriori state:

$$\mathbf{M}: \ \hat{\rho} \longrightarrow \hat{\rho}(\sigma) = \frac{\hat{\Pi}^{1/2}(\sigma)\hat{\rho}\hat{\Pi}^{1/2}(\sigma)}{\mathrm{tr}\left[\hat{\Pi}(\sigma)\hat{\rho}\right]} \ , \tag{5}$$

where σ is the random outcome of the measurement. It may take any real value with the normalized probability density

$$p(\sigma) = \mathrm{tr}\left[\hat{\Pi}(\sigma)\hat{\rho}\right] . \tag{6}$$

The theory of (non-projective) measurements does not imply a theory for the estimate $\hat{\rho}'$. One could mistakenly think the aposteriori state $\hat{\rho}(\sigma)$ a reasonable estimate for the apriori state $\hat{\rho}$. Unfortunately, the experimenter has no access to it. He/she infers the measured value σ and it is, contrary to the projective measurement (2), not enough to derive the aposteriori state. It is only sufficient to identify the approximate projector $\hat{\Pi}(\sigma)$. Its normalized form can be a reasonable estimate:

$$\hat{\rho}'(\sigma) = \frac{\hat{\Pi}(\sigma)}{\mathrm{tr}\hat{\Pi}(\sigma)} . \tag{7}$$

This is a mixed state. If the apriori states $\hat{\rho}$ are unknown pure states then the estimate should also be pure. To this end, the experimenter must refine his/her first choice (7). The estimate will be one of the pure eigenstates of the mixed state estimate (7), chosen randomly with probability equal to the corresponding eigenvalue. (The optimum estimate would be the most probable eigenstate [6].) The bilinearity of fidelity $\mathrm{tr}[\hat{\rho}'\hat{\rho}]$, valid originally between two pure states, will be preserved for the expected fidelity of our estimates:

$$F = \int \mathrm{tr}\left[\hat{\rho}'(\sigma)\hat{\rho}\right] p(\sigma) d\sigma \equiv \mathrm{E}\ \mathrm{tr}[\hat{\rho}'(\sigma)\hat{\rho}] \ , \tag{8}$$

where $\hat{\rho}'$ is defined by (7) and E stands for stochastic expectation value.

3 A useful trick

The expected fidelity (8) of our estimate (7) has been expressed in terms of the apriori $\hat{\rho}$ and the estimate state $\hat{\rho}'(\sigma)$. There is an alternative expression depending on a *hypothetic* aposteriori state $\hat{\rho}^?(\sigma)$ and on the true apriori state $\hat{\rho}$. The trick is that the true estimate state $\hat{\rho}'(\sigma)$ can be identified by the hypothetic aposteriori state:

$$\hat{\rho}'(\sigma) = \hat{\rho}^?(\sigma) \ , \tag{9}$$

where the state $\hat{\rho}^?(\sigma)$ results from a hypothetic measurement of the POVM $\hat{\Pi}(\sigma)$ on a completely mixed (hypothetic) apriori state $\hat{\rho}^? = \hat{I}/2$. Indeed, the measurement (5) yields

$$\mathbf{M}: \ \hat{\rho}^? = \frac{\hat{I}}{2} \longrightarrow \hat{\rho}^?(\sigma) = \frac{\hat{\Pi}(\sigma)}{\mathrm{tr}\hat{\Pi}(\sigma)} \ , \tag{10}$$

while

$$p^?(\sigma) = \mathrm{tr}\left[\hat{\Pi}(\sigma)\hat{\rho}^?\right] = \frac{1}{2}\mathrm{tr}\hat{\Pi}(\sigma) \tag{11}$$

is the probability distribution of the outcome. In the expression (8) of fidelity we can thus replace $\hat{\rho}'(\sigma)$ by $\hat{\rho}^?(\sigma)$ and $p(\sigma)$ by $2p^?(\sigma)\mathrm{tr}[\hat{\rho}^?(\sigma)\hat{\rho}]$ yielding:

$$F = 2\int \left(\mathrm{tr}\left[\hat{\rho}^?(\sigma)\hat{\rho}\right]\right)^2 p^?(\sigma)d\sigma \equiv 2\mathrm{E}\left(\mathrm{tr}\left[\hat{\rho}^?(\sigma)\hat{\rho}\right]\right)^2 \ . \tag{12}$$

Note that the stochastic average is to be taken with the hypothetical probability distribution $p^?(\sigma)$ instead of the true $p(\sigma)$. The new expression (12) contains the (hypothetical) aposteriori state while the old formula (8) contained the (true) estimate state. Finally, we have to average the fidelity (12) over random pure qubit states $\hat{\rho}$:

$$\bar{F} = \frac{1}{3} + \frac{1}{3}\mathrm{E}\ \mathrm{tr}[\hat{\rho}^?(\sigma)]^2 \ . \tag{13}$$

This formula of the average fidelity depends completely on the *purity* $\mathrm{tr}[\hat{\rho}^?(\sigma)]^2$ of the hypothetic aposteriori state $\hat{\rho}^?(\sigma)$. Purity's minimum value is $1/2$ for the totally mixed state while its maximum is 1 for a pure state.

4 Sequential inference

According to Fig. 2, we apply n unsharp measurements $\hat{\Pi}_1(\sigma_1), \dots, \hat{\Pi}_n(\sigma_n)$ (3-6) of the respective observables $\hat{\sigma}_1, \dots, \hat{\sigma}_n$ which are polarizations along independent random directions. Using the shorthand notation $(\sigma_1, \dots, \sigma_n) = (\sigma_.)$ for the measurement outcomes, the aposteriori and the estimate states will be denoted by $\hat{\rho}_n(\sigma_.)$ and $\hat{\rho}'_n(\sigma_.)$, respectively. To estimate the state from the measurement outcomes, we can follow the recipe of single inference. The above sequence of n measurements constitute a single (complicated) measurement. It has its POVM $\hat{\Pi}_n(\sigma_.)$ [10]. Extending the estimation strategy from single measurement, we introduce the mixed state estimate

$$\hat{\rho}'_n(\sigma_.) = \frac{\hat{\Pi}_n(\sigma_.)}{\mathrm{tr}\hat{\Pi}_n(\sigma_.)} \tag{14}$$

whose eigenstates, like in case of (7), will be the pure state estimates. Same considerations that led to fidelities (8,13) apply invariably. We can, for instance, write the average fidelity in terms of the aposteriori state $\hat{\rho}^?_n(\sigma_.)$ emerging from a hypothetical apriori qubit state $\hat{\rho}^? \equiv \hat{\rho}^?_0 = \hat{I}/2$:

$$\bar{F}_n = \frac{1}{3} + \frac{1}{3}\mathrm{E}\ \mathrm{tr}[\hat{\rho}^?_n(\sigma_.)]^2 \ . \tag{15}$$

It is obvious that $\bar{F}_0 = 1/2$, and we expect $\bar{F}_n$ is a monotone function of n. We shall prove that $\hat{\rho}^?_n(\sigma_.)$ tends to be pure for large n hence $\bar{F}_n$ attends the optimum (8).

5 Continuous inference

We assume long sequences of very unsharp measurements:

$$n \gg 1\ ,\ \Delta \gg 1\ . \tag{16}$$

The asymptotic limit [11, 15]

$$n, \Delta \longrightarrow \infty\ ,\ \frac{n}{\Delta^2} = \text{const} \tag{17}$$

will be called the 'continuum limit'. Formally, let us count the succession of measurements as if they happened at constant rate $\nu = 12/\Delta^2$. Accordingly, we replace the discrete parameter n by the continuous time:

$$t = \frac{12n}{\Delta^2}\ . \tag{18}$$

We consider all quantities as continuous functions of t, coarse-grained on scales $\gg 1/\nu$ involving many measurements. In this limit an approximate theory emerges in the form of markovian stochastic differential equations. (The theory becomes exact in the continuum limit.) The aposteriori state satisfies the conditional (or selective) master equation:

$$\frac{d\hat{\rho}_t}{dt} = -\frac{1}{2}[\hat{\vec{\sigma}}, [\hat{\vec{\sigma}}, \hat{\rho}_t]\,] + \{\hat{\vec{\sigma}} - \langle\hat{\vec{\sigma}}\rangle_t, \hat{\rho}_t\}\vec{w}_t\ , \tag{19}$$

where $\langle\hat{\vec{\sigma}}\rangle_t = \mathrm{tr}[\hat{\vec{\sigma}}\hat{\rho}_t]$. We have suppressed denoting the functional dependence of $\hat{\rho}_t$ on the outcomes $\{\sigma_\tau; 0 \le \tau \le t\}$. The $\vec{w}_t$ is the standard isotropic white-noise and the equation must be interpreted in the sense of the Ito stochastic calculus. There is a second stochastic differential equation for the outcome:

$$\vec{\sigma}_t = \langle\hat{\vec{\sigma}}\rangle_t + \frac{1}{2}\vec{w}_t\ . \tag{20}$$

The features of the above equations have been well understood [12]. This is not yet achieved for the differential equations, coupled to (19,20) via the noise w_t, which govern the estimate $\hat{\rho}'_t$ [10]. Coming back to the solution $\hat{\rho}_t$, it is known that for all initial states $\hat{\rho}_0$, including mixed ones, the aposteriori state $\hat{\rho}_t$ becomes asymptotically pure for long times [17, 18]. This assures the saturation of average fidelity (15), as proven in the next section.

6 Saturation of fidelity

We calculate the average fidelity $\bar{F}_t$. It corresponds to the (coarse-grained) $n-$dependent fidelity $\bar{F}_n$ (15) via $t = 12n/\Delta^2$. The latter requires the knowledge of the hypothetical aposteriori state $\hat{\rho}^?_t$ evolving from the hypothetical initial apriori state $\hat{\rho} = \hat{\rho}^?_0 = \hat{I}/2$. The stochastic 'master' equation (19) yields a certain diffusion process for the purity $\mathrm{tr}[\hat{\rho}^?_t]^2$. For long times it will approach the unity, therefore the aposteriori state $\hat{\rho}^?_t$ becomes asymptotically pure. My Monte-Carlo calculations have shown that the purity $\mathrm{tr}[\hat{\rho}^?_t]^2$ is dominated by the drift term. Ignoring diffusion, the analytic solution is possible [10]:

$$\mathrm{tr}[\hat{\rho}^?_t]^2 = \frac{1}{2} + \frac{1}{2}\frac{e^{8t} - 1}{e^{8t} - 1/3}\ . \tag{21}$$

Let us restore the original variable $n = t\Delta^2/12$ and substitute the above result into the expression (15):

$$\bar{F}_n = \frac{1}{3} + \frac{1}{3}\mathrm{E}\ \mathrm{tr}[\hat{\rho}^?_t]^2 = \frac{1}{2} + \frac{1}{6}\frac{e^{96n/\Delta^2} - 1}{e^{96n/\Delta^2} - 1/3}\ . \tag{22}$$

The average fidelity approaches the optimum value $2/3$ after a characteristic number $n \sim \Delta^2/96$ of unsharp measurements. Recalling the conditions (16) we conclude that our result is valid for *very* unsharp measurements, i.e., Δ must be at least $\sim\!\sqrt{96}$-times the natural scale of polarization σ.

7 Concluding remarks

I have proven that a very long sequence of very unsharp polarization measurements on a single qubit will provide the optimum fidelity $2/3$ in estimating the unknown apriori (pure) state. The details, including the strategy of estimation, are given in [10]. The result represents the first steps in extending the theory of continuous quantum measurement to continuous quantum estimation which altogether may constitute a future theory of continuous quantum inference. (For a related concept, restricted for Gaussian states, see [18].) This may be of interest every time one is accumulating and analyzing information from low rate quantum inference (e.g.: eavesdroppers of secret quantum communication, tomography with low detection efficiency [19, 20], cloner of $n \gg 1$ identical qubits into $n + 1$). One might be able to extend the concept of continuous estimation for pure states of non-trivial apriori distribution, see e.g. the issues in [21]. Whether it offers optimum fidelities we do not know for the moment.

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