His Function and Its Environmental Decoherence

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Abstract. In 1983, Wigner outlined a modified Schrödinger–von-Neumann equation of motion for macroobjects, to describe their typical coupling to the environment. This equation has become a principal model of environmental decoherence which is believed responsible for the emergence of classicality in macroscopic quantum systems. Typically, this happens gradually and asymptotically after a certain characteristic decoherence time. For the Wigner function, however, one can prove that it evolves perfectly into a classical (non-negative) phase-space distribution after a finite time of decoherence.

1. Introduction

Wigner, in his paper *On the Quantum Correction for Thermodynamic Equilibrium* [1], constructs a map from the *quantum* state of a particle into a certain *classical* phase-space distribution:

$$\hat{\rho} \rightarrow W(x,p). \tag{1}$$

(Wigner adds in footnote that 'it was found by L. Szilard and the present author some years ago for another purpose.') The Wigner function $W(x,p)$ became the prototype of all subsequent trials to simulate a quantum state through a statistical distribution over the classical phase space. As it was already obvious to Wigner himself, the sign of $W(x,p)$ is indefinite. By now, it is widely accepted that the existence of domains where W is negative should mean that the state $\hat{\rho}$ is essentially quantum in a sense that it cannot be simulated by a statistical distribution $W(x,p)$ over the classical phase space. In the contrary case, when $\hat{\rho}$'s Wigner function is positive, one can say that $\hat{\rho}$ exhibits classicality in the above particular sense.

After half a century [2], Wigner was certainly the first among the most influential to support the concept later called environmental decoherence [3,4]. Wigner,

according to his *Review of the quantum mechanical measurement problem*, was impressed [5] by the work of Zeh [6] claiming that a macroscopic body can actually not be a *closed* system of its microscopic degrees of freedom. Wigner adopts such a reality and emphasizes the need of a *new equation* for apparently non-isolated objects. He modifies the Schrödinger equation by adding a second term to the r.h.s.:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \text{irreversible term}. \tag{2}$$

The irreversible term models the inevitable interactions of the macroscopic body with its environment. Perhaps, this was the first closed equation with the explicit intention of modeling environmental decoherence.

Decoherence [3, 4] is responsible for the emergent classicality in quantized systems. In particular, the negative valued domains of the Wigner function $W(x, p)$ might become washed out and grow positive under the influence of a modified Schrödinger equation like Eq. (2). Kiefer and the present author have recently proved an even stronger theorem [7]. The positivity of $W(x, p)$ is achieved in finite time, contrary to our intuition that continuous variables would only asymptotically decohere.

Section 2 outlines the function and the equation proposed by Wigner in 1932 and 1983, respectively. In Section 3 the theorem of exact decoherence is recapitulated.

2. His Function (1932) and Equation (1983)

According to Wigner [1], the quantum state $\hat{\rho}$ can be mapped into a normalized phase-space distribution:

$$W(x, p) = \frac{1}{2\pi} \int \langle x - r/2 | \hat{\rho} | x + r/2 \rangle e^{ipr/\hbar} dr. \tag{3}$$

There is an equivalent form of the above map, see e.g. [8]:

$$W(x, p) = \text{tr}\left[\hat{\rho}\mathcal{S}\delta(x - \hat{x})\delta(p - \hat{p})\right], \tag{4}$$

where $\mathcal{S}$ stands for total symmetrization defined by successive application of the rules

$$\mathcal{S}\hat{x}\hat{f} = \frac{1}{2}\left(\hat{x}\hat{f} + \hat{f}\hat{x}\right), \qquad \mathcal{S}\hat{p}\hat{f} = \frac{1}{2}\left(\hat{p}\hat{f} + \hat{f}\hat{p}\right), \tag{5}$$

to the already $\mathcal{S}$-ordered function $\hat{f} = f(\hat{x}, \hat{p})$. The form (4) shows directly that the Wigner function has the correct quantum mechanical marginal distributions for both canonical variables separately:

$$\int W(x, p) dp = \text{tr}[\hat{\rho}\delta(x - \hat{x})], \tag{6}$$

$$\int W(x, p) dx = \text{tr}[\hat{\rho}\delta(p - \hat{p})]. \tag{7}$$

The form (4) of the Wigner function is covariant against *linear* canonical transformations:

$$\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} \quad \Rightarrow \quad W(x,p) \rightarrow W'(x',p') = W(x,p). \tag{8}$$

Wigner proposes the following modification of Schrödinger's equation in case of a macroscopic body [2]:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \sum_{\ell m} \varepsilon_\ell \left[\hat{L}_{\ell m}, [\hat{L}_{\ell m}, \hat{\rho}] \right] , \tag{9}$$

where $\hat{H}$ is the total Hamiltonian and $\hat{L}_{\ell m}$ are the multipole operators of the angular momentum. The second term is intended to decohere macroscopically different *multipole moments*. The strengths of their decoherence are being controlled by the rotation invariant parameters ε_ℓ. It turns out, however, that more typical is the decoherence between macroscopically different center of mass *positions* $\hat{x}$. The modified Schrödinger equation retains the mathematical structure of Wigner's one (9). Indeed, for a free object of mass m it is this simple [9]:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} \left[\frac{\hat{p}^2}{2m}, \hat{\rho} \right] - \frac{D}{2\hbar^2} [\hat{x}, [\hat{x}, \hat{\rho}]] . \tag{10}$$

This equation simplifies the environment as if it corresponded to a random external force $F(t) = \sqrt{D}w(t)$ where $w(t)$ is standard white noise. The solution of the standard Schrödinger equation with the randomly fluctuating potential $F(t)\hat{x}$, averaged over the noise w, yields the above modified Schrödinger equation [10].

The strength of decoherence is governed by a single parameter D. Assume a certain coherent width σ for the initial quantum state. Dimensional analysis of the modified Schrödinger equation shows that the Hamiltonian and the irreversible terms have opposite tendencies. The unitary term increases σ at a 'coherent broadening' characteristic time $m\sigma^2\hbar$. The irreversible term is tending to decrease σ at a time scale $\hbar^2/D\sigma^2$ of 'incoherent localization'. The two contrary effects may have a balance at the 'stationary coherence width' [11]

$$\sigma_0 = \left(\frac{\hbar^3}{Dm} \right)^{1/4} , \tag{11}$$

achieved typically after a characteristic 'decoherence time'

$$t_0 = \sqrt{\hbar m/D} . \tag{12}$$

Then the physical picture suggests that the quantum state has become a random (incoherent) mixture of wave packets whose characteristic width is the stationary value (11). The corresponding Wigner function must obviously be positive since each contributing wave packet is assumed to have a positive Wigner function.

3. Exact Decoherence

Let us consider the evolution of the quantum state $\hat{\rho}(t)$ under the influence of environmental decoherence as described by Eq. (10). The following theorem holds [7]. Independently of the initial state $\hat{\rho}(0)$, the solution $\hat{\rho}(t)$ will exhibit exact classicality for $t \geq t_D$ in a sense that the corresponding Wigner function becomes non-negative:

$$W(x, p; t) \geq 0 \quad \text{iff} \quad t \geq t_D . \tag{13}$$

The exact value of the decoherence time is calculable: $t_D = 3^{1/4} t_0$.

  The proof is the following. Substituting Eq. (3) into Eq. (10), we obtain the standard classical Fokker–Planck equation for the Wigner function (3):

$$\frac{dW}{dt} = -\frac{p}{m} \frac{\partial W}{\partial x} + \frac{D}{2} \frac{\partial^2 W}{\partial p^2} . \tag{14}$$

Consider the Gaussian of width $\sqrt{C}$ in both x and p:

$$g(x, p; C) = \frac{1}{2\pi C} \exp\left[-\frac{x^2 + p^2}{2C} \right] , \tag{15}$$

and use it to coarse-grain an arbitrarily chosen Wigner function. The result is always non-negative if and only if the coarse-graining scale is greater than one half:

$$g(x, p; C) \star W(x, p) \geq 0 \quad \text{iff} \quad C \geq 1/2 , \tag{16}$$

where $\star$ denotes convolution. Indeed, for $C = 1/2$ the coarse-graining yields the positive Husimi function [12]; greater values of C yield further coarse-graining which preserves the positivity of the Husimi function. Now we can generalize the lemma (16). Let us generalize the Gaussian profile (15) first:

$$g(x, p; \mathbf{C}) = \frac{1}{2\pi} |\mathbf{C}|^{-\frac{1}{2}} \exp\left[-(x, p) \frac{1}{2\mathbf{C}} \begin{pmatrix} x \\ p \end{pmatrix} \right] . \tag{17}$$

The following lemma holds, see also Ref. [13] by Khalfin and Tsirelson (1992):

$$g(x, p; \mathbf{C}) \star W(x, p) \geq 0 \quad \text{iff} \quad |\mathbf{C}| \geq 1/4 . \tag{18}$$

The correlation matrix $\mathbf{C}$ can always be transformed into the form $C\mathbf{I}$ with $C = \sqrt{|\mathbf{C}|}$. Hence, due to the covariance (8) of the Wigner function, the lemma (18) follows from the special case (16).

  Coming back to the Fokker–Planck equation (14), its solution can be written as the following time-dependent Gaussian coarse-graining:

$$W(x, p; t) = g(x, p; \mathbf{C}_W(t)) \star W(x - pt/m, p; 0) . \tag{19}$$

The correlation matrix of the coarse-graining profile is time-dependent:

$$\mathbf{C}_W(t) = Dt \begin{pmatrix} t^2/3m^2 & t/2m \\ t/2m & 1 \end{pmatrix} , \tag{20}$$

as it can be inspected if we insert it into Eq. (19) which we substitute into (14). The determinant yields:

$$|\mathbf{C}_W(t)| = \frac{D^2 t^4}{12m^2}\,. \tag{21}$$

The proof of the theorem (13) culminates in the observation that the determinant is monotone function, i.e. the Wigner function $W(x, p; t)$ results from progressive coarse-graining. Applying the lemma (18) to the determinant, we conclude that the Wigner function $W(x, p; t)$ is non-negative if $|\mathbf{C}_W(t)| \geq 1/4$. This condition is equivalent with $t \geq 3^{1/4} t_0 \equiv t_D$. This completes the exact proof of the theorem whose earlier versions can be found in Refs. [14].

4. Conclusions

I discussed two items among Wigner's remarkable contributions to the foundations of quantum mechanics: the first phase-space quasi-distribution (1932) and the first explicit equation of decoherence (1983), respectively. The simplified version of his decoherence equation leads to a theorem when applied to his function $W(x, p; t)$. I recapitulated the exact proof, given recently by Claus Kiefer and myself, guaranteeing the positivity of Wigner's function after a finite time of decoherence. The purpose of my talk was to demonstrate Wigner's continued impact even on the second century of quantum mechanics.

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Notes

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References

1. E. Wigner, *Phys. Rev.* **40** (1932) 749.
2. E. Wigner, *Quantum Optics, Experimental Gravitation and Measurement Theory*, eds. P. Meystre and M.O. Scully, Plenum, New York, 1983, p. 43.
3. W. Zurek, *Phys. Today* **44** (1991) 36.
4. D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu and H.D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory*, Springer, Berlin, 1996.
5. This writer's earlier belief that the physical apparatus' role can always be described by quantum mechanics [. . .] implied that the "collapse of the wave function" takes place only when the observation is made by a living being – a

being clearly outside of the scope of our quantum mechanics. The arguments which convinced me that quantum mechanics' validity has narrower limitation, that it is not applicable to the description of the detailed behavior of macroscopic bodies, is due to D. Zeh. (1971) [. . .]. The point is that a macroscopic body's inner structure, i.e. its wave function, is influenced by its environment in a rather short time even if it is in intergalactic space. Hence it can not be an isolated system [. . .]. Can an equation for the time-change of the state of the apparently not-isolated system be proposed?

6. H.D. Zeh, in *Foundations of Quantum Mechanics*, ed. B. d'Espagnat, Academic Press, New York, 1971, p. 263.
7. L. Diósi and C. Kiefer, *J. Phys.* **A35** (2002) 2675.
8. L. Diósi, *Quant. Semicl. Optics* **8** (1006) 309.
9. E. Joos and H.D. Zeh, *Z. Phys.* **B59** (1985) 223.
10. L. Diósi, *Phys. Lett.* **A112** (1985) 288.
11. L. Diósi, *Phys. Lett.* **A120** (1987) 377.
12. A. Peres, *Quantum Theory: Concepts and Methods*, Kluwer, Dordrecht, 1993.
13. L.A. Khalfin and B.S. Tsirelson, *Found. Phys.* **22** (1992) 879.
14. L. Diósi, *Phys. Lett.* **A122** (1987) 221; J.J. Halliwell and A. Zoupas, *Phys. Rev. A* **52** (1995) 7294; **55** (1997) 4697.