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Three-party pure quantum states are determined by two two-party reduced states

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We can uniquely calculate almost all entangled state vectors of tripartite systems ABC if we know the reduced states of any two bipartite subsystems, e.g., of AB and of BC. We construct the explicit solution.

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Generic multiparty composite quantum states establish complex multiparty correlations. In the particular case of pure composite states, however, recent evidence has shown that higher order correlations follow from lower order ones [1–3]. Such quantum features [4] are of central interest in the modern field of quantum information [5], as well as in the more traditional field of many-body physics [6].

Generic three-party pure quantum states have been shown to be uniquely determined by their two-party reduced states [1,2]. Consider, e.g., a composite pure state $\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ of three parties A, B, C of dimensions d_A, d_B, and d_C, respectively. Let ρ_{AB}, ρ_{BC}, and ρ_{AC} denote the two-party reduced states. In the case of three qubits, these three reduced states will uniquely determine the composite state ρ_{ABC} in almost all cases [1]. For higher dimensions, satisfying the "triangle inequality" $d_A \geqslant d_B + d_C - 1$, an alternative theorem holds: the two reduced states ρ_{AB} and ρ_{AC} are already sufficient to calculate the state ρ_{ABC} of the whole system [2]. Note that in both cases, one calculates ρ_{ABC} *without* assuming that ρ_{ABC} is pure. It comes out from the reduced states. If one assumes it, then a stronger statement holds. Then, as we shall prove in the present work, almost all pure composite states $|\psi_{ABC}\rangle$ can be uniquely calculated from the knowledge of any two of the two-party reduced states if one knows already that ρ_{ABC} is pure. This result holds in any finite dimensions. We present explicit equations for $|\psi_{ABC}\rangle$.

For concreteness, let us prove how a generic $|\psi_{ABC}\rangle$ is determined by ρ_{AB} and ρ_{BC}. Obviously, the latter two states determine the three single-party reduced states ρ_A, ρ_B, ρ_C as well. One shall diagonalize them, e.g.,

$$\rho_A = \sum_i p_A^i |i\rangle\langle i|, \quad p_A^i > 0. \tag{1}$$

Similarly, $|j\rangle$ and $|k\rangle$ stand for the eigenvectors with nonzero eigenvalues p_B^j, p_C^k of ρ_B and ρ_C, respectively. Note that p_A^i, p_B^j, and p_C^k are nonzero by definition. Since ρ_{ABC} is pure, the reduced state ρ_A shares its eigenvalues p_A^i with ρ_{BC}

$$\rho_{BC} = \sum_i p_A^i |i;BC\rangle\langle i;BC|, \tag{2}$$

where $|i;BC\rangle$ are the orthogonal eigenvectors of ρ_{BC} with nonzero eigenvalues. Similarly, we introduce the orthogonal decomposition of ρ_{AB} as well, with nonzero eigenvalues p_C^k and eigenvectors $|k;AB\rangle$. We may omit decomposition of ρ_{AC}: it is not required by the present proof. From the spectral decompositions (1) and (2), we can reconstruct the Schmidt-decomposition of *all* three-party pure states compatible with ρ_A and ρ_{BC}

$$|\psi_{ABC};\alpha\rangle = \sum_i \exp(\mathrm{i}\alpha_i)\sqrt{p_A^i}|i\rangle \otimes |i;BC\rangle, \tag{3}$$

where $\alpha \equiv \{\alpha_i\}$ is the set of phases to be specified later. From the spectral decompositions of ρ_{AB} and ρ_C, we have another family of *all* pure states compatible with ρ_{AB} and ρ_C

$$|\psi_{ABC};\gamma\rangle = \sum_k \exp(\mathrm{i}\gamma_k)\sqrt{p_C^k}|k;AB\rangle \otimes |k\rangle. \tag{4}$$

Since the true $|\psi_{ABC}\rangle$ is compatible with both ρ_{AB} and ρ_{BC} (and thus with ρ_A, ρ_C), therefore at least one solution *exists* for the α_i's and γ_k's such that

$$|\psi_{ABC};\alpha\rangle = |\psi_{ABC};\gamma\rangle. \tag{5}$$

We are going to prove that this solution is unique, hence the state (5), derived from ρ_{AB} and ρ_{BC}, will be the true $|\psi_{ABC}\rangle$.

First, we cast the vectorial equation (5) into equations for amplitudes. Let us calculate the following coefficients:

$$\mathcal{A}_{jk}^i = \langle jk|i;BC\rangle, \quad \mathcal{C}_{ij}^k = \langle ij|k;AB\rangle. \tag{6}$$

They are nonvanishing for a generic state $|\psi_{ABC}\rangle$. In fact, the eigenvectors (with nonzero eigenvalues) of a composite state are superpositions of the direct products formed by the eigenvectors (with nonzero eigenvalues) of the respective subsystem reduced states [7]. In our case, we use the following expansions:

$$|i;BC\rangle = \sum_{jk} \mathcal{A}_{jk}^i |jk\rangle, \quad |k;AB\rangle = \sum_{ij} \mathcal{C}_{ij}^k |ij\rangle. \tag{7}$$

Substituting them into Eqs. (3) and (4), considering orthogonality of the product states $|ijk\rangle$, we expand Eq. (5) into the following set of compatibility equations between α and γ:

$$\exp(\mathrm{i}\alpha_i)\sqrt{p_A^i}\mathcal{A}_{jk}^i = \exp(\mathrm{i}\gamma_k)\sqrt{p_C^k}\mathcal{C}_{ij}^k \tag{8}$$

for all i, j, k. Multiplying the left-hand side (l.h.s) by the complex conjugate of the right-hand-side (r.h.s.) and the r.h.s. by the complex conjugate of the l.h.s. will cancel the factors $\sqrt{p_A^i p_C^k}$, yielding

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$$\exp[\mathrm{i}(\alpha_i - \gamma_k)]\mathcal{A}^i_{jk}\overline{\mathcal{C}^k_{ij}} = \exp[-\mathrm{i}(\alpha_i - \gamma_k)]\overline{\mathcal{A}^i_{jk}}\mathcal{C}^k_{ij}. \quad (9)$$

Finally we obtain the following simple equations:

$$\alpha_i - \gamma_k = \arg\sum_j \overline{\mathcal{A}^i_{jk}}\mathcal{C}^k_{ij} \quad (10)$$

for all i and k. The solution α_i, γ_k is then trivial and unique up to an (irrelevant) constant phase shift $\alpha_i \to \alpha_i + \chi, \gamma_k \to \gamma_k + \chi$. The constant χ contributes to an irrelevant phase factor $\exp(\mathrm{i}\chi)$ in front of the pure state (5).

Reference [2] considers the generic pure state $|\psi\rangle$ of a large number of identical parties of dimension d each. The authors derived the upper bound $\alpha_U = 2/3$ on the fraction of parties whose reduced states enable one to reconstruct $|\psi\rangle$. The lower bound $\alpha_L = 1/2$ was obtained for large d. The conditions of this lower bound differ from the rest of the work [2], and follow the conditions assumed in the present paper: reconstruction has been restricted to pure states. At these conditions, the theorem of the present paper will sharpen the upper bound $\alpha_U = 2/3$. Let us group the parties into three subsystems A, B, and C where, e.g., A is a single d-state system, while B and C share the rest equally or almost equally. According to our theorem, ρ_{AB} and ρ_{AC} determine a generic pure state $|\psi\rangle$ of the whole system. This yields $\alpha_U = 1/2$ asymptotically. Observe the coincidence with the lower bound $\alpha_L = 1/2$ [2]. Accordingly, there must be an (almost) one-to-one mapping between the space of pure states of the whole system and (a certain region in) the space of the reduced states of all fractions $\sim 1/2$ of the whole.

We summarize the steps reconstructing a state vector $|\psi_{ABC}\rangle$ from two density matrices ρ_{AB} and ρ_{BC}. First we calculate ρ_A, ρ_B, and ρ_C. Then we diagonalize ρ_{AB}, ρ_{BC}, ρ_A, ρ_B, and ρ_C, and calculate the coefficients $\mathcal{A}^i_{jk}$, $\mathcal{C}^k_{ij}$ (6). The wanted pure state $|\psi_{ABC}\rangle$ takes the form (3) with

$$\alpha_i = \arg\sum_j \overline{\mathcal{A}^i_{jk}}\mathcal{C}^k_{ij}, \quad (11)$$

where k is set to any fixed value. Recall that the $\mathcal{A}^i_{jk}$'s and $\mathcal{C}^k_{ij}$'s are not independent at all. The above particular expression of α_i could well be replaced by a variety of equivalent, even simpler, expressions of them. One can, e.g., take any fixed value for j instead of the summation over j. The form (11) is preferred because it is explicitly invariant for the rephasing and relabeling of the basis vectors $|j\rangle$ of subsystem B. To display full representation invariance of the reconstruction, we need further investigations on the underlying geometric structure.

Finally, we mention a possible extension of the method for spatial tomography. Assume that we have to reconstruct a spatial wave function $\psi(xyz)$ from planar projections. Let us define the density matrices in the XY and YZ planes, e.g.,

$$\rho_{XY}(xy;x'y') = \int \psi(xyz)\overline{\psi}(x'y'z)dz, \quad (12)$$

and a similar equation for ρ_{YZ}. If our theorem remains valid for infinite dimensions as well, then the reduced states ρ_{XY} and ρ_{YZ} would determine the original spatial wave function $\psi(xyz)$. The choice of concrete equations of reconstruction would then require special care against numeric instabilities.

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[3] Y.-J. Han, Y.-S. Zhang, and G.-C. Guo, e-print quant-ph/0403151.

[4] In classical multiparty systems, the corresponding statement would be trivial since pure states are zero-uncertainty states described by multivariate delta functions.

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[7] One proves that $\mathcal{S}(\rho_{AB}) \subseteq \mathcal{S}(\rho_A) \otimes \mathcal{S}(\rho_B)$, where $\mathcal{S}(\rho)$ is the support of ρ. The relationship is trivial for pure ρ_{AB} like, e.g., $\rho^k_{AB} = |k;AB\rangle\langle k;AB|$. Then, for mixed ρ_{AB}, we can write:

$$\mathcal{S}(\rho_{AB}) = \otimes_k \mathcal{S}(\rho^k_{AB}) \subseteq \otimes_k[\mathcal{S}(\rho^k_A) \otimes \mathcal{S}(\rho^k_B)]$$
$$\subseteq \otimes_{k,k'}[\mathcal{S}(\rho^k_A) \otimes \mathcal{S}(\rho^{k'}_B)] = \mathcal{S}(\rho_A) \otimes \mathcal{S}(\rho_B)$$

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