Anomalies of Weakened Decoherence Criteria for Quantum Histories

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The theory of decoherent histories is checked for the requirement of statistical independence of subsystems. Strikingly, this is satisfied only when the decoherence functional is diagonal in both its real *and* imaginary parts. Although the weakened condition of consistency (or weak decoherence), allowing a nondiagonal imaginary part, is sufficient for the assignment of probabilities, it may easily violate the statistical independence of subsystems. Therefore, weakened consistency conditions and various related generalizations of the concept of decoherent histories appear to be ruled out. The same conclusion is obtained independently, by claiming a plausible dynamical robustness of decoherent histories.

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Introduction.—Deriving testable statistical predictions from a given quantum state ρ is possible via decoherent histories [1–4], without invoking von Neumann's theory [5] of quantum measurement and state reduction. Von Neumann's theory is known to be the only universal way to predict events and statistics. Yet, it requires a division between the quantum system in question and the rest of the unquantized world. Within standard theory, the definition of events and histories (sequences of events) in quantized systems is not possible without such a division. There are numerous theoretical efforts to relax the above dichotomy. The theory of decoherent histories is a unique one in that it focuses directly on the structure of histories and their statistics. First, Griffiths [1] and then Omnes [2] proposed so-called consistent histories for closed nonrelativistic quantum systems. Decoherent histories were, with one eye on quantum cosmology, introduced by Gell-Mann and Hartle [3]. It is usually agreed that consistent and decoherent histories are essentially the same, although different authors would stress cautious distinctions of terminologies and approaches. There are detailed reviews in Refs. [6–8] and in recent books by Omnes [9] and by Griffiths [10]. The definitive element of decoherent history (DH) theory is a minimum mathematical condition for the consistency of the probabilities p_α assigned to the histories α, as stressed first by Griffiths. There has been a consensus that postulating the so-called *weak* decoherence condition:

$$\mathrm{Re}\, D(\alpha', \alpha) = 0, \qquad \text{for all } \alpha \neq \alpha', \tag{1}$$

for the decoherence functional $D(\alpha, \alpha')$ will assure consistent probabilities. The term "weak" means we do not require the diagonality of $\mathrm{Im}\, D(\alpha, \alpha')$. Although most applications (e.g., [11]) are based on diagonal $D(\alpha, \alpha')$ advocated by [3], there is a logical option that weak DH and its further generalizations [12] might be relevant for quantum mechanics. This Letter, however, presents two evidences indicating that weakened decoherence conditions are problematic. The first evidence follows from trivial combination of statistically independent subsystems. The second one follows from trivial modification of the Hamiltonian dynamics.

Histories, decoherence.—To define a *history*, one introduces a sequence of binary events for a succession of instances $t_1 \langle t_2 \langle \ldots \langle t_n$:

$$P_{\alpha_1}(t_1), P_{\alpha_2}(t_2), \ldots, P_{\alpha_n}(t_n). \tag{2}$$

For each $k = 1, 2, \ldots, n$, the $\{P_{\alpha_k}(t_k)\}$ are various complete sets of orthogonal projectors:

$$\sum_{\alpha_k} P_{\alpha_k}(t_k) = I, \qquad P_{\alpha_k}(t_k) P_{\alpha'_k}(t_k) = \delta_{\alpha_k \alpha'_k} P_{\alpha_k}(t_k), \tag{3}$$

understood in Heisenberg representation. The history confining the events (2) will be labeled by $\alpha = (\alpha_1, \ldots, \alpha_n)$. All histories must be *consistent* in the sense that we can assign probabilities to them. Introducing the time-ordered class operators,

$$C_\alpha = P_{\alpha_n}(t_n) \ldots P_{\alpha_1}(t_1), \tag{4}$$

the following probability distribution is postulated:

$$p_\alpha = \langle C_\alpha^\dagger C_\alpha \rangle_\rho, \tag{5}$$

where $\langle\,\rangle_\rho$ stands for expectation values in state ρ. Concretely, consistency means the usual additivity of probabilities (see Refs. [3,4,6] for a standard explanation). If, in particular, we bunch two different histories α and α' into a coarse-grained one $\bar{\alpha}$,

$$\bar{C}_{\bar{\alpha}} = C_\alpha + C_{\alpha'}, \tag{6}$$

then the sum rule

$$\bar{p}_{\bar{\alpha}} = p_\alpha + p_{\alpha'} \tag{7}$$

must be satisfied for $\bar{p}_{\bar{\alpha}} = \langle \bar{C}_{\bar{\alpha}}^\dagger \bar{C}_{\bar{\alpha}} \rangle_\rho$. To guarantee this, one introduces [3] the *decoherence functional*:

$$D(\alpha', \alpha) = \langle C_{\alpha'}^\dagger C_\alpha \rangle_\rho. \tag{8}$$

If $D(\alpha', \alpha)$ is diagonal in its double argument,

$$D(\alpha', \alpha) = 0, \qquad \text{for all } \alpha \neq \alpha', \tag{9}$$

the consistency (7) of the probabilities (5) is guaranteed by the absence of interference terms between the contribution of the two histories α and α'. This is why Eq. (9) is called *decoherence condition* [3]. However, it is not a necessary condition. The same interference terms cancel if we require the *weak* decoherence condition (1) instead of the stronger condition (9).

While weak decoherence was considered a kind of sufficient and necessary condition of consistency, Goldstein and Page [12] suggested a radical generalization of DHs. In their theory of *linearly positive histories*, no constraint is postulated on the decoherence functional (8) and Eq. (5) does not assign probabilities. A new equation, linear in C_α, does:

$$p_\alpha = \mathrm{Re}\langle C_\alpha \rangle_\rho. \tag{10}$$

The only postulated constraint is the natural one:

$$\mathrm{Re}\langle C_\alpha \rangle_\rho \geq 0, \qquad \text{for all } \alpha, \tag{11}$$

simply for the sake of non-negativity of probabilities p_α (10). The consistency (7) of the probability assignment (11) is guaranteed by construction. This concept represents further loosening weak decoherence. As shown in Ref. [12], weak DHs form a subset of linearly positive histories.

Surprisingly, neither the weak decoherent nor the linear positive histories have thus far been checked against common tests such as system composition or trivial dynamic perturbations. This Letter shows that these tests indicate serious inconsistencies within the concept of weak decoherent and linearly positive histories.

Test of composition.—Assume two statistically independent quantum systems A and B with states ρ^A, ρ^B, respectively. Let us assume that the class operators C^A_α, C^B_β [cf. Eq. (4)] generate consistent histories for A and B, respectively. In ordinary quantum theory, a trivial composition of two statistically independent subsystems is always possible. In our case, the composite system's density operator is the direct product $\rho^A \otimes \rho^B$. It is plausible to expect that the operators $\{C^A_\alpha \otimes C^B_\beta\}$ will generate DHs for the composite system. This latter's decoherence functional factorizes; in obvious notations it reads

$$D^{AB}(\alpha'\beta', \alpha\beta) = D^A(\alpha', \alpha) D^B(\beta', \beta). \tag{12}$$

It is easy to see that the weak decoherence condition (1) for the statistically independent subsystems A and B does not imply the fulfillment of the same condition for the composite system. Using a weak decoherence criterium, it may thus happen that we have DHs in subsystem A and DHs in subsystem B while the composition of those DHs are, contrary to our expectations, not DHs. This anomaly follows from mere composition of the two subsystems, without any correlation or interaction between them. Exactly the same anomaly appears in the theory of linearly positive histories. The reason is that the fulfillment of the positivity condition (11) in the statistically independent subsystems cannot imply its fulfillment in the composite system.

Test of dynamical stability.—A given set of DHs may not persist if we alter the dynamics of the system. There are, nonetheless, situations when we expect them to persist. Rather than pursuing the general case, let us consider the simplest one. Consider a single k between 1 and n; introduce an interaction Hamiltonian $\delta H(t)$ acting at $t = t_k + 0^+$. We switch on a sudden external potential λ_{α_k} if the binary variable $P_{\alpha_k}(t_k)$ at time t_k takes value 1. Let the corresponding interaction Hamiltonian be

$$\delta H(t) = \delta(t - t_k - 0^+) \sum_{\alpha_k} \lambda_{\alpha_k} P_{\alpha_k}(t_k). \tag{13}$$

This Hamiltonian does not introduce coupling (coherence) between histories. We expect that consistency of histories is robust against it. Fortunately, an analytic treatment is possible. Under the perturbation (13), the time-ordered product (4) of Heisenberg operators changes in the following way:

$$C_\alpha \to e^{-i\lambda_{\alpha_k}} U_k^\dagger C_\alpha. \tag{14}$$

Here U_k is unitary transformation

$$U_k = U^\dagger(t_k) \exp\left(-i \sum_{\alpha_k} \lambda_{\alpha_k}(t_k) P_{\alpha_k}(t_k) \right) U(t_k), \tag{15}$$

caused by the Hamiltonian (13) at time $t_k + 0^+$, where $U(0) = 1$ and $U(t)$ is the solution of the equation $dU/dt = -iHU$ with the unperturbed Hamiltonian H. By virtue of Eq. (14), the decoherence functional (8) changes as follows:

$$D(\alpha', \alpha) \to e^{i(\lambda_{\alpha'_k} - \lambda_{\alpha_k})} D(\alpha', \alpha). \tag{16}$$

It is now seen that the original decoherence criterium (9) is preserved whereas the weakened one (1) is not. Therefore, the DHs will lose their robustness against the trivial external fields (13) if we use the loosened (weak) decoherence condition.

The same anomaly comes about for linearly positive histories. Satisfying the positivity condition (11) for the unperturbed class operator C cannot assure the positivity condition after perturbation. Under perturbation (13), the probabilities (10) would change as follows:

$$p_\alpha = \mathrm{Re}\langle C_\alpha \rangle_\rho \to \mathrm{Re}\, e^{-i\lambda_{\alpha_k}} \langle U_k^\dagger C_\alpha \rangle_\rho. \tag{17}$$

Obviously, the preservation of positivity is not guaranteed. Linearly positive histories may become lost under the influence of the trivial external field, which is contrary to our expectations.

Conclusion.—We have argued that the complex decoherence functional must be diagonal. The diagonality of its real part in itself is insufficient when checked for trivial composition of statistically independent subsystems. This anomaly seems to exclude the so-called weak decoherence condition and urges one to retain the stronger one. The linearly positive histories show the same anomaly in composite systems. The survival of decoherent as well as of linearly positive histories has also been tested under trivial perturbation of the Hamiltonian. Expected survival has been obtained only at the stronger decoherence condition.

The system composition evidence is likely to be an ultimate criticism. Any excuse should obviously question our standard notion of statistical independence. Once Bell inequalities changed our standard notion of statistical "dependence," without altering the more basic notion of statistical independence. This latter would be hard to challenge for usual closed dynamic systems where DH theory had originally been applied. The dynamical stability evidence is perhaps less convincing since the claim of stability might need further theoretical support. We emphasize that *consistency* of histories had traditionally been restricted to the request of additivity of probabilities in coarse-grained histories. Consistency in composition or in perturbation had not been targeted apart from an early work [13] of the present author.

Our analysis is independent from earlier criticisms [14–16] pointing out that DH theory *in itself* cannot single out classical histories since, e.g., the class of allowed histories is too large. Despite these criticisms, the concept of DH appears a sort of generalization of von Neumann's measurement theory and, with further constraints, a possible substitute for it. Obviously, we have to know the sensible limits of such a generalization, set basically by the concrete form of the so-called decoherence (consistency) condition. Our work narrows these limits down.

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