Application of continuous measurement theory to the current through quantum dots

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A simple theory of the detected current $I(t)$ flowing through charge qubits—quantum dots—is proposed in terms of standard continuous measurement theory. Applied to a double dot, our formalism easily confirms previous results on quantum Zeno effect, driven by growing ammeter performance γ. Due to the transparent formalism, we can calculate the exact fluctuation spectrum $S(\omega)$ of the detected current, containing a significant Lorentzian peak near the Rabi frequency of the double dot.

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Back-action of measurement on the measured object is an emblematic property of quantum systems. Quantum Zeno effect (QZE)—suppression of the object's coherent internal dynamics by measurement—is one of the marked scenarios for that back-action. In this Brief Report, we are going to discuss the effect in the framework of unsharp measurements, for which there is a well-developed theoretical framework called the time-continuous weak measurement theory (cf. Refs. 1 and 2 and references therein). The application we have in mind is the case of a double quantum dot (DQD), which is a semiconducting nanostructure, available in high quality due to massive progress in experimental technology.

There is a growing culture of indirect measurements on nanostructures by means of Coulomb-coupled quantum point contacts, single-electron transistors, or DQDs.[3–6] QZE is one of the effects studied from the beginning[3] (cf. also Ref. 7), and the concept of time-continuous measurement has penetrated the field[3–5] for a long time. All those studies assume the—sharp or unsharp—detection of the number $N(t)$ of electrons that have tunneled through the nanostructure, and the current $I(t)$ is defined as the stochastic mean $\langle dN(t)/dt\rangle$. The present work differs from those studies in assuming that detection is done by a tool usually considered as fully classical, an ammeter of high performance, monitoring the time-dependent current $I(t)$ flowing through that device.

The main parameter of the theory of unsharp measurement is detection performance, defined as

$$\gamma = (\Delta t)^{-1}(\Delta I)^{-2}, \tag{1}$$

where Δt is the time resolution (or, equivalently, the inverse bandwidth) of the ammeter and ΔI is the statistical error characterizing unsharp detection of the average current in the period Δt. The accuracies of commercial ammeters reach $pA \approx 10^7$ electron/s at a bandwidth of 10^4 Hz. Then, $\gamma = 10^{-10}$ s, which should be compared to the time scale of internal coherent dynamics of a DQD, characterized by the Rabi frequency $\Omega \approx 10^{10}$ Hz, which is also the order of magnitude of the steady-state current measured in electron/s units, viz., I/e. After all, $\gamma\Omega \approx 1$ can be reached through standard instrumentation. Below, we are going to show that this suffices for observing a continuous measurement version of QZE.

We rely upon the standard Markovian approximation, tracing out environmental variables referring to external leads relaxing on the fastest time scale of the problem, $\tau_r^{-1} \gg \Omega$. Transport is treated in terms of the reduced density matrix $\hat{\rho}$ of the DQD and a corresponding effective current operator $\hat{I}$ [see Eq. (17) below]. That gives a correct account for the mean current through the external tunnel barriers but neglects shot noise introduced by charge partitioning on those barriers. To recover that feature, many of the related papers apply the N-resolved technique[8] that considers $N(t)$ an additional dynamical variable. For the case of a single dot, we have checked that the procedure adds but a white noise background $I_\infty/2$ to the power spectrum; expecting a similar result in the general case, we refrain from using that technique.

The stochastic mean of the detected current is obtained as the quantum mechanical subsystem average,[9]

$$\langle I(t)\rangle = \langle \hat{I}\rangle_{\hat{\rho}(t)}. \tag{2}$$

Continuous measurement theory starts from this point and accounts for a double action of the ammeter of finite performance.

(1) Quantum mechanical back-action on the continuously measured quantum system, our main concern here, induces loss of coherence between eigenstates belonging to different eigenvalues of the measured quantity. As known from time-continuous measurement theory,[1,2] which is expressed by adding a Lindblad-type decoherence term proportional to γ to the master equation,[10]

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}\hat{\rho} - \frac{\gamma}{8}[\hat{I},[\hat{I},\hat{\rho}]] \equiv \mathcal{L}'\hat{\rho}. \tag{3}$$

Our analysis is based on the solution of this extended master equation, in which the Lindblad supermatrix $\mathcal{L}$ provides a Markovian description of decoherence and damping by fast-relaxing leads, as usual.

(2) Along with the quantum effect we expect to observe, inaccurate measurement generates classical white noise in the measured current, superposed on the average (2),

$$I(t) - \langle I(t) \rangle = g w(t). \tag{4}$$

Here, $w(t)$ is white noise on the scale of the Markovian dynamics (i.e., a flat spectrum noise of bandwidth much larger than the typical frequency range correctly accounted for by the Markovian approximation). The amplitude of the noise is related to the detection performance γ according to an inverse square-root law,

$$g = 1/\sqrt{\gamma}. \tag{5}$$

Let us introduce the stationary solution $\hat{\rho}_\infty$ of the extended master equation (3),

$$\mathcal{L}' \hat{\rho}_\infty = 0, \tag{6}$$

as well as the stationary current,[9]

$$I_\infty = \langle I(t) \rangle_\infty = \langle \hat{I} \rangle_{\hat{\rho}_\infty}, \tag{7}$$

where $\langle \cdot \rangle_\infty$ stands, in general, for stochastic mean of currents detected on $\hat{\rho}_\infty$. We also define the Heisenberg operator of the current for $t > 0$,

$$\frac{d\hat{I}(t)}{dt} = (\mathcal{L}')^\dagger \hat{I}(t), \quad \hat{I}(0) = \hat{I}. \tag{8}$$

The stationary correlation function of the fluctuating detected current turns out to be

$$\langle I(t)I(0) \rangle_\infty - \langle I(t) \rangle_\infty \langle I(0) \rangle_\infty = \tfrac{1}{2} \langle \{ \hat{I}(|t|), \hat{I} \} \rangle_{\hat{\rho}_\infty} - I_\infty^2 + g^2 \delta(t). \tag{9}$$

The nonsingular term on the right-hand side follows from the standard expression $\langle I(t)I(0) \rangle_\infty = \frac{1}{2} \langle \{ \hat{I}(|t|), \hat{I} \} \rangle_{\hat{\rho}\infty}$, valid for bulk quantum systems where the detector noise can be neglected. For our nanostructure, detection noise, as described by Eq. (4), gives rise to the $\delta(t)$ term. We have thus expressed the stationary correlation function of the detected classical current $I(t)$ in terms of the quantum correlation of the Heisenberg current $\hat{I}(t)$. The spectral density of the detected fluctuations will be defined as the Fourier transform[11] of the correlations (9), resulting in

$$S(\omega) = \int \langle I(t)I(0) \rangle_\infty e^{i\omega t} dt - 2\pi I_\infty^2 \delta(\omega), \tag{10}$$

which for high frequencies approaches the measurement-added white noise value,

$$S(\infty) = g^2 = 1/\gamma. \tag{11}$$

For *very high* values of ammeter performance γ, that noise level may get below the "shot-noise limit" $I_\infty/2$. In that case, ammeter and charge counter measurements may differ in characteristic ways not controlled by the present method of calculation; however, this is not our concern here, since the effect we envisage remains in the well-treated range of moderately high γ.

Now, we turn to the specific features of the double quantum dot, consisting of two potential wells ("dots:" left and right), with an internal barrier allowing coherent tunneling between them, and two external barriers, allowing incoherent tunneling between each dot and its joining lead. The Markovian approximation used is based on the assumption of thermalization in leads L and R being the fastest dynamical process present.

Because of intradot Coulomb blockade, at low temperature in each of the dots, there can be but one electron in the ground state, or none. Strong interdot Coulomb repulsion further reduces the set of available orthogonal basis states to the following three:

$$|0\rangle \equiv |0,0\rangle, \quad |L\rangle \equiv |1,0\rangle, \quad |R\rangle \equiv |0,1\rangle. \tag{12}$$

On the above basis, we introduce the following absorption and emission operators,

$$\hat{a}_L = |0\rangle\langle L|, \quad \hat{a}_R = |0\rangle\langle R|, \tag{13}$$

as well as the charge operators,

$$\hat{n}_L = |L\rangle\langle L|, \quad \hat{n}_R = |R\rangle\langle R|, \quad \hat{n} = \hat{n}_L + \hat{n}_R = 1 - |0\rangle\langle 0|. \tag{14}$$

With the above definitions, the Hamiltonian

$$\hat{H}_0 = \tfrac{1}{2}\delta(\hat{n}_L - \hat{n}_R) + \Omega(\hat{a}_L^\dagger \hat{a}_R + \text{H.c.}). \tag{15}$$

Neither the Hamiltonian nor any of the relevant observables [in particular, the current (see below)] has off-diagonal elements connecting the state $|0\rangle$ to the rest of the reduced Hilbert space of the DQD, imposing an effective charge superselection. Therefore, the off-diagonal elements $0L$, $0R$, $L0$, and $R0$ of the matrix $\hat{\rho}$ should be set to zero identically,

$$\hat{\rho}_\infty = \begin{pmatrix} \rho_{00} & 0 & 0 \\ 0 & \rho_{LL} & \rho_{LR} \\ 0 & \rho_{RL} & \rho_{RR} \end{pmatrix}. \tag{16}$$

The Markovian approximation yields two irreversible processes that modify the intrinsic Hamiltonian dynamics of the dots: tunneling of an electron from lead L to the left dot at rate Γ_L and tunneling of an electron from the right dot to lead R at rate Γ_R. The rates Γ_L, Γ_R depend on the details of the total Hamiltonian dynamics of $L+R+$dots, and we shall take their value for granted. The two processes give rise to two currents: $\hat{I}_L = \Gamma_L(1-\hat{n})$ and $\hat{I}_R = \Gamma_R \hat{n}_R$. Taking into account the Ramo-Shockley effect of fast screening in the leads,[12] on the slow Markovian time scale, there is a single time-local operator of the observable current flowing from L through dots and R to the ammeter,

$$\hat{I} = \frac{\Gamma_L(1-\hat{n}) + \Gamma_R \hat{n}_R}{2} = \frac{1}{2}\begin{pmatrix} \Gamma_L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Gamma_R \end{pmatrix}. \tag{17}$$

The Lindblad supermatrix $\mathcal{L}$ appearing in Eq. (3) contains the modification of Hamiltonian dynamics due to the two tunneling currents $\hat{I}_{L/R}$. Including Lamb shifts in the energy split δ of $\hat{H}_0$, one obtains

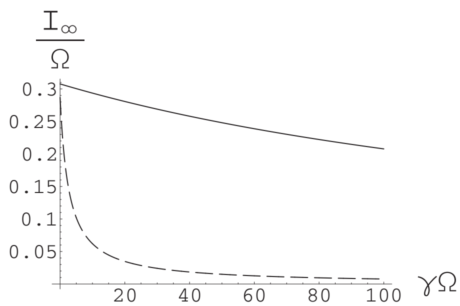

FIG. 1. Measured stationary current through an unbiased double quantum dot, as a function of ammeter performance, for symmetric tunneling rates $\Gamma_L = \Gamma_R = \Omega$ [Eq. (1)] (solid line) and asymmetric ones with $\Gamma_R = 5\Gamma_L = \Omega$ (dashed line). The shot-noise limit is at $\gamma\Omega = 6.7$.

$$\mathcal{L}\hat{\rho} \equiv -i[\hat{H}_0,\hat{\rho}] + \Gamma_L\left(\hat{a}_L^\dagger \hat{\rho}\hat{a}_L - \tfrac{1}{2}\{\hat{a}_L\hat{a}_L^\dagger,\hat{\rho}\}\right) + \Gamma_R\left(\hat{a}_R\hat{\rho}\hat{a}_R^\dagger - \tfrac{1}{2}\{\hat{a}_R^\dagger\hat{a}_R,\hat{\rho}\}\right). \tag{18}$$

Note that the continuity equation is always satisfied,[13]

$$\hat{I}_R - \hat{I}_L + \mathcal{L}^\dagger \hat{n} = 0, \tag{19}$$

where $\mathcal{L}^\dagger$ is the adjoint of $\mathcal{L}$.

Quantum measurement back-action enters through the modified Lindblad supermatrix $\mathcal{L}'$ introduced in Eq. (3). Then, the stationary state is obtained by solving Eq. (6), in which Eqs. (3), (15), (17), and (18) are used as inputs. The calculation is standard; surprisingly, we obtain exactly the same equations as those derived by Gurvitz[3] for the closely related but still different model of a DQD observed by an unsharp point-contact charge detector, if Gurvitz's clicking rate Γ_d is replaced by our combination $\gamma\Gamma_R^2/16$. We think the reason is that as long as Markovian approximation is valid, the effective current operator $\hat{I}$ is pinned down to the single-dot occupation operators $\hat{n}_{L/R}$ through Eq. (17). Then, a point-contact charge counting device, like the one discussed by Gurvitz, and a commercial ammeter, as we suggest, have no more freedom than to couple to the DQD in the same way.

The stationary electron number current reads

$$I_\infty = \frac{\Gamma_L\Gamma_R\Omega^2(1+y)}{\delta^2\Gamma_L + (2\Gamma_L+\Gamma_R)\Omega^2(1+y) + \frac{1}{4}\Gamma_L\Gamma_R^2(1+y)^2}, \tag{20}$$

where $y = \gamma\Gamma_R/16$. The above formula contains the way the quantum Zeno effect appears with growing measurement performance γ (see Fig. 1): in all cases, $I_\infty \to 0$ for $\gamma \to \infty$. By inspection of the density matrix, we learn that this is due to damping of the coherent interdot transport, causing increased occupation of the left dot, blocking the way of new electrons to enter. With no bias, $\delta=0$, the reduction of current is monotonous; with sufficiently strong bias, $\delta/\Omega > 2$, however, there is a range of small performances where the current *increases* with growing γ ("anti-Zeno effect"). Apparently, asymmetric occupation induced by decohering measurement and that caused by bias are competing with each other. The whole effect is small: for $\delta=0$, as the measurement performance passes the shot-noise limit $\gamma \approx 2/I_\infty$, it reaches a 3% reduction of current.

Of particular interest for the experiment is the case of asymmetric tunneling rates, $\Gamma_R > \Gamma_L$, resulting in much stronger QZE, reaching 30% for the example displayed in the figure. In this case, the right dot is strongly depleted, giving rise to higher sensitivity to measurement-induced blocking of the left-to-right tunneling. Asymmetry of the opposite sign has no marked effect on QZE.

Our simple theory offers the best of its performance in evaluating unsharp measurement back-action effects on the detected current fluctuation spectrum (10) through the operator correlation function (9), which has to be determined by means of our Markovian master equation. For the unbiased case $\delta=0$, we obtain

$$\mathcal{L}'(|L\rangle\langle R| + |R\rangle\langle L|) = -\tfrac{1}{2}\Gamma_R(1+\gamma\Gamma_R/16)(|L\rangle\langle R| + |R\rangle\langle L|), \tag{21}$$

hence the real part of the LR component of the Heisenberg current $\hat{I}(t)$ defined by Eq. (8) remains zero. So the matrix $\hat{I}(t)$ has four independent components instead of five. We can thus replace Eq. (8) by $d\hat{I}(t)/dt = \mathcal{L}_I'^\dagger \hat{I}(t)$, where $\mathcal{L}_I'$ is the restriction of $\mathcal{L}'$ on the four-dimensional subspace explored by $\hat{I}(t)$. Note, furthermore, that $\mathcal{L}'$ is degenerate also because $\mathcal{L}'\hat{\rho}_\infty = 0$ and this degeneracy is inherited by $\mathcal{L}_I'$, too. Therefore, the characteristic equation $|\mathcal{L}_I' + \lambda| = 0$ has a trivial zero root, and the rest of the characteristic equation is only cubic. For the special case $\Gamma_L = \Gamma_R = \Omega$, it turns out to be

$$\lambda^3 - \left(\frac{5}{2} + \frac{\gamma\Omega}{32}\right)\Omega\lambda^2 + \left(6 + \frac{\gamma\Omega}{16}\right)\Omega^2\lambda - \left(\frac{13}{2} + \frac{\gamma\Omega}{32}\right)\Omega^3 = 0. \tag{22}$$

It has one real positive root $\lambda = \Delta_0$ and two complex conjugate roots $\lambda = \Delta_1 \pm i\omega_R$ with positive Δ_1, implying that the solution $\hat{I}(t)$ consists of an exponentially decaying and an exponentially damped oscillatory part, the latter oscillating at the damped Rabi frequency $\omega_R \approx 2\Omega$. Let us have a look at the structure of the corresponding fluctuation spectrum (10),

$$S(\omega) = \frac{R_0}{\omega^2 + \Delta_0^2} + \frac{R_1 + \omega R_1'}{(\omega-\omega_R)^2 + \Delta_1^2} + \frac{R_1 - \omega R_1'}{(\omega+\omega_R)^2 + \Delta_1^2} + g^2. \tag{23}$$

One can obtain closed expressions for the parameters of the above expression. For low values of $\gamma\Omega$, we have

$$\omega_R/2\Omega = 1.025 + 0.002\gamma\Omega + \mathcal{O}(\gamma\Omega)^2,$$

$$\Delta_1/\Omega = 0.52 + 0.014\gamma\Omega + \mathcal{O}(\gamma\Omega)^2. \tag{24}$$

This spectrum has one local minimum at $\omega=0$ and two wide peaks at the Rabi frequency $\pm\omega_R$, the widths of the peaks are approximately one-fourth of the Rabi frequency, and on increasing the performance of the ammeter, their amplitudes start to decrease. For high values of γ, the Rabi peaks get overdamped and merge into a single peak around $\omega=0$ (see

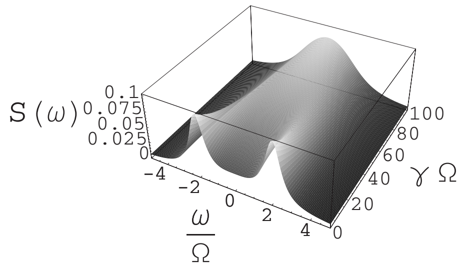

FIG. 2. Noise spectrum of the measured double-dot current as a function of the ammeter performance γ.

Fig. 2), where background noise has not been included. We stress that to observe the peaks, a good ammeter (high γ) is needed, since a poor ammeter introduces too much background white noise $g^2=1/\gamma$ to see anything of the coherent peak structure.

As a conclusion, we have derived explicit expressions for the back-action of an ammeter on measurable characteristics of a double quantum dot. The back-action has the character of quantum Zeno effect, counteracting coherent internal motion of the object, revealed both as reducing the mean transmitted current and damping—eventually, overdamping—Rabi oscillations, as observed in the noise spectrum. Appropriately tuned asymmetric tunneling barriers may strongly enhance the possibilities of observing the effect or eventually produce ranges of anti-Zeno effect. The paradoxical character of our results—quantum back-action propagating through rapidly decohering leads—can be resolved by considering that for the assembly of ammeter and its leads current is a robust "pointer variable," with its eigenstates avoiding decoherence. That reasoning, we think, is strong enough to warrant experimental check of the theory presented here.

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