

Wick theorem for all orderings of canonical operators

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Abstract

Wick's theorem, known for yielding normal ordered from time-ordered bosonic fields, may be generalized for a simple relationship between any two orderings that we define over canonical variables, in a broader sense than before. In this broad class of orderings, the general Wick theorem follows from the Baker–Campbell–Hausdorff identity. We point out that, generally, the characteristic function does not induce an unambiguous scheme to order the multiple products of the canonical operators although the value of the ordered product is unique. We construct a manifold of different schemes for each value of s of s -orderings of Cahill and Glauber.

Keywords: operator orderings, bosonic operators, Wick's theorem, Baker–Campbell–Hausdorff identity

1. Introduction

The problem of systematic ordering of canonical operators \hat{q}, \hat{p} appeared first in canonical quantization [1] and also the other way around: in classical phase-space representations of canonically quantized systems [2]. In a different realm, ordering and reordering of quantized fields became central to S-matrix theory [3–5].

A particular phase-space quasi-distribution $\rho(q, p)$ for a given density matrix $\hat{\rho}$,

$$\rho(q, p) = \text{tr}[\mathcal{O}\delta(q - \hat{q})\delta(p - \hat{p})\hat{\rho}], \quad (1)$$

requires a particular ordering \mathcal{O} of \hat{q} and \hat{p} . The Weyl–Wigner [1, 2], normal [5] and anti-normal orderings are special cases of s -orderings proposed by Cahill and Glauber [6], reviewed together with QP- and PQ-orderings e.g. in [7].

An ordering \mathcal{O} is traditionally defined by its action on the operator-valued characteristic function. In particular, the Weyl–Wigner ordering \mathcal{W} is defined by this relationship:

$$\mathcal{W}e^{a\hat{q}+b\hat{p}} = e^{a\hat{q}+b\hat{p}}, \quad (2)$$

with the arbitrary c -numbers a, b . The normal ordering \mathcal{N} is defined between the annihilation $\hat{c} = (\hat{x} + i\hat{p})/\sqrt{2}$ and creation \hat{c}^\dagger operators (we use units where $\hbar = 1$):

$$\mathcal{N}e^{\lambda^*\hat{c} + \lambda\hat{c}^\dagger} = e^{\lambda\hat{c}^\dagger} e^{\lambda^*\hat{c}}, \quad (3)$$

with an arbitrary complex number λ . The s -orderings are defined by

$$\mathcal{O}_s e^{\lambda^*\hat{c} + \lambda\hat{c}^\dagger} = e^{-s|\lambda|^2/2} e^{\lambda\hat{c}^\dagger + \lambda^*\hat{c}}, \quad s \in [-1, 1]. \quad (4)$$

They interpolate between the normal ordering ($s = 1$) and anti-normal ordering ($s = -1$), with the Weyl–Wigner ordering in the middle ($s = 0$).

Time-ordering \mathcal{T} (also called chronological or path-ordering) was proposed first by Dyson [4] in relativistic quantum field theory. The S-matrix is a time-ordered functional of free quantum field operators. If we transform it into the normal ordered form first, the evaluation of matrix elements becomes straightforward. Time- and normal orderings are related by Wick’s theorem [5]. For a free relativistic bosonic field $\hat{\varphi}(x)$ it can be written into the compact form, see [8]:

$$\mathcal{T} e^{\int J(x)\hat{\varphi}(x)dx} = e^C \mathcal{N} e^{\int J(x)\hat{\varphi}(x)dx}, \quad (5)$$

where $J(x)$ is an arbitrary c -number field. The exponent in the pre-factor is a c -number:

$$C = \int \int J(x)C(x, y)J(y)dx dy, \quad (6)$$

where $C(x, y)$ is called the kernel of chronological *contraction* (or pairing):

$$C(x, y) = \mathcal{T}\hat{\varphi}(x)\hat{\varphi}(y) - \mathcal{N}\hat{\varphi}(x)\hat{\varphi}(y) \equiv (\mathcal{T} - \mathcal{N})\hat{\varphi}(x)\hat{\varphi}(y). \quad (7)$$

The above form of Wick’s theorem suggests that a generalized theorem of the same structure holds for any pair \mathcal{O} and \mathcal{O}' of orderings.

In section 2 we define the notion of monomial and non-monomial orderings and we note that, in the latter class, the ordered characteristic function allows for ambiguous schemes of ordering. Section 3 suggests the general Wick theorem (GWT) for monomial orderings, with a tentative proof in section 4. Section 5 illustrates the theorem on simple applications in quantum mechanics and optics. Section 6 proves that an s -ordering of Cahill and Glauber is equivalent with a family of ‘redundant’ monomial path orderings, and we construct the corresponding family of ordering schemes.

2. Monomial and non-monomial orderings

Let us start with a collection $\{\hat{A}_\alpha, \alpha \in \Omega\}$ of operators to be ordered, where Ω is a (partially) ordered set of labels. (If $[\hat{A}_\alpha, \hat{A}_\beta] = 0$ for a certain pair $\alpha, \beta \in \Omega$ then the order of α and β may be left unspecified.) Consider an ordered subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} = \Omega' \subseteq \Omega$ of labels and define the corresponding ordered operator product:

$$\begin{aligned} \mathcal{O} \prod_{\alpha \in \Omega'} \hat{A}_\alpha &= \hat{A}_{\alpha_n} \dots \hat{A}_{\alpha_3} \hat{A}_{\alpha_2} \hat{A}_{\alpha_1} \\ &\equiv \left[\prod_{\alpha \in \Omega'} \hat{A}_\alpha \right]_P \end{aligned} \quad (8)$$

where \mathcal{O} stands for the given operator ordering to adopt the ordering $\alpha_n \succ \dots \succ \alpha_3 \succ \alpha_2 \succ \alpha_1$ of the labels; $[\dots]_P$ is alternative notation referring to the corresponding permutation P . The

collection of the operators may be redundant: a certain \hat{A}_α may coincide with a certain \hat{A}_β even for $\alpha \neq \beta$. Let us call (8) monomial ordering (or permutation). Ordering, in general, can be the weighted mixture of different monomial orderings:

$$\mathcal{O} \prod_{\alpha \in \Omega'} \hat{A}_\alpha = \sum_P w_P \left[\prod_{\alpha \in \Omega'} \hat{A}_\alpha \right]_P \tag{9}$$

where w_P are non-negative weights that sum up to one.

Alternatively, orderings used to be defined by their exponential characteristic functions

$$\hat{F}_\mathcal{O}(\lambda) = \mathcal{O} \prod_{\alpha \in \Omega} e^{\lambda_\alpha \hat{A}_\alpha} \tag{10}$$

at arbitrary arguments λ_α . For monomial orderings (8) they factorize:

$$\hat{F}_\mathcal{O}(\lambda) = \dots \exp(\lambda_{\alpha_3} \hat{A}_{\alpha_3}) \exp(\lambda_{\alpha_2} \hat{A}_{\alpha_2}) \exp(\lambda_{\alpha_1} \hat{A}_{\alpha_1}), \tag{11}$$

where $\dots \succ \alpha_3 \succ \alpha_2 \succ \alpha_1$ holds. As we see, the *value* of the characteristic function determines the *scheme* (8) of monomial ordering (and vice versa). This is not so for non-monomial orderings. The characteristic function does not determine the scheme (9) of ordering uniquely.

Typical non-monomial ordering is the Weyl–Wigner ordering \mathcal{W} defined by its characteristic function (2). It turns a monomial into weighted sum of ordered monomials. For instance:

$$\mathcal{W} \hat{q}^2 \hat{p} = \frac{\hat{q}^2 \hat{p} + \hat{p} \hat{q}^2 + \hat{q} \hat{p} \hat{q}}{3} = \frac{\hat{q}^2 \hat{p} + \hat{p} \hat{q}^2 + 2 \hat{q} \hat{p} \hat{q}}{4} \tag{12}$$

where we show two (of the many) schemes of weighted permutations (9) which correspond to the same non-monomial ordering \mathcal{W} . Still, we anticipate a trick (from section 6) to show that the Weyl–Wigner ordering can be squeezed into the class of monomial orderings. The trick is that we introduce the redundant collection $\{\hat{q}_\tau, \hat{p}_\tau; \tau \in [0, 1]\}$ of the canonical operators, where $\hat{q}_\tau = \hat{q}$ and $\hat{p}_\tau = \hat{p}$ for all $\tau \in [0, 1]$., i.e.: we assign the time-label τ formally. Then time-ordering, which is typical monomial ordering, leads to Weyl–Wigner ordering:

$$\begin{aligned} \mathcal{T} \exp \left(\int_0^1 (a \hat{q}_\tau + b \hat{p}_\tau) d\tau \right) &= \lim_{\epsilon \rightarrow +0} \left(e^{\epsilon (a \hat{q} + b \hat{p})} \right)^{[1/\epsilon]} = e^{a \hat{q} + b \hat{p}} \\ &= \mathcal{W} e^{a \hat{q} + b \hat{p}}. \end{aligned} \tag{13}$$

This is simplest redefinition of \mathcal{W} as a particular monomial time-ordering, there are infinite many other choices which we shall detail and extend for all *s*-orderings in section 6.

We state our general Wick’s theorem (section 3) for monomial orderings of canonical operators, and construct a tentative proof in section 4. The theorem can not be extended for non-monomial orderings in general. There is, however, a backdoor for some of them, like the above example of Weyl–Wigner ordering, and all *s*-orderings, see in section 6.

3. General Wick theorem

Consider a canonical system of n pairs of canonical variables $\hat{x}_k = \hat{q}_k$ and $\hat{x}_{n+k} = \hat{p}_k$ respectively, where $[\hat{q}_k, \hat{p}_l] = i$ for $k, l = 1, 2, \dots, n$. We introduce the linear combinations of the canonical variables:

$$\hat{X} = \sum_{k=1}^{2n} a_k \hat{x}_k, \quad (14)$$

with arbitrary coefficients $\{a_k\}$. Consider a collection of linear combinations

$$\hat{A}_\alpha = \sum_{k=1}^{2n} A_{\alpha k} \hat{x}_k, \quad \alpha \in \Omega, \quad (15)$$

which form a complete (or overcomplete) basis in the linear space $\{\hat{X}\}$. Let them be the operators to be ordered as given in section 2. If $\dots \succ \alpha_3 \succ \alpha_2 \succ \alpha_1$ holds then

$$\begin{aligned} \mathcal{O}e^{\hat{X}} &= \mathcal{O} \prod_{\alpha \in \Omega} e^{\lambda_\alpha \hat{A}_\alpha} \\ &= \dots \exp(\lambda_{\alpha_3} \hat{A}_{\alpha_3}) \exp(\lambda_{\alpha_2} \hat{A}_{\alpha_2}) \exp(\lambda_{\alpha_1} \hat{A}_{\alpha_1}), \end{aligned} \quad (16)$$

where

$$\hat{X} = \sum_{\alpha \in \Omega} \lambda_\alpha \hat{A}_\alpha. \quad (17)$$

It is important to anticipate that whenever we write $\mathcal{O}e^{\hat{X}}$ or $\mathcal{O}\hat{X}^2$, we understand the above linear combination in terms of the operators to be ordered. If $\{\hat{A}_\alpha\}$ forms an overcomplete basis, it matters for the ordering that the coefficients $\{\lambda_\alpha\}$ are not unique. In the case of simple redundancy $\hat{A}_\alpha = \hat{A}_\beta$ at $\alpha \neq \beta$ we can, if we wish to, remove this ambiguity by collapsing the coefficients: $\lambda_\alpha = \lambda_\beta$.

To prepare our GWT, we consider another collection of operators to be ordered another way \mathcal{O}' :

$$\hat{A}_a = \sum_{k=1}^{2n} A_{ak} \hat{x}_k, \quad a \in \mathcal{O}. \quad (18)$$

If $\dots \succ a_3 \succ a_2 \succ a_1$ holds then

$$\begin{aligned} \mathcal{O}'e^{\hat{X}} &= \mathcal{O}' \prod_{a \in \mathcal{O}} e^{\lambda_a \hat{A}_a} \\ &= \dots \exp(\lambda_{a_3} \hat{A}_{a_3}) \exp(\lambda_{a_2} \hat{A}_{a_2}) \exp(\lambda_{a_1} \hat{A}_{a_1}), \end{aligned} \quad (19)$$

where

$$\hat{X} = \sum_{a \in \mathcal{O}} \lambda_a \hat{A}_a. \quad (20)$$

We propose the following relationship between the two orderings:

$$\mathcal{O}'e^{\hat{X}} = e^C \mathcal{O}e^{\hat{X}}, \quad (21)$$

where the pre-factor is a c -number because the exponent is c -number:

$$C = \frac{1}{2}(\mathcal{O}' - \mathcal{O})\hat{X}^2. \quad (22)$$

We call it the *general contraction* between \mathcal{O}' and \mathcal{O} . This is the GWT, our central result.

If we substitute $\mathcal{O}' = \mathcal{W}$ and the identity $\mathcal{W}\hat{X}^2 = \hat{X}^2$, we get a simple equivalent form of GWT:

$$\mathcal{O}e^{\hat{X}} = e^C e^{\hat{X}} \tag{23}$$

$$C = \frac{1}{2}(\mathcal{O}\hat{X}^2 - \hat{X}^2), \tag{24}$$

where C is a c -number, being the contraction between \mathcal{O} and \mathcal{W} . To express its detailed structure, we insert the decomposition (17) of \hat{X} :

$$\begin{aligned} C &= \frac{1}{2} \sum_{\alpha, \beta} \lambda_\alpha \lambda_\beta \left(\mathcal{O}\hat{A}_\alpha \hat{A}_\beta - \frac{1}{2} \{ \hat{A}_\alpha, \hat{A}_\beta \} \right) \\ &= \frac{1}{2} \sum_{\alpha > \beta} \lambda_\alpha \lambda_\beta [\hat{A}_\alpha, \hat{A}_\beta]. \end{aligned} \tag{25}$$

To further simplify the result, we use the expansion (15) of \hat{A}_α , with the new notations $B_{\alpha k} = A_{\alpha, n+k}$ for the coefficients of the momenta, yielding:

$$C = \frac{i}{2} \sum_{\alpha > \beta} \lambda_\alpha \lambda_\beta \sum_{k=1}^n (A_{\alpha k} B_{\beta k} - A_{\beta k} B_{\alpha k}). \tag{26}$$

We note that in the special case when \mathcal{O}' and \mathcal{O} are to order the same collection of operators (in two different ways, of course) then GWT (21) and (22) can be stated for the characteristic functions as well:

$$\begin{aligned} \hat{F}_{\mathcal{O}'}(\lambda) &= \exp \left(\frac{1}{2} \sum_{\alpha, \beta} C_{\alpha\beta} \lambda_\alpha \lambda_\beta \right) \hat{F}_{\mathcal{O}}(\lambda) \\ C_{\alpha\beta} &= \frac{\partial^2}{\partial \lambda_\alpha \partial \lambda_\beta} \left[\hat{F}_{\mathcal{O}'}(\lambda) - \hat{F}_{\mathcal{O}}(\lambda) \right]_{\lambda=0}. \end{aligned} \tag{27}$$

4. Tentative proof

Let us consider a given \hat{X} , as in (14), and its decompositions (17) and (20):

$$\hat{X} = \sum_{k=1}^{2n} a_k \hat{x}_k = \sum_{\alpha \in \Omega} \lambda_\alpha \hat{A}_\alpha = \sum_{a \in \mathcal{O}} \lambda_a \hat{A}_a. \tag{28}$$

The proof of GWT (21) and (22) simplifies if we restrict ourselves for the case when all coefficients $a_k, A_{\alpha k}, A_{ak}$ are non-negative. Then we construct the asymptotic form of the maximally refined decomposition of \hat{X} :

$$\hat{X} = \lim_{\epsilon \rightarrow +0} \sum_{k=1}^{2n} \underbrace{(\epsilon \hat{x}_k + \epsilon \hat{x}_k + \dots + \epsilon \hat{x}_k)}_{[a_k/\epsilon]}. \tag{29}$$

If we approach the limit $\epsilon \rightarrow 0$, the two orderings in question act on the *same* (yet unordered) product of exponentials:

$$\begin{aligned} \mathcal{O}e^{\hat{X}} &= \lim_{\epsilon \rightarrow +0} \mathcal{O} \left\{ \prod_{k=1}^{2n} \underbrace{(e^{\epsilon \hat{x}_k} \times e^{\epsilon \hat{x}_k} \times \dots \times e^{\epsilon \hat{x}_k})}_{[a_k/\epsilon]} \right\}, \\ \mathcal{O}'e^{\hat{X}} &= \lim_{\epsilon \rightarrow +0} \mathcal{O}' \left\{ \prod_{k=1}^{2n} \underbrace{(e^{\epsilon \hat{x}_k} \times e^{\epsilon \hat{x}_k} \times \dots \times e^{\epsilon \hat{x}_k})}_{[a_k/\epsilon]} \right\}, \end{aligned} \tag{30}$$

where the number of factors $e^{\epsilon \hat{x}_k}$ is $[a_k/\epsilon]$ for all k . It will be the order of the factors, and nothing else, that distinguishes \mathcal{O} and \mathcal{O}' . From one to the other, we can go by a finite sequence $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_K$ of K switches between neighboring factors. ($K \rightarrow \infty$ for $\epsilon \rightarrow +0$.) Namely,

$$\mathcal{O}' e^{\hat{X}} = \mathcal{P}_K \dots \mathcal{P}_2 \mathcal{P}_1 \mathcal{O} e^{\hat{X}}, \quad (31)$$

where we understand the same (redundant) factorization (30) for $e^{\hat{X}}$. Let us consider the rhs of the above equation and suppose \mathcal{P}_K inverts the order of neighbors $e^{\epsilon \hat{x}_k} e^{\epsilon \hat{x}_l}$ in $\mathcal{P}_{K-1} \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}}$:

$$\mathcal{P}_K e^{\epsilon \hat{x}_k} e^{\epsilon \hat{x}_l} = e^{\epsilon \hat{x}_l} e^{\epsilon \hat{x}_k} \equiv e^{\epsilon^2 [\hat{x}_l, \hat{x}_k]} e^{\epsilon \hat{x}_k} e^{\epsilon \hat{x}_l}, \quad (32)$$

where we used the simple Baker–Campbell–Hausdorff [9] identity. We can write:

$$\mathcal{P}_K \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}} = e^{\epsilon^2 [\hat{x}_l, \hat{x}_k]} \mathcal{P}_{K-1} \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}}. \quad (33)$$

Observe that

$$\mathcal{P}_K \dots \mathcal{P}_1 \mathcal{O} \hat{X}^2 - \mathcal{P}_{K-1} \dots \mathcal{P}_1 \mathcal{O} \hat{X}^2 = 2\epsilon^2 [\hat{x}_l, \hat{x}_k]. \quad (34)$$

Using this, the rhs of (33) reads:

$$e^{\frac{1}{2}(\mathcal{P}_K \dots \mathcal{P}_1 - \mathcal{P}_{K-1} \dots \mathcal{P}_1) \mathcal{O} \hat{X}^2} \mathcal{P}_{K-1} \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}}. \quad (35)$$

We repeat the same identity transformation on the factor $\mathcal{P}_{K-1} \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}}$, yielding

$$e^{\frac{1}{2}(\mathcal{P}_K \dots \mathcal{P}_1 - \mathcal{P}_{K-2} \dots \mathcal{P}_1) \mathcal{O} \hat{X}^2} \mathcal{P}_{K-2} \dots \mathcal{P}_1 \mathcal{O} e^{\hat{X}}. \quad (36)$$

After K steps, we get

$$e^{\frac{1}{2}(\mathcal{P}_K \dots \mathcal{P}_1 \mathcal{O} - \mathcal{O}) \hat{X}^2} \mathcal{O} e^{\hat{X}} = e^{\frac{1}{2}(\mathcal{O}' - \mathcal{O}) \hat{X}^2} \mathcal{O} e^{\hat{X}} \quad (37)$$

where we used $\mathcal{O}' = \mathcal{P}_K \dots \mathcal{P}_1 \mathcal{O}$ from (31). This is our ultimate expression for the rhs of (21). Our GWT is confirmed.

Recall that we restricted the proof: the operators to be ordered had to be linear combinations of the canonical variables with non-negative coefficients. The maximally refined decomposition (29) contained terms with the unique positive coefficient ϵ . If we want to extend the proof for orderings of operators without the above restriction, we have to construct the maximally refined decomposition for all \hat{X} . There will be four types of coefficients: $\pm\epsilon, \pm i\epsilon$ instead of the positive ones ϵ . Once we have constructed such a maximally refined decomposition to host both \mathcal{O} and \mathcal{O}' , the proof follows the same steps as before—and needs a definitely more intricate book-keeping of the four types of exponentials. Intuition says, nonetheless, that the validity of our theorem extends from positive parameters for all complex ones just by the theorem's analytic form.

5. Examples: \mathcal{T} versus QP versus \mathcal{N} orderings

We are going to illustrate the flexibility of GWT in solving simple tasks in elementary quantum mechanics and quantum optics.

Consider the Schrödinger equation $\dot{\psi}_t = -\hat{H}_t \psi_t$ of a mass m , where the Hamiltonian $\hat{H}_t = (\hat{p}^2/2m) - F_t \hat{q}$ contains a given time-dependent force. Let us find the solution $\psi_t^I = \hat{U}_t \psi_0$ in the interaction picture where $\hat{q}_t = \hat{q} + \hat{p}t/m$ and $\hat{p}_t = \hat{p}$. (As is known, the solution ψ_t in Schrödinger picture is obtained if we solve the force-free Schrödinger equation with ψ_t^I as the initial state.) The evolution operator in interaction picture is a \mathcal{T} -ordered exponential:

$$\hat{U}_t = \mathcal{T}e^{\hat{X}_t}, \quad (38)$$

where \hat{X}_t is linear combination of the canonical variables:

$$\hat{X}_t = i \int_0^t F_\tau \hat{q}_\tau d\tau. \quad (39)$$

The rhs is the *chronological* decomposition of \hat{X}_t : the summation label $\alpha \in \Omega$ in (17) became the integral variable $\tau \in (0, t)$, the set $\{\hat{A}_\alpha\}$ became $\{\hat{q}_\tau\}$, the coefficients λ_α became iF_τ . To evaluate \hat{U}_t , we shall consider its QP-ordering, hence we need a *canonical* decomposition of \hat{X}_t :

$$\hat{X}_t = i \int_0^t F_\tau d\tau \hat{q} + \frac{i}{m} \int_0^t F_\tau \tau d\tau \hat{p} \equiv i\Delta p_t \hat{q} - i\Delta q_t \hat{p}, \quad (40)$$

where we introduced the momentum and coordinate shifts $\Delta p_t, \Delta q_t$ for notational convenience. The label a in (20) takes two values only: $\hat{A}_{a=1} = \hat{q}$ and $\hat{A}_{a=2} = \hat{p}$, also $\lambda_{a=1} = i\Delta p_t$ and $\lambda_{a=2} = -i\Delta q_t$. The QP-ordering \mathcal{O}_{QP} pushes all \hat{q} to the left of all \hat{p} . Now we apply our GWT (21)–(38),

$$\hat{U}_t = \mathcal{T}e^{\hat{X}_t} = e^{C_t} e^{i\Delta p_t \hat{q}} e^{-i\Delta q_t \hat{p}}, \quad (41)$$

with the contraction (22):

$$C_t \equiv \frac{1}{2}(\mathcal{T} - \mathcal{O}_{QP})\hat{X}_t^2 = \frac{i}{2} \int_0^t \int_0^t F_\tau F_\sigma |\tau - \sigma| d\tau d\sigma. \quad (42)$$

$\hat{U}_t \psi_0$ provides the following explicit solution for the wave function in the interaction picture:

$$\psi_t^I(q) = e^{C_t} e^{i\Delta p_t q} \psi_0(q - \Delta q_t). \quad (43)$$

A similar standard task is the electromagnetic cavity oscillator of frequency ω under external driving. The Hamiltonian reads: $\hat{H}_t = \omega \hat{c}^\dagger \hat{c} + (E_t^* \hat{c} - \text{H.c.})$. In interaction picture, where $\hat{c}_t = e^{-i\omega t} \hat{c}$ and $\hat{c}_t^\dagger = e^{i\omega t} \hat{c}^\dagger$, the evolution operator is (38) again, with

$$\hat{X}_t = \int_0^t (E_\tau^* \hat{c}_\tau - \text{H.c.}) d\tau \equiv \Delta c_t^* \hat{c} - \text{H.c.} \quad (44)$$

We can rewrite the evolution operator \hat{U}_t into the normal ordered form, applying the GWT (21) and (22):

$$\hat{U}_t = e^{\frac{1}{2}(\mathcal{T} - \mathcal{N})\hat{X}_t^2} \mathcal{N}e^{\hat{X}_t} = e^{C_t} e^{-\Delta c_t \hat{c}^\dagger} e^{\Delta c_t^* \hat{c}}, \quad (45)$$

$$C_t = - \int_0^t \int_0^t \theta(\tau - \sigma) E_\tau^* E_\sigma e^{-i\omega(\tau - \sigma)} d\tau d\sigma. \quad (46)$$

In quantum optics, there is a spectacularly simple definition of the squeezing operator with squeezing parameter μ)0:

$$\hat{S} = \frac{1}{\sqrt{\mu}} \int |q/\mu\rangle \langle q| dq = \mathcal{O}_{PQ} e^{i(1-1/\mu)\hat{p}\hat{q}}, \quad (47)$$

where the first expression was proposed in [10] while the second compact form is our finding based on it. To achieve the normal ordered form, which was the task in [10] as well, we first unravel the bilinear form $\hat{p}\hat{q}$ in the exponent. Introduce $\kappa = 1 - 1/\mu$ and consider

$$\hat{S} = \mathcal{O}_{PQ} e^{i\kappa \hat{p} \hat{q}} = \int \mathcal{O}_{PQ} e^{iz\hat{p} \pm z^* \hat{q}} \exp\left(-\frac{|z|^2}{|\kappa|}\right) \frac{d^2z}{\pi|\kappa|}, \quad (48)$$

where the sign \pm is the sign of κ . Now we can apply our GWT (21)–(22):

$$\mathcal{O}_{PQ} e^{iz\hat{p} \pm z^* \hat{q}} = e^{\frac{1}{4}(z^*)^2 - \frac{1}{4}z^2 \pm \frac{1}{2}|z|^2} \mathcal{N} e^{iz\hat{p} \pm z^* \hat{q}}. \quad (49)$$

Inserting this in (48) and performing the complex integral, we obtain:

$$\begin{aligned} \hat{S} &= \sqrt{\frac{2\mu}{1+\mu^2}} \mathcal{N} \exp\left\{\frac{-i(1-\mu^2)\hat{p}\hat{q} - \frac{1}{2}(1-\mu)^2(\hat{p}^2 + \hat{q}^2)}{1+\mu^2}\right\} \\ &= \sqrt{\frac{2\mu}{1+\mu^2}} \mathcal{N} \exp\left\{\frac{-\frac{1}{2}(1-\mu^2)(\hat{c}^2 - \hat{c}^{\dagger 2}) + (1-\mu)^2 \hat{c}^\dagger \hat{c}}{1+\mu^2}\right\}. \end{aligned} \quad (50)$$

6. s-Ordering

The s -orderings are defined by their characteristic function (4). Apart from the two marginal cases $s = \pm 1$ they are non-monomial and the induced scheme (9) of ordering is not unique. First we redefine a given s -ordering as a path (i.e.: monomial) ordering of a redundant collection of the canonical operators, then we can construct the corresponding scheme of their s -ordering.

6.1. s-Ordering as path ordering

We are going to show that an s -ordering is equivalent with path-orderings \mathcal{T} of a (redundant) collection of operators $\{\hat{c}_\tau, \hat{c}_\tau^\dagger; \tau \in [0, 1]\}$ where we take $\hat{c}_\tau \equiv \hat{c}$ and $\hat{c}_\tau^\dagger \equiv \hat{c}^\dagger$ after the path-ordering only. To define a concrete ordering, we must first introduce a decomposition of the exponent $\hat{X} = \lambda^* \hat{c} + \lambda \hat{c}^\dagger$. Let us choose the following structure:

$$\hat{X} = \int_0^1 (\lambda^* \hat{c}_\tau d\tau + \lambda \hat{c}_\tau^\dagger d\chi_\tau), \quad (51)$$

where $\chi_0 = 0$ and $\chi_1 = 1$, also χ_τ must be monotonous. As we said, we cancel the label τ of \hat{c}, \hat{c}^\dagger after the path-ordering. Along the path, the rate of \hat{c}^\dagger 's versus the constant rate of \hat{c} 's is ruled by χ_τ . We prove below that this path-ordering is equivalent with s -ordering (4):

$$\mathcal{O}_s e^{\lambda^* \hat{c} + \lambda \hat{c}^\dagger} = \mathcal{T} \exp\left\{\int_0^1 (\lambda^* \hat{c}_\tau d\tau + \lambda \hat{c}_\tau^\dagger d\chi_\tau)\right\} \Big|_{\hat{c}_\tau = \hat{c}}, \quad (52)$$

provided χ_τ satisfies

$$s = 1 - 2 \int_0^1 \chi_\tau d\tau. \quad (53)$$

The proof goes like this. Considering the decomposition (51) of \hat{X} , we write the rhs of (52) as $\mathcal{T} e^{\hat{X}}$ and apply (23) and (24) to it:

$$\mathcal{T} e^{\hat{X}} = e^C e^{\lambda^* \hat{c} + \lambda \hat{c}^\dagger}, \quad (54)$$

where

$$C = \frac{1}{2} (\mathcal{T} \hat{X}^2 - \hat{X}^2) = |\lambda|^2 \left(\int_0^1 \chi_\tau d\tau - \frac{1}{2} \right). \quad (55)$$

The steps that led to this expression of C were the following.

$$\mathcal{T}\hat{X}^2 = \int_0^1 \int_0^1 \mathcal{T} (\lambda^* \hat{c}_\tau d\tau + \lambda \hat{c}_\tau^\dagger d\chi_\tau) (\lambda^* \hat{c}_\sigma d\sigma + \lambda \hat{c}_\sigma^\dagger d\chi_\sigma). \quad (56)$$

Let us write \hat{X}^2 into the similar integral form:

$$\hat{X}^2 = \int_0^1 \int_0^1 (\lambda^* \hat{c} d\tau + \lambda \hat{c}^\dagger d\chi_\tau) (\lambda^* \hat{c} d\sigma + \lambda \hat{c}^\dagger d\chi_\sigma). \quad (57)$$

Then we get

$$\begin{aligned} 2C &= \mathcal{T}\hat{X}^2 - \hat{X}^2 \\ &= |\lambda|^2 \int_0^1 \int_0^1 \{ (\mathcal{T}\hat{c}_\tau \hat{c}_\sigma^\dagger - \hat{c}\hat{c}^\dagger) d\tau d\chi_\sigma + (\mathcal{T}\hat{c}_\tau^\dagger \hat{c}_\sigma - \hat{c}^\dagger \hat{c}) d\chi_\tau d\sigma \} \\ &= |\lambda|^2 \int_0^1 \int_0^1 \theta(\sigma - \tau) (d\chi_\tau d\sigma - d\tau d\chi_\sigma) \\ &= |\lambda|^2 \left(2 \int_0^1 \chi_\sigma d\sigma - 1 \right). \end{aligned} \quad (58)$$

Therefore (54), i.e. the rhs of (52), reads:

$$\mathcal{T}e^{\hat{X}} = \exp \left(|\lambda|^2 \int_0^1 \chi_\tau d\tau - \frac{|\lambda|^2}{2} \right) e^{\lambda^* \hat{c} + \lambda \hat{c}^\dagger}. \quad (59)$$

This, at the condition (53), becomes $\mathcal{O}_s e^{\lambda^* \hat{c} + \lambda \hat{c}^\dagger}$ as defined by (4).

6.2. Non-monomial schemes induced by s -ordering

Derivatives of the characteristic function (4) yield the s -ordered products of the canonical operators \hat{c}, \hat{c}^\dagger :

$$\begin{aligned} \mathcal{O}_s \hat{c}^n (\hat{c}^\dagger)^m &= \left(\frac{\partial}{\partial \lambda^*} \right)^n \left(\frac{\partial}{\partial \lambda} \right)^m \mathcal{O}_s e^{\lambda^* \hat{c} + \lambda \hat{c}^\dagger} \Big|_{\lambda=0} \\ &= \left(\frac{\partial}{\partial \lambda^*} \right)^n \left(\frac{\partial}{\partial \lambda} \right)^m e^{-s|\lambda|^2/2} e^{\lambda \hat{c}^\dagger + \lambda^* \hat{c}} \Big|_{\lambda=0}, \end{aligned} \quad (60)$$

for $n, m = 0, 1, 2, \dots$. Beyond the value $\mathcal{O}_s \hat{c}^n (\hat{c}^\dagger)^m$, our point of interest is the *scheme* (9) of ordering:

$$\mathcal{O}_s \hat{c}^n (\hat{c}^\dagger)^m = \sum_{P=1}^{(n+m)!} w_P [\hat{c}^n (\hat{c}^\dagger)^m]_P, \quad (61)$$

where $w_P \geq 0$ is the weight of the P th permutation, the weights are normalized; $[\hat{c}^n (\hat{c}^\dagger)^m]_P$ stands for the P th permutation of the factors.

We shall consider $\mathcal{O}_s \hat{c}^n (\hat{c}^\dagger)^m$ for $n = m = 1$ first. If we wish to bring it into the form on the rhs of (61), we get the following unique result:

$$\mathcal{O}_s \hat{c} \hat{c}^\dagger = \frac{1+s}{2} \hat{c}^\dagger \hat{c} + \frac{1-s}{2} \hat{c} \hat{c}^\dagger. \quad (62)$$

As expected, $s \in [-1, 1]$ interpolates between normal and anti-normal orderings. Next, let us take $n = 2, m = 1$. It is easy to derive, e.g. the following two forms (out of an infinite family):

$$\mathcal{O}_s \hat{c}^2 \hat{c}^\dagger = \hat{c}^\dagger \hat{c}^2 + (1-s)\hat{c} = \hat{c}^2 \hat{c}^\dagger - (1+s)\hat{c}. \quad (63)$$

Using the identity $[\hat{c}, \hat{c}^\dagger] = 1$, we could bring these two to two different forms, resp., both of them conform with the rhs of (61). But the issue grows unmanageable for higher powers n, m unless we find a systematic way. Let us use our concept of orderings and the GTW (section 3).

Inserting (52) into (60), we obtain the general expression

$$\begin{aligned} \mathcal{O}_s \hat{c}^n (\hat{c}^\dagger)^m &= \mathcal{T} \left(\int_0^1 \hat{c}_\tau d\tau \right)^n \left(\int_0^1 \hat{c}_\tau^\dagger d\chi_\tau \right)^m \Big|_{\hat{c}_\tau = \hat{c}} \\ &= \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_n \int_0^1 d\chi_{\sigma_1} \cdots \int_0^1 d\chi_{\sigma_m} \mathcal{T} \hat{c}_{\tau_1} \cdots \hat{c}_{\tau_n} \hat{c}_{\sigma_1}^\dagger \cdots \hat{c}_{\sigma_m}^\dagger \Big|_{\hat{c}_\tau = \hat{c}}, \end{aligned} \quad (64)$$

which is already conform with the structure on the rhs of (61). Here we work out the special case $n = 2, m = 1$ only:

$$\begin{aligned} \mathcal{O}_s \hat{c}^2 \hat{c}^\dagger &= \int_0^1 d\chi_\tau \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \mathcal{T} \hat{c}_\tau^\dagger \hat{c}_{\sigma_1} \hat{c}_{\sigma_2} \Big|_{\hat{c}_\tau = \hat{c}} \\ &= w_1 \hat{c}^\dagger \hat{c} \hat{c} + w_2 \hat{c} \hat{c}^\dagger \hat{c} + w_3 \hat{c} \hat{c} \hat{c}^\dagger, \end{aligned} \quad (65)$$

with

$$\begin{aligned} w_1 &= \int_0^1 d\chi_\tau \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \theta(\tau - \sigma_1) \theta(\tau - \sigma_2) \\ &= \int_0^1 d\chi_\tau \tau^2, \\ w_2 &= 2 \int_0^1 d\chi_\tau \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \theta(\sigma_1 - \tau) \theta(\tau - \sigma_2) \\ &= 2 \int_0^1 d\tau \int_0^\tau d\chi_\tau \tau, \\ w_3 &= \int_0^1 d\chi_\tau \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \theta(\sigma_1 - \tau) \theta(\sigma_2 - \tau) \\ &= 2 \int_0^1 d\tau \int_0^\tau d\tau \chi_\tau. \end{aligned} \quad (66)$$

The constraint $w_1 + w_2 + w_3 = 1$ is satisfied by construction. The three integrals (66) depend on the choice of χ_τ . If we take a simplest function $\chi_\tau = \tau^\kappa$ with $\kappa = \frac{1+s}{1-s} \geq 1$, covering the values $s \in [0, 1)$, we get

$$\mathcal{O}_s \hat{c}^\dagger \hat{c}^2 = \frac{(1-s)\hat{c}^\dagger \hat{c} \hat{c} + (1-s^2)\hat{c} \hat{c}^\dagger \hat{c} + (1+s)^2 \hat{c} \hat{c} \hat{c}^\dagger}{3+s}. \quad (67)$$

Obviously, other functions χ_τ would have resulted in different weights w_1, w_2, w_3 , i.e. in different schemes (61) of the same non-monomial s -ordering.

7. Concluding remarks

In 1950, Wick's theorem between normal and chronological orderings of relativistic quantum fields was invented for and became the robust tool of S-matrix theory of interactions. The present work started from the recognition that Wick's theorem, at least the bosonic one, is the specific case of a simple relationship between operator orderings in general. That is our general Wick theorem (GWT). Wick's original contraction is replaced by the general contraction, a form which is fairly straightforward and transparent. A little more involved, yet plausible, is the proposed definition of operator orderings. It covers all common orderings, typical in quantum optics and quantum field theory, and opens a wide perspective toward new ones. The general Wick theorem (GWT) clarifies the universal structure of orderings. Contrary to naive expectations, the general Wick theorem (GWT) seems to hold for monomial orderings (permutations) only. Fortunately, we could show that s -orderings of Cahill and Glauber (including the Weyl–Wigner ordering) are reductions of redundant monomial path orderings, hence the general Wick theorem (GWT) extends for them.

Simple form of the general Wick theorem (GWT) facilitates analytic calculations. We showed plain explanatory applications in forced quantum mechanical motion, a short derivation of the normal-ordered squeezing operator and a less trivial application to elucidate s -orderings of Cahill and Glauber. The power of general orderings and their Wick theorem would get confirmed further by their future non-trivial utilizations.

It should be admitted on one hand that our proof of the general Wick theorem (GWT) is tentative. On the other hand, more important is the non-triviality of this proof which tells us about the theorem's depths unless a simpler proof strategy pops up in the future.

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