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Schrödinger–Newton Equation with Spontaneous Wave Function Collapse

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Abstract: Based on the assumption that the standard Schrödinger equation becomes gravitationally modified for massive macroscopic objects, two independent proposals have survived from the 1980s. The Schrödinger–Newton equation (1984) provides well-localized solitons for free macro-objects but lacks the mechanism of how extended wave functions collapse on solitons. The gravity-related stochastic Schrödinger equation (1989) provides the spontaneous collapse, but the resulting solitons undergo a tiny diffusion, leading to an inconvenient steady increase in the kinetic energy. We propose the stochastic Schrödinger–Newton equation, which contains the above two gravity-related modifications together. Then, the wave functions of free macroscopic bodies will gradually and stochastically collapse to solitons, which perform inertial motion without momentum diffusion: conservation of momentum and energy is restored.

Keywords: Schrödinger–Newton equation; spontaneous wave function collapse; stochastic Schrödinger–Newton equation; solitons; energy-momentum conservation

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1. Introduction

The conjectured validity of quantum theory in the macroscopic world is famously problematic. This is shown usually on the elementary example: the free motion of an isolated macroscopic mass M. Its center of mass (c.o.m.) wave packet widens eternally, while we would expect a certain stationary localization. The most spectacular paradox concerns the Schrödinger cat states, which are superpositions of two distant wave packets, legitimate in the microworld but problematic for a macroscopic body.

A Newtonian semiclassical (mean-field) modification of the Schrödinger equation to ensure stationary localization was proposed first [1]. This Schrödinger–Newton equation (SNE) yields well-localized soliton wave packets for the mass M. However, the Schrödinger cat states remain legitimate solutions, and an independent mechanism is required to destroy them, as stated in ref. [1]. To this end, another Newton-gravity-related (G-related) mechanism was proposed in two steps. The G-related master equation (G-ME) decoheres the cat states [2]. Its unraveling, the G-related stochastic Schrödinger equation (G-SSE) [3], collapses them spontaneously to one or the other wave packet and drives this component into a soliton—however, with one annoying effect. The c.o.m. motion of the soliton never becomes stationary; it remains subject to tiny stochastic fluctuations generating kinetic energy at a constant rate.

Such spontaneous universal gain of energy from nothing, i.e., the robust violation of energy conservation, denies the existence of stationary free motion under the G-SSE, while it was possible for the solitons of the SNE. Our goal here is to show that the SNE and the G-SSE together will facilitate the issue so as to conserve energy and momentum at least in the free motion of massive objects.

Here, we propose the combination of the SNE and the G-SSE, obtaining the new stochastic SNE (SSNE). We show that the momentum diffusion of solitons of massive isolated objects is canceled by the interplay between the stochasticity of the collapse mechanism and the semiclassical Newtonian self-interaction. The remainder of the stochasticity is merely an extreme small universal coordinate diffusion proportional to $\hbar/M$. Our SSNE partially realizes the concept of ref. [4], where the diffusion effects of a stereotypical collapse equation (our Equation (11) below) were completely eliminated by imposing frame drag. Additionally, a related vision of induced gravity was put forward.

Sections 2 and 3 recapitulate the basics of the SNE and the G-SSE, invoking the results of refs. [1–3]. The combination of the two equations is introduced in Section 4. Then, Section 5 proves that the soliton momenta of SSNE are constant, so their diffusion is canceled. Section 6 contains final remarks.

2. Semiclassical Schrödinger–Newton Equation

Consider the standard Schrödinger equation $d|\Psi\rangle/dt = -(i/\hbar)\hat{H}|\Psi\rangle$ of a massive non-relativistic many-body system, with the Hamiltonian $\hat{H}$ not including the Newtonian pair potential. Let $\hat{\varrho}(\mathbf{r})$ denote the operator of mass density at the location $\mathbf{r}$, normalized to the total mass M. Instead of the Newton pair potential, one postulates that the classical gravitational field is sourced by the mean-field $\langle\hat{\varrho}(\mathbf{r})\rangle = \langle\Psi|\hat{\varrho}(\mathbf{r})|\Psi\rangle$. The semiclassical SNE reads

$$\begin{aligned} \frac{d|\Psi\rangle}{dt} &= -\frac{i}{\hbar}\hat{H}|\Psi\rangle + \frac{i}{\hbar}G\int\int\frac{\hat{\varrho}(\mathbf{r})\langle\hat{\varrho}(\mathbf{s})\rangle d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}|\Psi\rangle \\ &\equiv -\frac{i}{\hbar}(\hat{H}+\hat{V}_\Psi)|\Psi\rangle, \end{aligned} \tag{1}$$

where $\hat{V}_\Psi$ is the Ψ-dependent Newtonian semiclassical (mean-field) interaction.

The SNE possesses soliton solutions for the c.o.m. wave function of isolated bodies. Consider the motion of a single spherical symmetric rigid mass M; the canonical position and momentum operators are $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$, respectively. The mass density operator can be written as

$$\hat{\varrho}(\mathbf{r}) = Mf(\mathbf{r}-\hat{\mathbf{x}}) \tag{2}$$

where f is a normalized non-negative spherical symmetric function. Let us introduce the frequency parameter ω_G as defined in ref. [5]:

$$\omega_G^2 = \frac{4\pi}{3}GM\int f^2(\mathbf{r})d\mathbf{r}. \tag{3}$$

This sets an effective strength of Newtonian self-attraction (of the G-related spontaneous collapse, too, cf. Sections 3–5) when the position uncertainty Δx in state $|\Psi\rangle$ is much smaller than the characteristic length scale(s) of $f(\mathbf{r})$ [6]. Then, ignoring higher (than 2nd) order terms in $\hat{\mathbf{x}}_c = \hat{\mathbf{x}} - \langle\hat{\mathbf{x}}\rangle$, we can write the SNE (1) into the simple form (cf. Appendix A):

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle - \frac{i}{2\hbar}M\omega_G^2\hat{\mathbf{x}}_c^2|\Psi\rangle. \tag{4}$$

If $\hat{H} = \hat{\mathbf{p}}^2/2M$ and $\langle\hat{\mathbf{x}}\rangle = 0$, then the ground state coincides with the ground state of a central harmonic oscillator of frequency ω_G:

$$\Psi_0(\mathbf{x}) = \mathcal{N}\exp\left(-\frac{M\omega_G\mathbf{x}^2}{2\hbar}\right). \tag{5}$$

This is a soliton standing at the origin. The soliton and its inertially traveling versions are suitably representing the stationary localization of free macro objects.

However, there is no mechanism that drives larger solitons and large macroscopically extended wave functions toward the basic soliton states of shape (5). Consider, e.g., a

Schrödinger cat state which is a superposition of two solitons at a large distance ℓ from each other:

$$|\text{Cat}\rangle = \frac{|\text{left}\rangle + |\text{right}\rangle}{\sqrt{2}}. \tag{6}$$

They attract each other via $\hat{V}_\Psi$, can even form a Kepler system, but cannot collapse to a single soliton. Ref. [1] emphasizes that collapse needs an additional mechanism not present in the SNE.

3. Gravity-Related Wavefunction Collapse

Alternatively to the SNE (1), the G-SSE considers the following stochastic modification of the standard Schrödinger equation:

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle - \frac{G}{2\hbar}\int\int\frac{\hat{\varrho}_c(\mathbf{r})\hat{\varrho}_c(\mathbf{s})d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}|\Psi\rangle + \frac{1}{\hbar}\int\hat{\varrho}_c(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r}|\Psi\rangle, \tag{7}$$

where $\hat{\varrho}_c(\mathbf{r}) = \hat{\varrho}(\mathbf{r}) - \langle\hat{\varrho}(\mathbf{r})\rangle$, and $\Phi(\mathbf{r},t)$ is a white-noise field of spatial correlation

$$\mathbf{M}\Phi(\mathbf{r},t)\Phi(\mathbf{s},\tau) = \frac{G\hbar}{|\mathbf{r}-\mathbf{s}|}\delta(t-\tau), \tag{8}$$

$\mathbf{M}$ stands for the stochastic mean. The white-noise is to be taken in terms of the Ito differential calculus. This G-SSE yields the spontaneous collapse of massive macroscopic spatial superpositions of the Schrödinger cat states in particular. The average state, i.e., the density matrix $\hat{\rho} = \mathbf{M}|\Psi\rangle\langle\Psi|$, satisfies the G-ME:

$$\begin{aligned}\frac{d\hat{\rho}}{dt} &= -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{G}{2\hbar}\int\int\frac{[\hat{\varrho}(\mathbf{r}),[\hat{\varrho}(\mathbf{s}),\hat{\rho}]]d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\\ &\equiv -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{D}\hat{\rho}.\end{aligned} \tag{9}$$

This suppresses the coherence of macroscopically distinct superpositions of the mass density, yielding their statistical mixture without the collapse. The G-SSE (7) also adds the collapse to the decoherence. The two distant wave packets of the Schrödinger cat state (6) collapse onto one of them randomly at the rate

$$\frac{\Delta E_G}{\hbar}, \tag{10}$$

where ΔE_G denotes how much the collapse reduces the gravitational energy of the cat [7–9].

The G-SSE (7) becomes simple for a single mass in the regime of small Δx (cf. Appendices B and C):

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle - \frac{1}{2\hbar}M\omega_G^2\hat{\mathbf{x}}_c^2|\Psi\rangle + \sqrt{\frac{M}{\hbar}}\omega_G\hat{\mathbf{x}}_c\mathbf{w}|\Psi\rangle, \tag{11}$$

where $\mathbf{w}$ is the vector of three independent standard white noises. (The correspondence with notations in ref. [3] is $\Gamma = 2M\omega_G^2/\hbar$ and $d\xi = \sqrt{M/\hbar}\omega_G\mathbf{w}$.) Note that the Hermitian self-attracting potential in the SNE (4) becomes anti-Hermitian here. For free bodies ($\hat{H} = \hat{\mathbf{p}}^2/2M$), the solutions converge to solitons of the steady shape

$$\Psi_0(\mathbf{x}) = \mathcal{N}\exp\left(-(1-i)\frac{M\omega_G\mathbf{x}^2}{2\hbar}\right). \tag{12}$$

This is slightly different from (5) due to the new complex factor $1-i$, which leaves the spread $\Delta x = \sqrt{(\hbar/2M\omega_G)}$ unchanged, simply creating the correlation $\hbar/2$ between $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. However, the major difference is in the c.o.m. motion, which is plagued by a certain correlated diffusion of both the position and the momentum:

$$\frac{d}{dt}\langle\hat{\mathbf{x}}\rangle = \frac{\langle\hat{\mathbf{p}}\rangle}{M} + \sqrt{\hbar/M}\mathbf{w}, \tag{13}$$

$$\frac{d}{dt}\langle\hat{\mathbf{p}}\rangle = \sqrt{\hbar M}\omega_G\mathbf{w}. \tag{14}$$

The momentum diffusion (14) increases the kinetic energy at the rate $\frac{1}{2}\hbar\omega_G^2$, which is probably an unphysical artifact of the collapse model. We dispose of it below.

4. Schrödinger–Newton Equation with Wave Function Collapse

We slightly alter the original 1989 version (7) of the G-SSE. We insert a factor $\exp(-i\pi/4) = (1-i)/\sqrt{2}$ in front of the anti-Hermitian stochastic potential:

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle - \frac{G}{2\hbar}\int\int\frac{\hat{\varrho}_c(\mathbf{r})\hat{\varrho}_c(\mathbf{s})d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}|\Psi\rangle + \frac{\mathrm{e}^{-i\pi/4}}{\hbar}\int\hat{\varrho}_c(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r}|\Psi\rangle. \tag{15}$$

This yields the same G-ME (9) for the density matrix and also encodes the collapse of macroscopic superpositions. The novel feature is the appearance of the Hermitian stochastic potential $\Phi(\mathbf{r})/\sqrt{2}$, which will cancel the momentum diffusion (14) of the solitons, provided we include the gravitational self-attraction contained in $\hat{V}_\Psi$ of the SNE.

Our new proposal is the following combination of the SNE (1) and the modified G-SSE (15):

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}(\hat{H}+\hat{V}_\Psi)|\Psi\rangle - \frac{G}{2\hbar}\int\int\frac{\hat{\varrho}_c(\mathbf{r})\hat{\varrho}_c(\mathbf{s})d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}|\Psi\rangle + \frac{\mathrm{e}^{-i\pi/4}}{\hbar}\int\hat{\varrho}_c(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r}|\Psi\rangle. \tag{16}$$

The ME of the average state reads:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}+\hat{V}_\Psi,\hat{\rho}] + \mathcal{D}\hat{\rho}. \tag{17}$$

This is not a closed equation for $\hat{\rho}$ since $\hat{V}_\Psi$ depends on the pure state $|\Psi\rangle$. The lack of a closed linear ME is the signature of anomalies [10] already troubling the SNE and inherited by our SSNE (16). There is, however, a difference. The spontaneous collapse might shadow the anomalies. One of them, the fake action-at-a-distance, is based on the attraction caused by $\hat{V}_\Psi$ between the two halves $|\mathrm{left}\rangle, |\mathrm{right}\rangle$ of the Schrödinger cat [11]. Unlike in the case of the SNE, the cat now has the finite lifetime $\hbar/\Delta E_G$, which may be too short to reach detectable shifts of the left or the right wave packets [12].

We do not mean that the collapse mechanism prevents the fake action-at-a-distance exactly, yet we mean that this anomaly is relaxed compared to its robust presence in the SNE itself. A systematic way out of the difficulties may come from the starting idea of ultimate space-time indeterminacy [2,7] behind the scene. In a more advanced theory, the metric indeterminacy may, e.g., exclude the observability of the anomalies of our gravity-related quantum non-linearity.

5. Solitons with Energy Conservation

We consider the soliton solutions of the new spontaneous collapse dynamics SSNE in the small-Δx approximation. The single body special case of the SSNE (16) takes this form:

$$\frac{d|\Psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}|\Psi\rangle - \frac{1+i}{2\hbar}M\omega_G^2\hat{\mathbf{x}}_c^2|\Psi\rangle + (1-i)\sqrt{\frac{M}{2\hbar}}\omega_G\hat{\mathbf{x}}_c\mathbf{w}|\Psi\rangle. \tag{18}$$

Note the important complex factors $(1+i)$ and $(1-i)$ compared to the small-Δx approximation (11) of the G-SSE model. The ancestor of this equation is Equation (11) in [4]. It eliminated both momentum and coordinate diffusion by construction. The basis of the present work was the unexpected realization that the above small-Δx limit of the SSNE (16) coincides with Equation (11) in the special case when $R = 1/2$ in the solitonic state and

$\sigma = 0$ to remove the terms that would compensate coordinate diffusion as well. Below, we show an independent proof that the the soliton's momentum diffusion (14) cancels.

For the time-dependent wave function of the soliton, we take the following ansatz:

$$\begin{aligned}\Psi_t(\mathbf{x}) &= \mathcal{N}\exp\left(-(1-i)\frac{|\mathbf{x}-\langle\hat{\mathbf{x}}\rangle_t|^2}{4\Delta\mathrm{x}^2}+\frac{i}{\hbar}\langle\hat{\mathbf{p}}\rangle\hat{\mathbf{x}}\right),\\ \Delta\mathrm{x}^2 &= \frac{\hbar}{\sqrt{2}M\omega_G},\\ \frac{d}{dt}\langle\hat{\mathbf{x}}\rangle_t &= \frac{\mathbf{p}}{M}+\sqrt{\hbar/M}\mathbf{w}_t.\end{aligned} \tag{19}$$

These solutions correspond to inertial c.o.m. motion at a constant momentum $\langle\hat{\mathbf{p}}\rangle$, apart from a minuscule diffusion of the coordinate.

We still must show that the wave function $\Psi_t(\mathbf{x})$ satisfies the Equation (18). It is sufficient if we prove it for the soliton initially at rest at the origin, i.e., for $\langle\hat{\mathbf{x}}\rangle_0 = 0$ and $\langle\hat{\mathbf{p}}\rangle = 0$. First, we apply Equation (18) to $\Psi_0(\mathbf{x})$:

$$\frac{d\Psi_0(\mathbf{x})}{dt} = \left(i\hbar\frac{\nabla^2}{2M}-\frac{1+i}{2\hbar}M\omega_G^2\mathbf{x}^2+(1-i)\sqrt{\frac{M}{2\hbar}}\omega_G\mathbf{x}\mathbf{w}\right)\Psi_0(\mathbf{x}). \tag{20}$$

Second, we derive $d\Psi_0(\mathbf{x})/dt$ from the ansatz (19):

$$\frac{d\Psi_0(\mathbf{x})}{dt} = \left(-\sqrt{\hbar/M}\mathbf{w}\nabla+\frac{\hbar}{2M}\nabla^2\right)\Psi_0(\mathbf{x}), \tag{21}$$

where the second term comes from the Ito correction $(\mathbf{x}\mathbf{w}dt\nabla)^2 = \mathbf{x}^2\nabla^2 dt$. We substitute the expression

$$\nabla\Psi_0 = -(1-i)\frac{\mathbf{x}}{2\Delta\mathrm{x}^2}\Psi_0 = -(1-i)\frac{\mathbf{x}}{2}\frac{\sqrt{2}M\omega_G}{\hbar}\Psi_0$$

and then observe that the stochastic term coincides with that of (20). We also substitute the expression

$$\nabla^2\Psi_0 = \left(-i\frac{\mathbf{x}^2}{2\Delta\mathrm{x}^4}-(1-i)\frac{1}{2\Delta\mathrm{x}^2}\right)\Psi_0$$

in both Equations (20) and (21). Then, after elementary algebraic steps, their deterministic parts will also coincide.

6. Final Remarks

Penrose also proposed the spontaneous collapse rate (10) of the Schrödinger cat as well as the SNE (1) to generate the stationary states after the collapse [7–9]. He is treating the G-related spontaneous collapse and the SNE together. However, he has been in a holding position regarding any concrete dynamics (such as the G-SSE or others) of the collapse.

With the goal of reaching a closed linear ME to avoid the anomalies of the SNE, ref. [13] proposed a formal completion of the SNE by stochastic terms. Ref. [14] showed that this goal can be achieved by combining the SNE with the G-SSE, of course differently from the present proposal SSNE, which, contrary to refs. [13,14], sacrifices the closed ME in favor of energy-momentum conservation—at least in the free c.o.m. motion of the macroscopic body.

The advantage of the newly proposed SSNE (16) compared to the G-SSE (7) is this: when the state of a free massive body is collapsing towards the localized soliton, the spontaneous gain of kinetic energy is gradually disappearing, and in the soliton states, the energy-momentum conservation becomes restored. The non-conservations by the G-SSE (and by other collapse models) are probably warnings of the infancy of the models. Yet, their related predictions are currently the only testable effects [15], since massive Schrödinger cat states are not available in the laboratory to date. To test the G-SSE, ref. [16]

searched for the predicted c.o.m. momentum diffusion in the super-precise data of the Lisa Pathfinder experiment. Ref. [17] was hunting the spontaneous radiation predicted by the momentum diffusion of the nuclei inside the detector in the super-low-background Gran Sasso laboratory. Both works put an upper bound on the strength of the G-related spontaneous collapse. If we adopt the new SSNE, the predicted c.o.m. momentum diffusion drops to zero; hence, the strength of the G-related collapse could not be constrained by the Lisa Pathfinder data. The upper bound obtained with the Gran Sasso data can be retained because the new SSNE model predicts the same spontaneous radiation. The SSNE does not remove the momentum diffusion of the microscopic constituents but of the c.o.m. of the macro-objects.

Our proposal is the first dynamical model of spontaneous collapse with the partial restoration of energy-momentum conservation at the price of typical anomalies of semiclassical theories, which might become partially masked by the collapse mechanism. Future works should aim at a more complete restoration of energy-momentum conservation and at a better understanding of whether the said anomalies would become completely neutralized.

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Appendix A

To derive $\hat{V}_\Psi$ of Equation (1) for small Δx, we start from

$$\hat{V}_\Psi = -GM^2 \int\int \frac{f(\mathbf{r}-\hat{\mathbf{x}})\langle f(\mathbf{s}-\hat{\mathbf{x}})\rangle d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}, \tag{A1}$$

and consider the expansions

$$\begin{aligned} f(\mathbf{r}-\hat{\mathbf{x}}) &= \left(1-(\hat{\mathbf{x}}_c\nabla)+\frac{1}{2}(\hat{\mathbf{x}}_c\nabla)^2\right)f(\mathbf{r}-\langle\hat{\mathbf{x}}\rangle) && \text{(A2)}\\ \langle f(\mathbf{r}-\hat{\mathbf{x}})\rangle &= \left(1+\frac{1}{2}(\hat{\mathbf{x}}_c\nabla)^2\right)f(\mathbf{r}-\langle\hat{\mathbf{x}}\rangle) && \text{(A3)}\end{aligned}$$

omitting higher-order terms in $\hat{\mathbf{x}}_c$. Translation invariance of $\hat{V}_\Psi$ allows us to set $\langle\hat{\mathbf{x}}\rangle = 0$.

$$f(\mathbf{r}-\hat{\mathbf{x}})\langle f(\mathbf{s}-\hat{\mathbf{x}})\rangle = \left[1-\hat{\mathbf{x}}_c\nabla_r+\frac{1}{2}(\hat{\mathbf{x}}_c\nabla_r)^2\right]f(\mathbf{r})\left[1+\frac{1}{2}(\hat{\mathbf{x}}_c\nabla_s)^2\right]f(\mathbf{s}). \tag{A4}$$

Rotational invariance of $\hat{V}_\Psi$ cancels the linear term and yields the identity $(\hat{\mathbf{x}}_c\nabla)^2 = (1/3)\hat{\mathbf{x}}_c^2\Delta$; hence,

$$f(\mathbf{r}-\hat{\mathbf{x}})\langle f(\mathbf{s}-\hat{\mathbf{x}})\rangle = f(\mathbf{r})f(\mathbf{s})+\frac{1}{6}\hat{\mathbf{x}}_c^2[f(\mathbf{s})\Delta f(\mathbf{r})+f(\mathbf{r})\Delta f(\mathbf{s})], \tag{A5}$$

ignoring higher orders of $\hat{\mathbf{x}}_c$. Using this in Equation (A1), we obtain:

$$\begin{aligned}\hat{V}_\Psi &= -GM^2\int\int\frac{f(\mathbf{r})f(\mathbf{s})d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}+\frac{GM^2\hat{\mathbf{x}}_c^2}{6}\int\int\frac{f(\mathbf{s})\Delta f(\mathbf{r})+f(\mathbf{r})\Delta f(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}d\mathbf{r}d\mathbf{s}\\ &= 2E_G+\frac{4\pi GM^2}{3}\hat{\mathbf{x}}_c^2\int f^2(\mathbf{r})d\mathbf{r} = 2E_G+\frac{1}{2}M\omega_G^2\hat{\mathbf{x}}_c^2, \end{aligned} \tag{A6}$$

where partial integrations and the indentity $\Delta|\mathbf{r}-\mathbf{s}|^{-1} = -4\pi\delta(\mathbf{r}-\mathbf{s})$ have been used. The constant E_G stands for the gravitational self-energy.

Appendix B

To derive the double integral in Equation (7) for small Δx, we start from

$$\frac{GM^2}{2}\int\int \frac{f_c(\mathbf{r}-\hat{\mathbf{x}})f_c(\mathbf{s}-\hat{\mathbf{x}}_c)d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}, \tag{A7}$$

and substitute the expansion

$$f_c(\mathbf{r}-\hat{\mathbf{x}}) = f(\mathbf{r}-\hat{\mathbf{x}}) - \langle f(\mathbf{r}-\hat{\mathbf{x}})\rangle = -(\hat{\mathbf{x}}_c\nabla)f(\mathbf{r}-\langle\hat{\mathbf{x}}\rangle). \tag{A8}$$

Again, we can take $\langle\hat{\mathbf{x}}\rangle = 0$, yielding

$$\frac{GM^2\hat{\mathbf{x}}_c^2}{6}\int\int \frac{\nabla_r f(\mathbf{r})\nabla_s f(\mathbf{s})d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|} = \frac{4\pi GM^2\hat{\mathbf{x}}_c^2}{6}\int f^2(\mathbf{r})d\mathbf{r} = \frac{1}{2}M\omega_G^2\hat{\mathbf{x}}_c^2. \tag{A9}$$

Appendix C

To derive the stochastic term in Equation (7) for small Δx, we write

$$\frac{1}{\hbar}\int \hat{\varrho}_c(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r} = \frac{M}{\hbar}\hat{\mathbf{x}}_c\int \nabla f(\mathbf{r})\Phi(\mathbf{r})d\mathbf{r} \tag{A10}$$

and introduce the new stochastic variable, linear in the old stochastic field $\Phi(\mathbf{r})$:

$$\mathbf{w} = \sqrt{\frac{M}{\hbar}}\omega_G^{-1}\int \nabla f(\mathbf{r}-\langle\hat{\mathbf{x}}\rangle)\Phi(\mathbf{r})d\mathbf{r}. \tag{A11}$$

For the correlation function, we obtain the following:

$$\begin{aligned}\mathbf{M}\mathbf{w}_t\circ\mathbf{w}_\tau &= \frac{M}{\hbar}\omega_G^{-2}\int\int(\nabla f(\mathbf{r})\circ\nabla f(\mathbf{s}))\mathbf{M}\Phi(\mathbf{r},t)\Phi(\mathbf{s},\tau)d\mathbf{r}d\mathbf{s}\\ &= \frac{M}{\hbar}\omega_G^{-2}\int\int f(\mathbf{r})f(\mathbf{s})\nabla_r\circ\nabla_s)\frac{\hbar G}{|\mathbf{r}-\mathbf{s}|}d\mathbf{r}d\mathbf{s}\delta(t-\tau)\\ &= \omega_G^{-2}\frac{4\pi}{3}I_{3\times3}GM\int f^2(\mathbf{r})d\mathbf{r}\delta(t-\tau) = I_{3\times3}\delta(t-\tau)\end{aligned} \tag{A12}$$

where $\mathbf{M}\Phi(\mathbf{r},t)\Phi(\mathbf{s},\tau)$ has been substituted by the expression (8).

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