Linear-friction many-body equation for dissipative spontaneous wave-function collapse

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We construct and study the simplest universal dissipative Lindblad master equation for many-body systems with the purpose of a new dissipative extension of existing nonrelativistic theories of fundamental spontaneous decoherence and spontaneous wave function collapse in nature. It is universal as it is written in terms of second-quantized mass density $\hat{\varrho}$ and current $\hat{\mathbf{J}}$, thus making it independent of the material structure and its parameters. Assuming linear friction in the current, we find that the dissipative structure is strictly constrained. Following the general structure of our dissipative Lindblad equation, we derive and analyze the dissipative extensions of the two most known spontaneous wave function collapse models, the Diósi-Penrose and the continuous spontaneous localization models.

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I. INTRODUCTION

Testable predictions of quantum theory assume the presence of measuring devices providing data on the quantum system in question. This process, called quantum measurement, yields the collapse of superpositions in accordance with random measurement outcomes. The concept of spontaneous collapse can be interpreted as the hypothesis that measurements are occurring spontaneously at each point of space and time according to some universal protocol, but without the presence of actual measuring devices [1]. For a comprehensive overview of the theory of spontaneous collapse models and their experimental testing, see reviews in Refs. [2–5]. Through a suitable choice of its structure and parameters, the protocol ensures that such spontaneous collapses of the wave functions ϕ become significant for macroscopic (e.g., massive) systems but remains negligible for microscopic ones. Currently, two nonrelativistic models, the Diósi-Penrose (DP) and the continuous spontaneous localization (CSL), have been crystallized [6–9], and they correspond to the spontaneous measurement of the mass spatial density operator $\hat{\varrho}(\mathbf{r},t)$ at all $\mathbf{r}$ and t. Correspondingly, the persistence of (macroscopic) superpositions is lost and the unitary dynamics is modified. Once the statistical average is considered, the corresponding dynamical equation for the *statistical operator* $\hat{\rho}$ describing such models is a Lindblad master equation borrowed from open quantum systems theory. The noise generated by such a dynamics leads to a low-rate spontaneous heating. Nonetheless, the accumulation of such a heat is problematic, even for a phenomenological model. The basic form of the DP and the CSL models' master equations lead to decoherence without dissipation. To dissipate the spontaneously generated heat, a dissipative mechanism needs to be included to the basic master equations. Attempts in such a direction were made [10–12] and some experiments were used to test the theory [13–16].

Here, we consider the simplest many-body dissipative Lindblad master equation where the friction term is linear in the second-quantized current $\hat{\mathbf{J}}(\mathbf{r},t)$. We show that our choice of the Lindblad collapse operator, which is independent from the details of the considered system, leads to the dissipation of the current and of the mean energy. We construct and analyze the corresponding dissipative extensions of the DP and CSL models.

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II. SPONTANEOUS DECOHERENCE MODELS

We start from a modified von Neumann–Schrödinger (master) equation

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{D}\hat{\rho}, \tag{1}$$

where $\hat{H}$ is the many-body Hamiltonian describing the Schrödinger dynamics and $\mathcal{D}$ is a term introducing the action of spontaneous decoherence. For the latter, we consider a simple Lindblad form corresponding to the spontaneous measurement of the second-quantized mass density $\hat{\varrho}(\mathbf{r}) = m\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})$ with $\hat{\psi}(\mathbf{r})$ being the (fermionic) annihilation field operator. Explicitly, it reads

$$\mathcal{D}\hat{\rho} = -\frac{1}{2\hbar^2}\iint d\mathbf{r}d\mathbf{s}D(\mathbf{r}-\mathbf{s})[\hat{\varrho}(\mathbf{r}),[\hat{\varrho}(\mathbf{s}),\hat{\rho}]],$$
$$= \frac{1}{\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}\left(\hat{\varrho}_{\mathbf{k}}\hat{\rho}\hat{\varrho}_{\mathbf{k}}^\dagger - \frac{1}{2}\{\hat{\varrho}_{\mathbf{k}}^\dagger\hat{\varrho}_{\mathbf{k}},\hat{\rho}\}\right), \tag{2}$$

where we introduced the Fourier transform of the mass density $\hat{\varrho}_{\mathbf{k}}$ and the kernel D. Depending on the explicit space (or momentum) dependence of the kernel, one might want to introduce a short-length regularization, typically in the form of a Gaussian smearing of the field $\hat{\varrho}(\mathbf{r})$. In the Fourier representation, a Gaussian smearing of scale σ takes a simple form for both the models (DP and CSL) we will consider:

$$D_{\mathbf{k}} = \exp(-\sigma^2k^2)\times\begin{cases}4\pi\hbar G/k^2 & \text{(DP)}\\ \hbar^2\gamma & \text{(CSL)}\end{cases}. \tag{3}$$

In the DP model, the decoherence rate is set by the Newton constant G and the kernel contains a $1/k^2$ factor in addition to the smearing prefactor. Using an alternative notation, $R_0 = \sigma$ is the spatial cutoff in the DP model and it is the only free parameter of the model. Current experimental bounds set the typical smearing at subatomic length scales $\sigma \geqslant 5\times10^{-11}$ m [17], although larger values can be also considered [18,19]. On the other hand, the CSL model can be described in terms of two free parameters being $\lambda = \gamma m_0^2/(\sqrt{4\pi}r_C)^3$ and $r_C = \sigma$, which are respectively the collapse rate and localization length of the model (m_0 is a reference mass chosen as that of a nucleon). Conversely to the DP model, the typically considered values of the spatial smearing are around $\sigma \simeq 10^{-7}$ m, well in the mesoscopic regime. A mapping between the two models can be introduced, and it is based on the simple relationship $D_{\rm CSL} = -\text{const}\times\partial D_{\rm DP}/\partial(\sigma^2)$ between the two kernels. Consequently, the decoherence term of the CSL model can be obtained from the DP one through

$$\mathcal{D}_{\rm CSL} = -\frac{\hbar\gamma}{4\pi G}\frac{\partial\mathcal{D}_{\rm DP}}{\partial(\sigma^2)}. \tag{4}$$

The inverse integral relation can be also simply derived. We anticipate that these relations will survive in the forthcoming dissipative generalization of the dissipator $\mathcal{D}$.

III. SPONTANEOUS HEATING

The smaller the cutoff σ, the larger the strength of the collapse effect and spontaneous heating [20]. The latter implies a continuous increase of the kinetic energy for each particle. We elucidate this mechanism on a single point-like, free particle of mass m and canonical variables $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$. The corresponding heating power P, i.e., the time derivative of the kinetic energy $\hat{H} = (\hat{\mathbf{p}}^2/2m)$, is obtained from the master equation (1)

$$P = \frac{d\langle\hat{H}\rangle}{dt} = \langle\mathcal{D}^\dagger\hat{H}\rangle = \frac{1}{2m}\langle\mathcal{D}^\dagger\hat{\mathbf{p}}^2\rangle, \tag{5}$$

where $\mathcal{D}^\dagger$ is given by

$$\mathcal{D}^\dagger\hat{O} = \frac{1}{\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}\left(\hat{\varrho}_{\mathbf{k}}^\dagger\hat{O}\hat{\varrho}_{\mathbf{k}} - \frac{1}{2}\{\hat{\varrho}_{\mathbf{k}}^\dagger\hat{\varrho}_{\mathbf{k}},\hat{O}\}\right). \tag{6}$$

For the case under study, the mass density and its Fourier transform read

$$\hat{\varrho}(\mathbf{r}) = m\delta(\mathbf{r}-\hat{\mathbf{x}}),\quad \hat{\varrho}_{\mathbf{k}} = me^{i\mathbf{k}\hat{\mathbf{x}}}. \tag{7}$$

We insert $\hat{\varrho}_{\mathbf{k}}$ in Eq. (6) and obtain the expression of the heating power

$$P = \frac{m}{2}\int\frac{d\,\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}k^2, \tag{8}$$

where we used the identity

$$e^{-i\mathbf{k}\hat{\mathbf{x}}}f(\hat{\mathbf{p}})e^{i\mathbf{k}\hat{\mathbf{x}}} = f(\hat{\mathbf{p}}+\hbar\mathbf{k}) \tag{9}$$

and the spherical symmetry of $D_{\mathbf{k}}$. The integral is characteristic for the regularized behavior of the kernel and we calculate it from the Eq. (3) for both models:

$$P = -\frac{m}{2}D''(\mathbf{r})\Big|_{\mathbf{r}=0} = \frac{1}{\sqrt{(2\pi)^3}}\begin{cases}4\pi\hbar G/\sigma^3 & \text{(DP)}\\ 3\hbar^2\gamma/\sigma^5 & \text{(CSL)}\end{cases}. \tag{10}$$

Hence, we get the following heating powers:

$$P^{\rm DP} = \frac{\hbar Gm}{4\sqrt{\pi}\sigma^3}, \tag{11a}$$

$$P^{\rm CSL} = \frac{3m\hbar^2\gamma}{32\pi^{3/2}\sigma^5}, \tag{11b}$$

which are related to each other in conformity with the mapping of Eq. (4).

To get insight into the underlying effective mechanism, we consider the dynamics of the momentum $\hat{\mathbf{p}}$ in details, i.e., the dynamics of arbitrary functions $f(\hat{\mathbf{p}})$ of momentum. It could be shown that the Heisenberg equation of motion of $f(\hat{\mathbf{p}})$ is closed, and this important feature has its parallel in the equivalent von Neumann–Schrödinger dynamics (1) of the state $\hat{\rho}$. Since we are not interested in the dynamics of the coordinate $\hat{\mathbf{x}}$ but of $\hat{\mathbf{p}}$, we can start with the specific form $\hat{\rho} = \rho(\hat{\mathbf{p}})$ of the state, then consider its evolution under the Fourier representation of the dissipator in Eq. (2). The specific form $\rho(\hat{\mathbf{p}})$, diagonal in momentum basis, is preserved:

$$\frac{d\rho(\hat{\mathbf{p}})}{dt} = \frac{m^2}{\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}(\rho(\hat{\mathbf{p}}-\hbar\mathbf{k})-\rho(\hat{\mathbf{p}})). \tag{12}$$

The result is a semiclassical single-particle kinetic equation. The effect of dissipator $\mathcal{D}$ is equivalent to the random jumps $\mathbf{p}\to\mathbf{p}+\hbar\mathbf{k}$ in momentum at the isotropic probability rate

$$\frac{m^2}{\hbar^2}\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}. \tag{13}$$

Since the kernel $D_{\mathbf{k}}$ contains the regularizing factor $\exp(-\sigma^2k^2)$, the elementary momentum and energy transfers are in a bounded range. We introduce the characteristic bound of the elementary energy transfer, which reads

$$E_\sigma = \frac{\hbar^2}{4m\sigma^2}. \tag{14}$$

The quantity E_σ is important when, in the rest of our work, we balance the spontaneous heating by a different dissipative mechanism to reach a balance equation:

$$\frac{d\,\langle \hat{H}\rangle}{dt} = P - \Gamma\,\langle \hat{H}\rangle\,, \tag{15}$$

with a dissipation rate $\Gamma > 0$. In such a way, a finite asymptotic (equilibrium) energy $\langle \hat{H}\rangle_\infty = P/\Gamma$ is reached for $\mathrm{d}\,\langle \hat{H}\rangle/\mathrm{d}\,t = 0$, and the corresponding effective temperature T is defined by the equipartition theorem $\langle \hat{H}\rangle_\infty = \frac{3}{2}k_\mathrm{B}T$. We underline that the single free-particle case is sufficient to show if the spontaneous heating effects lead to a divergence of the energy, or if they can be counterbalanced with a damping effect thus leading to a finite asymptotic energy. This is the same approach that was considered in Ref. [10] for the CSL model and in Ref. [11] for the DP model.

IV. DISSIPATIVE EXTENSION: AN EXERCISE

Before introducing our dissipative extension of the DP and the CSL models, it is instructive to understand the elementary Lindblad form of friction. We start by considering the master equation for a single free particle of Hamiltonian $\hat{H} = \hat{\mathbf{p}}^2/2m$ with a decoherence term of the form

$$\mathcal{D}\hat{\rho} = -\frac{D}{\hbar^2}[\hat{\mathbf{x}},[\hat{\mathbf{x}},\hat{\rho}]]. \tag{16}$$

This can be considered as a minimal model for the spontaneous measurement of $\hat{\mathbf{x}}(t)$. It yields a spatial decoherence at rate $D/\hbar^2$. Equivalently, it implies a momentum diffusion with diffusion constant $D/2$, yielding a constant heating at power $P = 3D/m$. A way to include dissipation is to replace the Hermitian Lindblad generator $\hat{\mathbf{x}}$ with the non-Hermitian operator [21,22]:

$$\hat{\mathbf{L}} = \hat{\mathbf{x}} + i\frac{\hbar\beta}{4m}\hat{\mathbf{p}}, \tag{17}$$

where β will turn out to be the inverse equilibrium temperature. Correspondingly, the Lindbladian term of the master equation takes the form

$$\mathcal{D}\hat{\rho} = \frac{2D}{\hbar^2}\left(\hat{\mathbf{L}}\hat{\rho}\hat{\mathbf{L}}^\dagger - \frac{1}{2}\{\hat{\mathbf{L}}^\dagger\hat{\mathbf{L}},\hat{\rho}\}\right). \tag{18}$$

By expanding $\hat{\mathbf{L}}$, we find

$$\mathcal{D}\hat{\rho} = \frac{i}{\hbar}\left[D\frac{\beta}{4m}\{\hat{\mathbf{x}},\hat{\mathbf{p}}\},\hat{\rho}\right] - \frac{D}{\hbar^2}\Bigg([\hat{\mathbf{x}},[\hat{\mathbf{x}},\hat{\rho}]] + i\frac{\hbar\beta}{2m}[\hat{\mathbf{x}},\{\hat{\mathbf{p}},\hat{\rho}\}] + \frac{\hbar^2\beta^2}{16m^2}[\hat{\mathbf{p}},[\hat{\mathbf{p}},\hat{\rho}]]\Bigg), \tag{19}$$

where we can cancel the first Hamiltonian term with a counterterm in $\hat{H}$. The first term of the second line generates the heating power P as before. The second term yields to the standard mechanical friction: via the Heisenberg equation $\mathcal{D}^\dagger\hat{\mathbf{p}} = -\eta\hat{\mathbf{p}}$ with friction coefficient $\eta = 2\beta D/m$. The second term imposes a relaxation mechanism also for $\hat{\mathbf{p}}^2$. For the mean kinetic energy we get

$$\frac{d\,\langle \hat{H}\rangle}{dt} = \frac{3D}{m} - \frac{2\beta D}{m}\,\langle \hat{H}\rangle\,. \tag{20}$$

This is the balance expressed in Eq. (15) with power $P = 3D/m$ and positive dissipation rate $\Gamma = 2\beta D/m$. We get a finite equilibrium energy $\langle \hat{H}\rangle_\infty = \frac{3}{2}\beta^{-1}$. We conclude that the effective temperature is $T = 1/(k_\mathrm{B}\beta)$. Fortunately, it is known that the equilibrium state of the master equation with the dissipator in Eq. (19) is the exact Gibbs state [21,22]:

$$\hat{\rho}_\beta = \mathcal{N}\mathrm{e}^{-\beta\hat{\mathbf{p}}^2/2m}, \tag{21}$$

and hence $1/(k_\mathrm{B}\beta)$ is not only an effective temperature but the true one.

V. THE MANY-BODY MASTER EQUATION OF LINEAR FRICTION

Now we introduce our model. In analogy with the single-particle linear friction in the previous section, we replace the Hermitian Lindblad generators $\hat{\varrho}(\mathbf{r})$ in the dissipator of Eq. (2) with a non-Hermitian operator of the form

$$\hat{L}(\mathbf{r}) = \hat{\varrho}(\mathbf{r}) - i\frac{\hbar\beta}{4}\nabla_\mathbf{r}\hat{\mathbf{J}}(\mathbf{r}), \tag{22}$$

whose Fourier representation is

$$\hat{L}_\mathbf{k} = \hat{\varrho}_\mathbf{k} + \frac{\hbar\beta}{4}\mathbf{k}\hat{\mathbf{J}}_\mathbf{k}, \tag{23}$$

and where we introduced the current

$$\hat{\mathbf{J}}(\mathbf{r}) = -i\frac{\hbar}{2}(\hat{\psi}^\dagger(\mathbf{r})\nabla_\mathbf{r}\hat{\psi}(\mathbf{r}) - \nabla_\mathbf{r}\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})). \tag{24}$$

We note that the anti-Hermitian part of $\hat{L}(\mathbf{r})$ needs to be a scalar, like the Hermitian part $\hat{\varrho}(\mathbf{r})$. Indeed, one cannot just take the current $\hat{\mathbf{J}}(\mathbf{r})$, but needs to take its divergence. With this choice for the Lindblad generator, the master equation becomes

$$\begin{aligned}\mathcal{D}\hat{\rho} &= \frac{1}{\hbar^2}\int\int d\mathbf{r}d\mathbf{s}\,D(\mathbf{r}-\mathbf{s})\left(\hat{L}(\mathbf{r})\hat{\rho}\hat{L}^\dagger(\mathbf{s}) - \frac{1}{2}\{\hat{L}^\dagger(\mathbf{s})\hat{L}(\mathbf{r}),\hat{\rho}\}\right),\\ &= \frac{1}{\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_\mathbf{k}\left(\hat{L}_\mathbf{k}\hat{\rho}\hat{L}_\mathbf{k}^\dagger - \frac{1}{2}\{\hat{L}_\mathbf{k}^\dagger\hat{L}_\mathbf{k},\hat{\rho}\}\right).\end{aligned} \tag{25}$$

A possible unravelling of Eq. (25), which provides the stochastic and nonlinear dynamical equation for the wave function and describes its collapse, can be derived by following the prescription highlighted in Eq. (5) of Ref. [14]. Alternatively, one can construct such an unravelling starting from the structure in Eq. (4) of Ref. [10]. Merging the expression in Eq. (25) with the definition of $\hat{L}(\mathbf{r})$, we get a Hamiltonian term, which is suitably reabsorbed, and the following equivalent structures are obtained:

$$\begin{aligned}\mathcal{D}\hat{\varrho} &= -\frac{1}{2\hbar^2}\iint d\mathbf{r}d\mathbf{s}\, D(\mathbf{r}-\mathbf{s})\left([\hat{\varrho}(\mathbf{r}),[\hat{\varrho}(\mathbf{s}),\hat{\rho}]] - \frac{i\hbar\beta}{2}[\hat{\varrho}(\mathbf{r}),\{\nabla_\mathbf{s}\hat{\mathbf{J}}(\mathbf{s}),\hat{\rho}\}] + \frac{\hbar^2\beta^2}{16}[\nabla_\mathbf{r}\hat{\mathbf{J}}(\mathbf{r}),[\nabla_\mathbf{s}\hat{\mathbf{J}}(\mathbf{s}),\hat{\rho}]]\right),\\ &= -\frac{1}{2\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_\mathbf{k}\left([\hat{\varrho}_{-\mathbf{k}},[\hat{\varrho}_\mathbf{k},\hat{\rho}]] - \frac{\hbar\beta}{2}[\hat{\varrho}_{-\mathbf{k}},\{\mathbf{k}\hat{\mathbf{J}}_\mathbf{k},\hat{\rho}\}] + \frac{\hbar^2\beta^2}{16}[\mathbf{k}\hat{\mathbf{J}}_{-\mathbf{k}},[\mathbf{k}\hat{\mathbf{J}}_\mathbf{k},\hat{\rho}]]\right),\end{aligned} \tag{26}$$

where the three terms of $\mathcal{D}$ are respectively responsible for the decoherence in mass density $\hat{\varrho}$, the damping of the current $\hat{\mathbf{J}}$ (dissipation), and the decoherence in the (divergence) of the current $\hat{\mathbf{J}}$, respectively. The corresponding Heisenberg equation of motion for an arbitrary observable $\hat{O}$ can be obtained from the adjoint of the master equation $\dot{\hat{O}} = \frac{i}{\hbar}[\hat{H},\hat{O}] + \mathcal{D}^\dagger\hat{O}$, where

$$\begin{aligned}\mathcal{D}^\dagger\hat{O} &= -\frac{1}{2\hbar^2}\iint d\mathbf{r}d\mathbf{s}D(\mathbf{r}-\mathbf{s})\left([\hat{\varrho}(\mathbf{r}),[\hat{\varrho}(\mathbf{s}),\hat{O}]] + \frac{i\hbar\beta}{2}\{\nabla_\mathbf{s}\hat{\mathbf{J}}(\mathbf{s}),[\hat{\varrho}(\mathbf{r}),\hat{O}]\} + \frac{\hbar^2\beta^2}{16}[\nabla_\mathbf{r}\hat{\mathbf{J}}(\mathbf{r}),[\nabla_\mathbf{s}\hat{\mathbf{J}}(\mathbf{s}),\hat{O}]]\right),\\ &= -\frac{1}{2\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_\mathbf{k}\left([\hat{\varrho}_{-\mathbf{k}},[\hat{\varrho}_\mathbf{k},\hat{O}]] + \frac{\hbar\beta}{2}\{\mathbf{k}\hat{\mathbf{J}}_\mathbf{k},[\hat{\varrho}_{-\mathbf{k}},\hat{O}]\} + \frac{\hbar^2\beta^2}{16}[\mathbf{k}\hat{\mathbf{J}}_{-\mathbf{k}},[\mathbf{k}\hat{\mathbf{J}}_\mathbf{k},\hat{O}]]\right),\end{aligned} \tag{27}$$

It is central to this work to confirm that the second term in Eq. (27) is indeed a damping of the current. The corresponding contribution to the evolution of $\hat{\mathbf{J}}(\mathbf{r})$ is given by

$$\dot{\hat{\mathbf{J}}}(\mathbf{r})|_2 = -\frac{i\beta}{4\hbar}\int\int d\mathbf{s}d\mathbf{s}'\, D(\mathbf{s}-\mathbf{s}')\{\nabla_\mathbf{s}\hat{\mathbf{J}}(\mathbf{s}),[\hat{\varrho}(\mathbf{s}'),\hat{\mathbf{J}}(\mathbf{r})]\}. \tag{28}$$

We can calculate the commutator of the second quantized (fermionic) density and current:

$$[\hat{\varrho}(\mathbf{s}'),\hat{\mathbf{J}}(\mathbf{r})] = -i\hbar\nabla_{\mathbf{s}'}[\delta(\mathbf{s}'-\mathbf{r})\hat{\varrho}(\mathbf{r})]. \tag{29}$$

By inserting it in Eq. (28) and integrating the latter by parts, we get

$$\dot{\hat{\mathbf{J}}}(\mathbf{r})|_2 = -\frac{\beta}{4}\int \mathrm{d}\,\mathbf{s}\nabla_\mathbf{r}\circ\nabla_\mathbf{s}D(\mathbf{s}-\mathbf{r})\{\hat{\mathbf{J}}(\mathbf{s}),\hat{\varrho}(\mathbf{r})\}, \tag{30}$$

where $\circ$ indicates the tensor product. We expand the anticommutator of the fermionic density and current:

$$\begin{aligned}&\frac{1}{2}\{\hat{\varrho}(\mathbf{r}),\hat{\mathbf{J}}(\mathbf{s})\}\\ &\quad= m\delta(\mathbf{r}-\mathbf{s})\hat{\mathbf{J}}(\mathbf{r}) - \frac{i\hbar}{2}m(\hat{\psi}^\dagger(\mathbf{r})\nabla_\mathbf{s}\hat{\psi}^\dagger(\mathbf{s})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{s}) - \text{H.c.}).\end{aligned} \tag{31}$$

Using this, we find that

$$\dot{\hat{\mathbf{J}}}(\mathbf{r})|_2 = -\eta\hat{\mathbf{J}}(\mathbf{r}) + i(\hat{\mathbf{Y}}(\mathbf{r}) - \text{H.c.}), \tag{32}$$

where

$$\hat{\mathbf{Y}}(\mathbf{r}) = \frac{\beta\hbar m}{4}\int d\mathbf{s}\nabla_\mathbf{r}\circ\nabla_\mathbf{s}D(\mathbf{s}-\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\nabla_\mathbf{s}\hat{\psi}^\dagger(\mathbf{s})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{s}). \tag{33}$$

The latter contributes only when multiple fermions are present, while it vanishes when applied to a single fermion state. As a first-order approximation, we neglect $\hat{\mathbf{Y}}(\mathbf{r})$ contributions. Then, the current effectively decays with a friction rate

$$\eta = -\frac{\beta m}{2}D''(\mathbf{r})|_{\mathbf{r}=0}, \tag{34}$$

which depends crucially on the parameter σ that regularizes $D(\mathbf{r})$ at $\mathbf{r}=0$; see expressions in Eq. (10). In the Appendix, we show that, in the case of a single particle, Eq. (32) holds with no approximations.

Since the methods to infer exact analytic features of the dissipative master equation are limited, we turn to the special single-particle case. In such a case the mechanism of dissipation is transparent, and exact analytic calculations are possible. Moreover, in the case of both the standard CSL and DP models, i.e., with no dissipation included, the heating rate is independent from the presence of interactions or external potentials. This has been well addressed in Ref. [23]. The inclusion of dissipative effects, however, breaks this simple feature in dense interactive fermionic matter. To include interaction and fermionic exchange, one should employ perturbative methods such those used in Ref. [24], which can be applied independently of the Hamiltonian structure. However, this goes beyond the scope of this paper. Nonetheless, as long as our single-fermion approximation is valid, the thermodynamics remains trivial as in the nondissipative case, while the rates and the equilibrium temperature are calculable exactly as for single fermions.

VI. SINGLE-PARTICLE DISSIPATIVE MECHANISM

In case of the single particle, it is most convenient to work in the Fourier representation of the mass density and the current:

$$\hat{\varrho}_\mathbf{k} = me^{i\mathbf{k}\hat{\mathbf{x}}}, \quad \hat{\mathbf{J}}_\mathbf{k} = \tfrac{1}{2}\{\hat{\mathbf{p}},e^{i\mathbf{k}\hat{\mathbf{x}}}\}. \tag{35}$$

Then, the Lindblad generator, leading to the many-body dissipator $\mathcal{D}$ shown in Eq. (26), reduces to simple alternative forms:

$$\begin{aligned}\hat{L}_\mathbf{k} &= \left(m - \frac{\hbar^2k^2\beta}{8} + \frac{\hbar\mathbf{k}\beta}{4}\hat{\mathbf{p}}\right)e^{i\mathbf{k}\hat{\mathbf{x}}},\\ &= e^{i\mathbf{k}\hat{\mathbf{x}}}\left(m + \frac{\hbar^2k^2\beta}{8} + \frac{\hbar\mathbf{k}\beta}{4}\hat{\mathbf{p}}\right).\end{aligned} \tag{36}$$

Let us see how the kinetic equation of the momenta differs from that of the standard DP and CSL in Sec. III. We use both forms in Eq. (36) of the Lindblad generator in the dissipator in Eq. (25), to yield

$$\frac{d\rho(\hat{\mathbf{p}})}{dt} = \frac{1}{\hbar^2}\int \frac{d\mathbf{k}}{(2\pi)^3} D_{\mathbf{k}}\left(\left(m - \frac{\beta\hbar^2 k^2}{8} + \frac{\beta\hbar\mathbf{k}}{4}\hat{\mathbf{p}}\right)^2 \rho(\hat{\mathbf{p}} - \hbar\mathbf{k}) - \left(m + \frac{\beta\hbar^2 k^2}{8} + \frac{\beta\hbar\mathbf{k}}{4}\hat{\mathbf{p}}\right)^2 \rho(\hat{\mathbf{p}})\right). \tag{37}$$

This is equivalent with a classical kinetic equation, and therefore $\mathbf{p}$ instead of $\hat{\mathbf{p}}$ can be written. According to this kinetic equation, the momentum jumps like $\mathbf{p} \to \mathbf{p} + \hbar\mathbf{k}$ at probability rate

$$\frac{m^2}{\hbar^2}\frac{d\mathbf{k}}{(2\pi)^3} D_{\mathbf{k}}\left(1 + \frac{\beta}{8m}[\mathbf{p}^2 - (\mathbf{p} - \hbar\mathbf{k})^2]\right)^2. \tag{38}$$

This jump rate, unlike in standard DP and CSL, is not isotropic, and the anisotropy can generate the desired friction. Nevertheless, the above rate is subtle. Consider for simplicity a momentum transfer of $\hbar\mathbf{k} = \mp\kappa\mathbf{p}$ where $0 \leqslant \kappa \leqslant 1$, where upper and lower signs correspond to damping and heating respectively. The difference between damping and heating rates is proportional to the following expression:

$$\left(1 - \frac{\beta\hbar^2 k^2}{8m} + \kappa\frac{\beta p^2}{4m}\right)^2 - \left(1 - \frac{\beta\hbar^2 k^2}{8m} - \kappa\frac{\beta p^2}{4m}\right)^2 = \kappa\frac{\beta p^2}{m}\left(1 - \frac{\beta\hbar^2 k^2}{8m}\right). \tag{39}$$

Damping dominates as long as $\beta(\hbar^2 k^2/8m) < 1$ and heating takes over otherwise. Earlier, when we defined E_σ in Eq. (14), we noticed that the range of k is $1/\sigma$, and hence $(\hbar^2 k^2/4m) \sim E_\sigma$. Accordingly for damping, the largest value of $1/\beta$ is about $2E_\sigma$. The forthcoming analytic calculation shows that, indeed, there is an exact critical value of $1/\beta$ above which dissipation gives way to heating.

In order to prove that the model exhibits the expected dissipative mechanism described in Eq. (15) for $\langle\hat{H}\rangle = \langle\hat{\mathbf{p}}^2/2m\rangle$, we derive the time derivative of $\langle\hat{\mathbf{p}}^2\rangle$. Since the dynamics of $\hat{\mathbf{p}}$ is semiclassical, we can derive the time derivative of the equivalent semiclassical mean $\langle\mathbf{p}^2\rangle$. According to the above discussion, the expression in Eq. (38) is the rate of random jumps $\mathbf{p} \to \mathbf{p} + \hbar\mathbf{k}$, each of which leads to a change of $\hbar^2\mathbf{k}^2 + 2\hbar\mathbf{k}\mathbf{p}$ for $\mathbf{p}^2$. Hence, we can write

$$\frac{d}{dt}\langle\mathbf{p}^2\rangle = \frac{m^2}{\hbar^2}\int\frac{d\mathbf{k}}{(2\pi)^3}\int d\mathbf{p} D_{\mathbf{k}}\rho(\mathbf{p}) \times (\hbar^2\mathbf{k}^2 + 2\hbar\mathbf{p}\mathbf{k})\left(1 + \frac{\beta}{8m}[\mathbf{p}^2 - (\mathbf{p} - \hbar\mathbf{k})^2]\right)^2. \tag{40}$$

We temporarily set $\hbar = 1$ and introduce $a^2 = (\beta/4m)$. We then rewrite the following expression

$$(\mathbf{k}^2 + 2\mathbf{p}\mathbf{k})\left(1 - \tfrac{1}{2}a^2\mathbf{k}^2 - a^2\mathbf{k}\mathbf{p}\right)^2 \tag{41}$$

$$= k^2\left(1 - a^2k^2 + \tfrac{1}{4}a^4k^4\right) + a^2(3a^2k^2 - 4)(\mathbf{k}\mathbf{p})^2 + \cdots$$

$$\Rightarrow k^2\left(1 - a^2k^2 + \tfrac{1}{4}a^4k^4\right) + a^2k^2\left(a^2k^2 - \tfrac{4}{3}\right)p^2, \tag{42}$$

where the ellipsis stands for odd powers of $\mathbf{k}$ to be canceled when intergrating, and we replaced $(\mathbf{k}\mathbf{p})^2$ by $\frac{1}{3}k^2p^2$ also because of the isotropy of $D_{\mathbf{k}}$ in the integral. We insert in Eq. (40) the bottom line of the above expansion and perform the integral in $\mathbf{p}$. We get the balance equation (15), where the power P and the dissipation Γ rates are respectively

$$P = \frac{m}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}k^2\left(1 - \frac{\beta\hbar^2}{4m}k^2 + \frac{\beta^2\hbar^4}{64m^2}k^4\right),$$

$$\Gamma = \frac{\beta m}{3}\int\frac{d\mathbf{k}}{(2\pi)^3}D_{\mathbf{k}}k^2\left(1 - \frac{3\beta\hbar^2}{16m}k^2\right), \tag{43}$$

where we have restored $\hbar$ and $a^2 = \beta/4m$. The obtained result shows that the energy of the system is dissipated, as expected, through the anisotropic process described by Eq. (38). Moreover, we show in the Appendix that such a result is independent of the specific form of the state $\hat{\rho}$ and is valid beyond the assumption of having $\hat{\rho} = \rho(\hat{\mathbf{p}})$.

The integrals in Eq. (43) can be calculated analytically for the two models. They respectively read

$$P^{\mathrm{DP}} = \frac{\hbar m G}{4\sqrt{\pi}\sigma^3}\left(1 - \frac{3}{4}x_\beta^2 + \frac{15}{64}x_\beta^4\right),$$

$$\Gamma^{\mathrm{DP}} = \beta\frac{\hbar m G}{6\sqrt{\pi}\sigma^3}\left(1 - \frac{9}{16}x_\beta^2\right). \tag{44}$$

and

$$P^{\mathrm{CSL}} = \frac{3m\gamma\hbar^2}{32\pi^{3/2}\sigma^5}\left(1 - \frac{5}{4}x_\beta^2 + \frac{35}{64}x_\beta^4\right),$$

$$\Gamma^{\mathrm{CSL}} = \beta\frac{m\gamma\hbar^2}{16\pi^{3/2}\sigma^5}\left(1 - \frac{15}{16}x_\beta^2\right), \tag{45}$$

where we defined the dimensionless parameter

$$x_\beta^2 = 2\beta E_\sigma = \frac{\hbar^2\beta}{2m\sigma^2}, \tag{46}$$

which is the ratio of the elementary energy transfer E_σ defined in Eq. (14) to $1/(2\beta)$, the latter being the equilibrium thermal kinetic energy at high temperatures. Note that, according to the mapping in Eq. (4), the relation $P^{\mathrm{CSL}} = -(\hbar\gamma/4\pi G)\partial P^{\mathrm{DP}}/\partial(\sigma^2)$ holds, and similarly between Γ^{CSL} and Γ^{DP}. The dissipative rates Γ become negative if x_β^2, which is proportional to the parameter β, is larger than a critical value, which is different for the two models.

It is in order now to interpret our results, shown in Eqs. (43)–(45), that exhibit the dissipation mechanism postulated by the balance Eq. (15). The equilibrium energy is obtained as $\langle\hat{H}\rangle_\infty = P/\Gamma$. Following the equipartition theorem, we define the effective temperature as $T = \frac{2}{3}\langle\hat{H}\rangle_\infty/k_{\mathrm{B}}$

and we introduce the parameter $T_\beta = 1/\beta$ in place of β. Then, we have $x_\beta^2 = 2E_\sigma/T_\beta$, and we can express the effective temperatures of the two models as

$$T^{\rm DP} = T_\beta \frac{1 - \frac{3}{2}(E_\sigma/k_{\rm B}T_\beta) + \frac{15}{16}(E_\sigma/k_{\rm B}T_\beta)^2}{1 - \frac{9}{8}(E_\sigma/k_{\rm B}T_\beta)} \tag{47}$$

and

$$T^{\rm CSL} = T_\beta \frac{1 - \frac{5}{2}(E_\sigma/k_{\rm B}T_\beta) + \frac{35}{16}(E_\sigma/k_{\rm B}T_\beta)^2}{1 - \frac{15}{8}E_\sigma/T_\beta}. \tag{48}$$

In the regime $k_{\rm B}T_\beta \gg E_\sigma$, the effective temperature T asymptotically coincides with the parameter T_β, which justifies our choice of parametrizing the dissipator in Eq. (26) by $\beta = 1/T_\beta$. When lowering the parameter temperature T_β, the effective temperature T is also lowering. But the dissipation rate Γ is also decreasing and at a point the effective T is no longer lowering together with T_β, but it is growing again and becomes infinite when the dissipation rate reduces to zero, i.e., at $k_{\rm B}T_\beta = (9/8)E_\sigma$ in the DP model and at $k_{\rm B}T_\beta = (15/8)E_\sigma$ in the CSL model. Below these critical temperatures, the negative dissipative rate Γ is contributing to a higher heating power P rather than balancing it. This effect follows from what we noticed about the subtlety of momentum jump rate in Eq. (38).

The standard DP and CSL models correspond to $T = T_\beta = \infty$, where dissipative rates Γ vanish, and the powers P reduce to the expressions in Eqs. (11a) and (11b), respectively. The kinetic energy $\langle \hat{H} \rangle_t$ goes to infinity and, from a theoretical viewpoint, this looks unphysical. In practice, however, we face a different situation. The predicted powers in Eqs. (11a) and (11b) are extremely small and are typically masked by the environmental effects (see experimental investigations summarized in Ref. [5]). Clearly, studying experimentally a system under the action of a collapse mechanism, but otherwise isolated, is impossible. Indeed, there will be always a coupling of the system with its surrounding environment. This might be the residual gas in the vacuum chamber, the blackbody radiation, or the noises (e.g., seismic or electronic) that shake, and thus heat, the experiment (see, for instance, Refs. [25,26]). As a matter of fact, current laboratory efforts of isolation are not yet able to exclude the values $T = T_\beta = \infty$ either for the DP or the CSL model, however unphysical they would theoretically be.

For this reason, we now consider the case of a system undergoing simultaneously to the dissipative collapse mechanism and the interaction of an external thermal environment. Let $T_{\rm E}$ be the temperature of the environment, and let us define the power $P_{\rm E}$ and the dissipative rate $\Gamma_{\rm E}$ to model the environmental effect on our particle, where $\frac{3}{2}k_{\rm B}T_{\rm E} = P_{\rm E}/\Gamma_{\rm E}$ is satisfied. One can straightforwardly derive the evolution of the mean energy of the system, which reads

$$\frac{d}{dt}\langle H \rangle_t = P + P_{\rm E} - (\Gamma + \Gamma_E)\langle H \rangle_t, \tag{49}$$

where P and Γ are those defined in Eq. (43). Consequently, the asymptotic mean energy is

$$\langle \hat{H} \rangle_\infty = \frac{P + P_{\rm E}}{\Gamma + \Gamma_{\rm E}}. \tag{50}$$

According to the equipartition theorem, the asymptotic (equilibrium) temperature of our particle is $T_{\rm eff} = \frac{2}{3}k_{\rm B}^{-1}\langle \hat{H} \rangle_\infty$, i.e.,

$$T_{\rm eff} = \frac{\Gamma T + \Gamma_{\rm E}T_{\rm E}}{\Gamma + \Gamma_{\rm E}}, \tag{51}$$

where $T = \frac{2}{3}k_{\rm B}^{-1}P/\Gamma$ is the effective temperature obtained in Eqs. (47) and (48) of the collapse noise and $T_{\rm E} = \frac{2}{3}k_{\rm B}^{-1}P_{\rm E}/\Gamma_{\rm E}$ is that of the thermal environment. The relation in Eq. (51) is fundamental when it comes to experiments. Indeed, it provides the experimental requirement to be reached in terms of $\Gamma_{\rm E}$ and $T_{\rm E}$ to be able to measure the temperature T of the collapse noise.

VII. CONCLUSION

We introduced a simple and universal dissipative extension of the DP and the CSL models. Contrary to previous attempts [10,11], our model modifies the collapse operator by adding (instead of a multiplying) a new term leading to dissipative effects. A similar method has been considered for a different gravity-related model in Ref. [27]. Such a term is proportional to the divergence of the current and is parametrized by the constant β [cf. Eq. (22)]. We demonstrate that the model dissipates the current and leads to the thermalization of the system's energy to the asymptotic value of $\langle \hat{H} \rangle_\infty = \frac{3}{2}k_{\rm B}T$, where the expression for T is given in Eq. (47) for the DP model and in Eq. (48) for the CSL model.

We find a threshold temperature T_0, which is defined as

$$T_0 = \frac{\hbar^2}{mk_{\rm B}\sigma^2}, \tag{52}$$

and is determined by the cutoff length σ of the DP and the CSL models. For $k_{\rm B}/\beta$ much higher than T_0, the system's mean energy asymptotically converges to $1/(2\beta)$, which suggests that β can be interpreted as the inverse temperature of the collapse noise. Nevertheless, for generic values of β, the latter enters nontrivially in $\langle \hat{H} \rangle_\infty$. The noise temperature T becomes different from $k_{\rm B}/\beta$ when the latter approaches T_0 from above. At a certain point, the noise temperature T inverts its trend with respect to $k_{\rm B}/\beta$ and increases to infinity at T_0. It is thus impossible to draw a clear one to one connection between the temperature T of the collapse noise and the parameter β. In general, the temperature of the collapse noise does not coincide with $\beta^{-1}/k_{\rm B}$, and the latter plays the role of a parameter in the master equation that is detached from its familiar statistical mechanical interpretation.

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APPENDIX: SINGLE-PARTICLE DISSIPATIVE MECHANISM IN THE HEISENBERG PICTURE

We rederive here the results appearing in Secs. V and VI in the Heisenberg picture for a generic state $\hat{\rho}$. In particular, we show that the second term of $\mathcal{D}$ in Eq. (26) leads to the dissipation of the current, and that the energy follows the balance equation displayed in Eq. (15). For the sake of simplicity, we focus on the case of a single particle. In such a case, the explicit forms of the Fourier transform of the mass density $\hat{\varrho}_{\mathbf{k}}$ and the current $\hat{\mathbf{J}}_{\mathbf{k}}$ are

$$\hat{\varrho}_{\mathbf{k}} = m e^{i\mathbf{k}\hat{\mathbf{x}}}, \quad \hat{\mathbf{J}}_{\mathbf{k}} = \tfrac{1}{2}\{\hat{\mathbf{p}}, e^{i\mathbf{k}\hat{\mathbf{x}}}\}. \tag{A1}$$

From these and from Eq. (9), one can compute their commutator and anticommutator, which respectively read

$$\begin{aligned}[\hat{\varrho}_{\mathbf{q}}, \hat{\mathbf{J}}_{\mathbf{k}}] &= -m\hbar\mathbf{q}e^{i(\mathbf{k}+\mathbf{q})\hat{\mathbf{x}}} = -\hbar\mathbf{q}\hat{\varrho}_{\mathbf{k}+\mathbf{q}} \quad \text{and}\\ \{\hat{\varrho}_{\mathbf{q}}, \hat{\mathbf{J}}_{\mathbf{k}}\} &= m e^{i(\mathbf{k}+\mathbf{q})\hat{\mathbf{x}}}(2\hat{\mathbf{p}} + \hbar(\mathbf{k}+\mathbf{q})) = 2m\hat{\mathbf{J}}_{\mathbf{k}+\mathbf{q}}.\end{aligned} \tag{A2}$$

The dynamics of the current due to the second term in Eq. (26) is given, equivalently, by the following two expressions:

$$\dot{\hat{\mathbf{J}}}(\mathbf{r})|_2 = -\frac{i\beta}{4\hbar}\int\int d\mathbf{s}d\mathbf{s}'\, D(\mathbf{s}-\mathbf{s}')\{\nabla_{\mathbf{s}}\hat{\mathbf{J}}(\mathbf{s}), [\hat{\varrho}(\mathbf{s}'), \hat{\mathbf{J}}(\mathbf{r})]\}, \tag{A3a}$$

$$= -\frac{\beta}{4\hbar}\int \frac{d\mathbf{k}'}{(2\pi)^3} e^{-i\mathbf{k}'\mathbf{r}} \int \frac{d\mathbf{k}}{(2\pi)^3} D_{\mathbf{k}}\{\mathbf{k}\hat{\mathbf{J}}_{\mathbf{k}}, [\hat{\varrho}_{-\mathbf{k}}, \hat{\mathbf{J}}_{\mathbf{k}'}]\}. \tag{A3b}$$

By employing the second of these expressions and merging it with Eq. (A2), one straightforwardly finds

$$\dot{\hat{\mathbf{J}}}(\mathbf{r})|_2 = -\eta\hat{\mathbf{J}}(\mathbf{r}), \tag{A4}$$

where

$$\eta = \frac{\beta m}{2}\int \frac{d\mathbf{k}}{(2\pi)^3} D_{\mathbf{k}}\mathbf{k}^2, \tag{A5}$$

which shows explicitly that the contributions due $\hat{\mathbf{Y}}(\mathbf{r})$ in Eq. (32) vanishes exactly in the single-fermion case. Similarly, one computes the following commutators of $\hat{\varrho}_{\mathbf{k}}$ and $\hat{\mathbf{J}}_{\mathbf{k}}$ with $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m}$:

$$\begin{aligned}[\hat{\varrho}_{\mathbf{k}}, \hat{H}] &= -\frac{e^{i\mathbf{k}\hat{\mathbf{x}}}}{2}(\hbar^2 k^2 + 2\hbar\mathbf{k}\hat{\mathbf{p}}) \quad \text{and}\\ [\mathbf{k}\hat{\mathbf{J}}_{\mathbf{k}}, \hat{H}] &= -\frac{\hbar e^{i\mathbf{k}\hat{\mathbf{x}}}}{4m}(\hbar k^2 + 2\mathbf{k}\hat{\mathbf{p}})^2,\end{aligned} \tag{A6}$$

from which one obtains

$$[\hat{\varrho}_{-\mathbf{k}}, [\hat{\varrho}_{\mathbf{k}}, \hat{H}]] = -m\hbar^2 k^2, \tag{A7}$$

$$\{\mathbf{k}\hat{\mathbf{J}}_{\mathbf{k}}, [\hat{\varrho}_{-\mathbf{k}}, \hat{H}]\} = 2\hbar(\mathbf{k}\hat{\mathbf{p}})^2 + \frac{\hbar^3 k^4}{2}, \tag{A8}$$

$$[\mathbf{k}\hat{\mathbf{J}}_{-\mathbf{k}}, [\mathbf{k}\hat{\mathbf{J}}_{\mathbf{k}}, \hat{H}]] = -\frac{3\hbar^2 k^2(\mathbf{k}\hat{\mathbf{p}})^2}{m} - \frac{\hbar^4 k^6}{4m}. \tag{A9}$$

Owning that, for any spherically symmetric kernel $D_{\mathbf{k}} = D_k$, the following holds,

$$\begin{aligned}\int d\mathbf{k}D_k(\mathbf{k}\hat{\mathbf{p}})^2 &= \int d\mathbf{k}D_k(k\hat{p}\cos\theta)^2\\ &= \hat{p}^2\frac{4\pi}{3}\int dk D_k k^4\\ &= \int d\mathbf{k}D_k\left(\frac{k^2\hat{p}^2}{3}\right),\end{aligned} \tag{A10}$$

we can substitute $(\mathbf{k}\hat{\mathbf{p}})^2$ with $(k^2\hat{p}^2)/3$ [this has been used also in Eq. (42)]. Then, from Eq. (27) we obtain the dynamics for the Hamiltonian:

$$\mathcal{D}^\dagger\hat{H} = P - \Gamma\hat{H}, \tag{A11}$$

where the explicit form of P and Γ is given in Eq. (43). We underline that such an equation is state independent, and thus it can be straightforwardly used to evaluate the expectation value of the energy for any state, also beyond the assumption of $\hat{\rho} = \rho(\hat{\mathbf{p}})$, which has been considered in the main text.

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