Classical-quantum hybrid canonical dynamics and its difficulties with special and general relativity

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We discuss the Hamiltonian hybrid coupling between a classical and a quantum subsystem. If applicable to classical gravity coupled to quantized matter, this hybrid theory might realize a captivating "postquantum" alternative to full quantum gravity. We summarize the nonrelativistic hybrid dynamics in improved formalism adequate to Hamiltonian systems. The mandatory decoherence and diffusion terms become divergent in special and general relativistic extensions. It is not yet known if any renormalization method might reconcile Markovian decoherence and diffusion with relativity. Postquantum gravity could previously only be realized in the Newtonian approximation. We argue that pending problems of the recently proposed general relativistic postquantum theory will not be solved if Markovian diffusion/decoherence are truly incompatible with relativity.

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I. INTRODUCTION

The dynamical coupling between a classical and a quantum subsystem is of multiple interests, e.g., in mathematical physics, in heuristic models, and particularly in foundations. If gravity were fundamentally classical, then its hybridized dynamics with quantized matter would replace the mean-field (semiclassical) approximation [1,2] and the famously inconclusive versions of quantum gravity. Such a captivating idea has been kept alive from an episodic suggestion [3,4]—based on incorrect nonrelativistic (NR) hybrid dynamics [5]—through works by the present author and by others [6–12] until culminating in the *postquantum* gravity of Oppenheim and coworkers [13–16].

Parallel to the fundamental concept, the underlying mathematical tool has been researched persistently along important milestones [8–10,17–28]. The central technical issue that has been solved nonrelativistically is the following. Suppose the hybrid Hamiltonian contains in turn the classical Hamilton function of the classical subsystem, the Hamilton operator of the quantum subsystem, and the coupling between them:

$$\hat{H}(q,p) = H_{\mathrm{C}}(q,p) + \hat{H}_{\mathrm{Q}} + \hat{H}_{\mathrm{CQ}}(q,p). \tag{1}$$

The evolution equation of the state vector of the quantum subsystem is the Schrödinger equation $i\hbar|\Psi\rangle/dt = \hat{H}(q,p)|\Psi\rangle$. The *backaction* of the quantum subsystem on the classical one is nontrivial. Toward the solution of interest, we introduce the hybrid state, represented by the hybrid density $\hat{\rho}(q,p) \geq 0$, which is a combination of the density operator $\hat{\rho}_{\mathrm{Q}} = \int \hat{\rho}(q,p)dqdp$ of the quantum subsystem and the phase-space density $\rho_{\mathrm{C}}(q,p) = \mathrm{tr}\hat{\rho}(q,p)$ of the classical one. Assume the following combination of the classical and quantum dynamics [17]:

$$\begin{aligned}\frac{d\hat{\rho}(q,p)}{dt} &= -\frac{i}{\hbar}[\hat{H}(q,p),\hat{\rho}] + \mathbb{H}\{\hat{H}(q,p),\hat{\rho}(q,p)\}\\ &\equiv \{\hat{H}(q,p),\hat{\rho}(q,p)\}_{\mathrm{A}},\end{aligned} \tag{2}$$

where $\{,\}$ stands for the Poisson bracket. The term $\mathbb{H}\{\hat{H}_{CQ}(q,p),\hat{\rho}(q,p)\}$ represents the backaction, and the symbol $\mathbb{H}$ means the Hermitian part. If it is zero, we get the standard classical and quantum dynamics separately for the two subsystems, as we should. But the seemingly plausible dynamics (2) is not yet mathematically correct; it does not preserve the positivity of $\hat{\rho}(q,p)$. Additional decoherence and diffusion mechanisms are mandatory, and they are subject to trade-off: stronger decoherence allows for weaker diffusion and vice versa [8]. The ultimate general form of hybrid NR dynamics appeared in Refs. [26–29].

Instead of a master equation for $\hat{\rho}(q,p)$, the stochastic differential equations for the pure quantum state $\hat{P}$ and the classical variables (q,p) offer an equivalent alternative. As an analogy, remember, for example, that the classical Fokker-Planck equation is equivalent to the Langevin stochastic differential equation. In the hybrid case, the backaction is realized by time-continuous quantum measurement—*monitoring*—of the quantum subsystem and

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feedback of the measured signal into the classical subsystem. The importance of this formalism is emphasized especially in Refs. [28,30]. Compared to the master equation of hybrid canonical coupling, the modular monitoring-plus-feedback construction gives better intuition as observed in Ref. [30].

Undoubtedly, hybrid coupling is not possible without compromises. For example, there are two fundamental issues with the semiclassical approximation: fake action-at-a-distance and breakdown of Born's statistical interpretation (cf., e.g., notes [31] and references therein). These two issues would be unacceptable in a fundamental theory. But, it is crucial that they are NR effects absolutely unrelated to relativity or gravity, related only to the nonlinearity of the semiclassical hybrid equations. It was shown that linearity can be maintained by assuming a well-defined minimum noise in the hybrid coupling [8]. We thus have a linear NR hybrid dynamics [26–29] with tractable compromise (noisiness) instead of the two fundamental defects mentioned above. The question now is whether there is a relativistic extension of this hybrid dynamics.

Our goal is threefold: a convenient introduction to the mathematics of NR hybrid canonical dynamics, the assessment of its application in postquantum gravity, and a discussion if it could have surpassed its old Newtonian "forerunner."

Section II recapitulates state-of-the-art knowledge of NR hybrid canonical dynamics. Section III explains the locality condition of relativistic invariance and the resulting divergences. Section IV tests the special relativistic extension on the simplest example of hybrid coupling between a classical and a quantized scalar field. Section V revisits the effort toward general relativistic postquantum gravity, extending the NR hybrid dynamics for general relativity. Section VI recapitulates the NR forerunner of postquantum general relativity. Final remarks and our conclusion are given in Sec. VII.

II. THE NONRELATIVISTIC CANONICAL HYBRID DYNAMICS

Our hybrid system of interest consists of a NR classical canonical subsystem and a NR quantized subsystem. To model their coupled dynamics we start from the naive combination (2). In addition to the Dirac and Poisson brackets, there are mandatory decoherence and diffusion terms that will necessitate the postulation of a Riemann metric on the phase-space manifold (or on its submanifold). The resulting irreversible dynamics obtain the form of the hybrid master equation (HME) which is the combination of the classical Fokker-Planck and the quantum Lindblad equations (Sec. II A). This irreversible dynamics is equivalent with the coupled stochastic processes in the classical phase space and the Hilbert space, respectively, and represented by a couple of hybrid stochastic differential equations (HSDEs) in Sec. II B. In physics, the special case is of interest when the classical coordinates are coupled to the quantum subsystem but the classical momenta are not (Sec. II C). The material presented here is based primarily on Refs. [26–29], deduced basically from [28] (cf. Appendix A), and improved by the Riemann metric interpretation of the decoherence and diffusion kernels. It is important that we treat the HME and HSDE formalisms as equivalent; both have their own conceptual universality.

A. Hybrid master equation

Let $\hat{H}(x) \equiv \hat{H}(q,p)$ be our hybrid Hamiltonian where the classical subsystem is canonical. The first N canonical variables $\{x^n; n=1,\dots,N\}$ are the coordinates and the second N ones $\{x^n; n=N+1,\dots,2N\}$ are the momenta:

$$x^n = \begin{cases} q_n; & n=1,2,\dots N \\ p^n; & n=N+1,N+2,\dots,2N. \end{cases} \tag{3}$$

The HME of the hybrid density $\hat{\rho}(q,p)=\hat{\rho}(x)$ takes this form:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathbb{H}\{\hat{H},\hat{\rho}\} + \mathcal{D}\hat{\rho} \equiv \{\hat{H},\hat{\rho}\}_{\mathrm{A}} + \mathcal{D}\hat{\rho}, \tag{4}$$

where $\{,\}_{\mathrm{A}}$ is the Aleksandrov hybrid bracket, $\mathcal{D}$ is the superoperator of decoherence and diffusion (D&D). The notation of x-dependences of $\hat{H},\hat{\rho},\mathcal{D}$ are spared. The classical canonical Poisson bracket is defined by

$$\{\hat{A},\hat{B}\} = \hat{A}_{,n}\epsilon^{nm}\hat{B}_{,m} = (\hat{A}\epsilon^{nm}\hat{B}_{,m})_{,n} = (\hat{A}_{,n}\epsilon^{nm}\hat{B})_{,m}, \tag{5}$$

where ϵ^{nm} is the $2N\times 2N$ symplectic matrix. We introduced the shorthand notation for partial derivatives such as $\partial\hat{A}/\partial x^n = \hat{A}_{,n}$ as well as Einstein's convention for the summation of repeated indices. If we define the canonical velocity operators

$$\hat{v}^n = \{x^n,\hat{H}\} = -\epsilon^{nm}\hat{H}_{,m}, \tag{6}$$

then the Poisson bracket will have the useful equivalent expression

$$\mathbb{H}\{\hat{H},\hat{\rho}\} = \mathbb{H}\hat{v}^n\hat{\rho}_{,n}. \tag{7}$$

To construct the canonical HME we impose a Riemann metric structure in addition to the symplectic structure of the phase space, via the arbitrary choice of the $2N\times 2N$ covariant metric tensor $\gamma_{nm}(x)$. The D&D terms are the following:

$$\mathcal{D}\hat{\rho} = -\frac{\gamma_{nm}}{8}[\hat{v}^n,[\hat{v}^m,\hat{\rho}]] + \frac{1}{2}(\gamma^{nm}\hat{\rho})_{,nm} \equiv \mathcal{D}_{\mathrm{Q}}\hat{\rho} + \mathcal{D}_{\mathrm{C}}\hat{\rho}, \tag{8}$$

where we assume that the velocities $\hat{v}^n(x)$ are linearly independent operators, also independent from any c-number functions. That is, we assume the equation

$$\lambda_n(x)\hat{v}^n(x) = \varphi(x) \tag{9}$$

is satisfied only for vanishing λ_n and φ.

B. Hybrid stochastic differential equations

The canonical HME (4) with D&D (8) is equivalent to two coupled stochastic processes, one for the diffusion of the pure state $\hat{P}_t \equiv |\Psi_t\rangle\langle\Psi_t|$ in the Hilbert space, the other one for the diffusion of x_t in the phase space, meaning in fact the statistical interpretation of the HME. Also called stochastic *unraveling* of the HME, the processes are defined by the coupled HSDEs:

$$\frac{d\hat{P}}{dt} = -\frac{i}{\hbar}[\hat{H}(x),\hat{P}] + \mathcal{D}_{\mathrm{Q}}(x)\hat{P} + \mathbb{H}(\hat{v}^n(x) - \langle\hat{v}^n(x)\rangle)\hat{P}w_n(x), \tag{10}$$

$$\frac{dx^n}{dt} = \langle\hat{v}^n(x)\rangle + w^n(x), \tag{11}$$

where $\langle\hat{v}^n(x)\rangle = \mathrm{tr}(\hat{v}^n(x)\hat{P})$. Both SDEs are driven by the same white noise $w_n = \gamma_{nm}w^m$ whose correlations are determined by the metric

$$\begin{aligned}
\mathbb{M}w^n(x,t)w^m(x,\tau) &= \gamma^{nm}(x)\delta(t-\tau),\\
\mathbb{M}w_n(x,t)w_m(x,\tau) &= \gamma_{nm}(x)\delta(t-\tau),\\
\mathbb{M}w^n(x,t)w_m(x,\tau) &= \delta^n_m\delta(t-\tau).
\end{aligned} \tag{12}$$

The symbol $\mathbb{M}$ stands for the stochastic mean.

In this formalism of the hybrid dynamics the back-action follows from the monitoring-plus-feedback mechanism. Equation (10) coincides with the stochastic master equation of time-continuous simultaneous quantum measurements—monitoring—of the observables $\hat{v}^n$. The measured signal $\langle\hat{v}^n\rangle + w^n$ will then control feedback in the equation of motion (11) of the classical phase-space variables x^n. Note that this SDE can be written as

$$\frac{dx^n}{dt} = \{x^n, \langle\hat{H}(x)\rangle\} + w^n, \tag{13}$$

which is the mean-field (semiclassical) backaction plus our mandatory white noise. Observe that unlike white noises, the phase-space coordinates $x^n(t)$ are continuous functions, containing the integrals of the white noises $w^n(t)$. The path in phase space is a (generalized) Wiener process.

C. Coordinate coupling

The D&D terms (8) correspond to the minimum noise dynamics if the $2N$ velocities $\hat{v}^n(x)$ are independent operator fields on the phase space. However, they are not so in many concrete hybrid systems. Suppose K is the maximum number of independent constraints (9):

$$\lambda^a_n(x)\hat{v}^n(x) = \varphi^a(x) \quad (a = 1,2,\ldots,K) \tag{14}$$

with K linear independent vector fields $\lambda^a_n \neq 0$. Then we can always find a coordinate transformation $x^n \Rightarrow f^n(x)$ such that the first $2N-K$ velocities $\hat{v}^n$ become independent operators and the rest of them are c-numbers: $\hat{v}^n = v^n\hat{I}$ for $n\rangle 2N-K$. Then the minimum noise D&D corresponds to the same structure (8) but the indices run from 1 to $2N-K$. The $(2N-K)\times(2N-K)$ metric tensor γ_{nm} defines a Riemann structure on the first $2N-K$ coordinates while it depends parametrically on the rest of them.

An important special case is coordinate coupling when $\partial\hat{H}/\partial q^n$ are independent operators but $\partial\hat{H}/\partial p^n$ are zeros or c-number functions. We impose the Riemann metric structure on the subspace of canonical coordinates only. The $N\times N$ metric tensor $\gamma_{nm}(q,p)$ will be the metric for the coordinates q; still it may parametrically depend on the momenta p as well. With the hybrid part of momentum velocity operators

$$\hat{v}^n = -\frac{\partial\hat{H}_{\mathrm{CQ}}}{\partial q_n}, \tag{15}$$

the D&D terms take this form:

$$\mathcal{D}\hat{\rho} = -\frac{\gamma_{nm}}{8}\left[\frac{\partial\hat{H}_{\mathrm{CQ}}}{\partial q_n},\left[\frac{\partial\hat{H}_{\mathrm{CQ}}}{\partial q_m},\hat{\rho}\right]\right] + \frac{1}{2}\frac{\partial^2(\gamma^{nm}\hat{\rho})}{\partial p^n\partial p^m}. \tag{16}$$

As we see, momentum velocity operators $\hat{v}^n$ are actors of decoherence and classical momenta p^n are subjects of diffusion.

The HSDEs (11) and (10) of the equivalent stochastic processes become the following:

$$\begin{aligned}
\frac{d\hat{P}}{dt} = &-\frac{i}{\hbar}[\hat{H}(q,p),\hat{P}] + \mathcal{D}(q,p)\hat{P}\\
&+ \mathbb{H}(\hat{v}^n(q,p) - \langle\hat{v}^n(q,p)\rangle)\hat{P}w_n(q,p),
\end{aligned} \tag{17}$$

$$\frac{dq_n}{dt} = \frac{\partial\langle\hat{H}(q,p)\rangle}{\partial p^n}, \tag{18}$$

$$\overline{dp^n}\,dt = -\frac{\partial\langle\hat{H}(q,p)\rangle}{\partial q_n} + w^n(q,p). \tag{19}$$

As in Eq. (12), the noise $w^n = \gamma^{nm}w_m$ satisfies

$$\begin{aligned}
\mathbb{M}w^n(q,p,t)w^m(q,p,\tau) &= \gamma^{nm}(q,p)\delta(t-\tau),\\
\mathbb{M}w_n(q,p,t)w_m(q,p,\tau) &= \gamma_{nm}(q,p)\delta(t-\tau),\\
\mathbb{M}w^n(q,p,t)w_m(q,p,\tau) &= \delta^n_m\delta(t-\tau).
\end{aligned} \tag{20}$$

This is the minimum-noise D&D term of general coordinate coupling provided the derivatives $\partial\hat{H}/\partial q_n$ are N independent operators.

III. LOCALITY CONDITION OF RELATIVISTIC CONTINUUM DYNAMICS

Let us consider the Markovian dynamics $d\rho/dt = L\rho$ where ρ is classical, quantum, or hybrid state, and L is the generator of time evolution, respectively, of Fokker-Planck, Lindblad, or hybrid field dynamics. For relativistic invariance, L must be the zeroth component of a four-vector. This condition on L is, however, not sufficient [32]. It must be the spatial integral of the generator density $\mathcal{L}(\mathbf{r})$:

$$L = \int \mathcal{L}(\mathbf{r})d\mathbf{r}, \tag{21}$$

and $\mathcal{L}(\mathbf{r})$ must satisfy the locality condition

$$[\mathcal{L}(\mathbf{r}), \mathcal{L}(\mathbf{s})] = 0. \tag{22}$$

Then, given the state on the hypersurface σ_1, it maps to another hypersurface as follows:

$$\rho(\sigma_2) = \exp\left(\int_{\sigma_2 \succ (t,\mathbf{r}) \succ \sigma_1} \mathcal{L}(\mathbf{r})d\mathbf{r}dt\right)\rho(\sigma_1). \tag{23}$$

Without the locality condition, this relationship does not exist, and we miss the map between states on two different hypersurfaces. Of course, the map between Lorentz frames is also impossible.

In standard relativistic field theories, classical or quantum, the generator field reads $\mathcal{L} = \{\mathcal{H},\}$ or $\mathcal{L} = -(i/\hbar)[\hat{\mathcal{H}},]$, respectively, and is local since the Hamiltonian densities $\mathcal{H}, \hat{\mathcal{H}}$ are local. Locality of the generator $\mathcal{L}$ survives in effective field theories. If, however, the effective theory contains diffusion (or decoherence), then we face difficulties. To retain locality of the generator $\mathcal{L}$ the diffusion (decoherence) kernel must be local, i.e., proportional to $\delta(\mathbf{r}-\mathbf{s})$, and then, unfortunately, the theory yields infinities. Take, for instance, the Fokker-Planck equation of a scalar field with the local diffusion kernel $\gamma\delta(\mathbf{r}-\mathbf{s})$. It yields an infinite rate kinetic energy production at each point $\mathbf{r}$. It is not known whether relativistic Fokker-Planck field equations are renormalizable or are not. The same concern applies to the Lindblad and hybrid dynamics.

IV. ON SPECIAL RELATIVISTIC HYBRID FIELD DYNAMICS

We test the NR hybrid classical-quantum theory (Sec. II) in coordinate coupling (Sec. II C) of special relativistic fields. The coordinates and momenta become functions $q(\mathbf{r}), p(\mathbf{r})$, and the discrete labels n, m become the continuous spatial vectors $\mathbf{r}$, $\mathbf{s}$, respectively. Sums over indices become spatial integrals, Kronecker deltas become Dirac deltas, derivations, e.g., $\partial/\partial q_n$, become functional derivations $\delta/\delta q(\mathbf{r})$.

Consider the coupling of the free classical scalar field $q(\mathbf{r})$ [with canonical momentum $p(\mathbf{r})$] to the free quantized boson field $\hat{\phi}(\mathbf{r})$ [with canonical momentum $\hat{\pi}(\mathbf{r})$]:

$$\hat{H}_{\mathrm{CQ}}[q] = \kappa\int q(\mathbf{r})\hat{\phi}(\mathbf{r})d\mathbf{r}. \tag{24}$$

This coupling is independent of the classical canonical momentum $p(\mathbf{r})$, and we can apply Eq. (16) with

$$\frac{\delta\hat{H}_{\mathrm{CQ}}}{\delta q(\mathbf{r})} = -\kappa\hat{\phi}(\mathbf{r}). \tag{25}$$

The D&D terms depend on the metric that can in general be a functional kernel $\gamma_{[q,q']}$. At the same time, we should damp remote correlation in decoherence as well as in diffusion. The metric must have a spatial damping factor. In the simplest case, we choose a flat metric $\gamma_{\mathbf{rr'}}$ without the functional dependencies. The covariant and contravariant kernels are inverses of each other:

$$\int \gamma_{\mathbf{rs'}}\gamma^{\mathbf{s's}}d\mathbf{s'} = \delta(\mathbf{r}-\mathbf{s}). \tag{26}$$

Then the D&D terms (16) take the following form:

$$\mathcal{D}\hat{\rho} = -\frac{\kappa^2}{8}\int\int \gamma_{\mathbf{rs}}[\hat{\phi}(\mathbf{r}), [\hat{\phi}(\mathbf{s}), \hat{\rho}]]d\mathbf{r}d\mathbf{s} + \frac{1}{2}\int\int \gamma^{\mathbf{rs}}\frac{\delta^2(\hat{\rho})}{\delta p(\mathbf{r})\delta p(\mathbf{s})}d\mathbf{r}d\mathbf{s}. \tag{27}$$

Both D&D terms violate the special relativistic invariance unless the kernel itself is invariant. It is easy to ensure Galilean invariance if $\gamma_{\mathbf{rs}}$ is a function of $|\mathbf{r}-\mathbf{s}|$. The only kernels that ensure relativistic invariance are the singular local ones:

$$\gamma_{\mathbf{rs}} = \gamma\delta(\mathbf{r}-\mathbf{s}), \qquad \gamma^{\mathbf{rs}} = \gamma^{-1}\delta(\mathbf{r}-\mathbf{s}). \tag{28}$$

But they lead to untractable divergences of the kinetic energy density $\mathcal{K} = \frac{1}{2}(\hat{\pi}^2 + p^2)$:

$$\frac{d\mathcal{K}(\mathbf{r})}{dt} = \frac{1}{2}\mathcal{D}_{\mathrm{Q}}^{\dagger}\pi^2(\mathbf{r}) + \frac{1}{2}\mathcal{D}_{\mathrm{C}}^{\dagger}p^2(\mathbf{r}) = \left(\frac{\gamma}{4\hbar^2} + \frac{1}{\gamma}\right)\delta(\mathbf{0}). \tag{29}$$

The D&D terms (27) yield an infinite rate of heating at each location in the quantized bosonic as well as in the classical scalar field subsystems. Allowing functional dependence of the metric does not help since the relativistic invariance of spatial damping requires the presence of the spatial δ function singularity.

These divergences are different from the usual divergences in relativistic field theory. Either we invent their renormalization, if it is possible at all, or we are losing special relativistic invariance, and we are left with the NR hybrid calculus.

V. ON HYBRID GENERAL RELATIVITY

Instead of full quantum gravity, it would be of great simplification if we could keep the spacetime classical. Accordingly, we take a chance to extend the NR hybrid dynamics of Sec. II for coupling between a classical canonical form of general relativity and quantized relativistic matter. In the canonical form of Einstein's general relativity, $(3+1)$-dimensional diffeomorphism invariance is encoded by the combination of three-dimensional spatial diffeomorphism (sDM) invariance and time-reparametrization (tRP) invariance. Following Refs. [15,16], we build up the formal sDM and tRP invariant hybrid equations (Sec. V A). We are *going to the wall* to ensure both these invariances but that remains a problem (Sec. V B).

A. Equivalent formalisms: HME and HSDE

The canonical coordinates are the configurations of the 3×3 metric tensor field $g_{ik}(\mathbf{r})$, satisfying the canonical commutation relationship with the canonical momenta $\pi^{ik}(\mathbf{r})$:

$$\{g_{ij}(\mathbf{r}),\pi^{kl}(\mathbf{s})\}=\delta_{ij}^{kl}\delta(\mathbf{r},\mathbf{s}),\tag{30}$$

where $\delta_{ij}^{kl}=\frac{1}{2}(\delta_i^k\delta_j^l+\delta_i^l\delta_j^k)$ and we use the covariant delta function

$$\delta(\mathbf{r},\mathbf{s})=\frac{1}{\sqrt{g(\mathbf{r})}}\delta(\mathbf{r}-\mathbf{s}),\tag{31}$$

where $g=\det g_{ij}$. The covariant Poisson bracket is defined by

$$\{\hat{A},\hat{B}\}=\int\left(\frac{\delta\hat{A}}{\delta g_{ij}(\mathbf{r})}\frac{\delta\hat{B}}{\delta\pi^{ij}(\mathbf{r})}-\frac{\delta\hat{A}}{\delta\pi^{ij}(\mathbf{r})}\frac{\delta\hat{B}}{\delta g_{ij}(\mathbf{r})}\right)dV,\tag{32}$$

where $dV=dV_{\mathbf{r}}=\sqrt{g(\mathbf{r})}d\mathbf{r}$. Through this section, the functional derivatives are the covariant ones, i.e., $1/\sqrt{g}$ times the common ones.

The hybrid Hamiltonian reads

$$\hat{H}[g,\pi;N,\vec{N}]=H_{\mathrm{G}}[g,\pi;N,\vec{N}]+\hat{H}_{\mathrm{M}}[g;N,\vec{N}],\tag{33}$$

where $H_{\mathrm{G}}[g,\pi;N,\vec{N}]$ is the classical Hamilton function of gravity and $\hat{H}_{\mathrm{M}}[g;N,\vec{N}]$ is the Hamiltonian of the quantized matter fields, coupled only to g_{ik} and not to π^{ik}. They depend on the freely chosen lapse N and shift N_i:

$$H_{\mathrm{G}}[g,\pi;N,\vec{N}]=\int(N(\mathbf{r})\mathcal{H}_{\mathrm{G}}(\mathbf{r})+N_i(\mathbf{r})\mathcal{P}_G^i(\mathbf{r}))dV,\tag{34}$$

$$\hat{H}_{\mathrm{M}}[g;N,\vec{N}]=\int(N(\mathbf{r})\hat{\mathcal{H}}_{\mathrm{M}}(\mathbf{r})+N_i\mathcal{P}_M^i(\mathbf{r}))dV.\tag{35}$$

$\mathcal{H}_{\mathrm{G}}(\mathbf{r})$ and $\hat{\mathcal{H}}_{\mathrm{M}}(\mathbf{r})$ are the Hamiltonian densities of gravity and matter, respectively, and $\mathcal{P}_G^i$ is the momentum density of gravity:

$$\mathcal{P}_G^i=-2\nabla_i\pi^{ij}(\mathbf{r}),\tag{36}$$

where ∇_j denotes covariant derivation. The gravity's Hamiltonian density reads

$$\mathcal{H}_G=\frac{16\pi G}{c^2}\frac{1}{g}\left(\pi^{ij}\pi_{ij}-\frac{1}{2}(\pi_i^i)^2\right)-\frac{c^4}{16\pi G}R,\tag{37}$$

with the scalar curvature R. The matter's Hamiltonian $\hat{\mathcal{H}}_M(\mathbf{r})$ and momentum density $\mathcal{P}_M^i$ depend on the matter fields. Remember that they should not depend on π^{ik}.

In the hybrid Hamiltonian (33), the lapse N multiplies the Hamiltonian constraint, the shift N_i multiplies the diffeomorphism constraint which we impose on the hybrid state:

$$(\mathcal{H}_{\mathrm{G}}(\mathbf{r})+\hat{\mathcal{H}}_{\mathrm{M}}(\mathbf{r}))\hat{\rho}[g,\pi]=0,\tag{38}$$

$$(\mathcal{P}_G^i(\mathbf{r})+\hat{\mathcal{P}}_M^i(\mathbf{r}))\hat{\rho}[g,\pi]=0.\tag{39}$$

These might ensure tRP and sDM invariances, respectively. The conditional phrase is of reason. If both gravity and matter were quantized (or classical), then the above constraints would guarantee the said invariances under pure classical canonical (or pure unitary) dynamics. Their compatibility and applicability in hybrid dynamics are not yet clear. Moreover, hybrid dynamics are not necessarily compatible with tRP and sDM invariances, as we see below.

To construct the hybrid coupling and the D&D terms, we need the momentum velocity operators (15):

$$\hat{v}^{ik}(\mathbf{r})=-\frac{\delta\hat{H}_M}{\delta g_{ik}(\mathbf{r})}=-N(\mathbf{r})\left(\frac{\partial\hat{\mathcal{H}}_M(\mathbf{r})}{\partial g_{ik}(\mathbf{r})}+\frac{1}{2}g^{ik}(\mathbf{r})\hat{\mathcal{H}}_M(\mathbf{r})\right).\tag{40}$$

The HME (4) of the state $\hat{\rho}[g,\pi]$ takes this form:

$$\frac{d\hat{\rho}}{dt}=-\frac{i}{\hbar}[\hat{H}_M,\hat{\rho}]+\{H_G,\hat{\rho}\}-\mathbb{H}\int\hat{v}^{ik}\frac{\delta\hat{\rho}}{\delta\pi^{ik}}dV+\mathcal{D}\hat{\rho}.\tag{41}$$

While the hybrid Hamiltonian parts are unique, the D&D term $\mathcal{D}\hat{\rho}$ is not, and its consistent choice is nontrivial (see Sec. V B).

The HME (41) has its alternative stochastic representation in terms of SHDEs. We apply Eqs. (18)–(20):

$$\frac{d\hat{P}}{dt} = -\frac{i}{\hbar}[\hat{H}_{\rm M}, \hat{P}] + \mathcal{D}_{\rm Q}\hat{P} + \mathbb{H}\int (\hat{v}^{ij} - \langle\hat{v}^{ij}\rangle)\hat{P} w_{ij} dV, \tag{42}$$

$$\frac{dg_{ij}}{dt} = \frac{\delta H_{\rm G}}{\delta\pi^{ij}}, \tag{43}$$

$$\frac{d\pi^{ij}}{dt} = -\frac{\delta H_{\rm G}}{\delta g_{ij}} + \langle\hat{v}^{ij}\rangle + w^{ij}, \tag{44}$$

where the noises satisfy

$$\begin{aligned}
\mathbb{M} w^{ij}(\mathbf{r},t) w^{kl}(\mathbf{s},\tau) &= \gamma^{ij|kl}(\mathbf{r}|\mathbf{s})\delta(t-\tau),\\
\mathbb{M} w_{ij}(\mathbf{r},t) w_{kl}(\mathbf{s},\tau) &= \gamma_{ij|kl}(\mathbf{r}|\mathbf{s})\delta(t-\tau),\\
\mathbb{M} w^{ij}(\mathbf{r},t) w_{kl}(\mathbf{s},\tau) &= \delta^{ij}_{kl}\delta(t-\tau),
\end{aligned} \tag{45}$$

and $\mathcal{D}_{\rm Q}$ will be discussed in Sec. V B.

As we said in Sec. II B, Eq. (42) corresponds to the quantum monitoring of the velocity operators $\hat{v}^{ij} = -\delta\hat{H}_{\rm M}/\delta g_{ij}$, and the noisy measured signal $\langle\hat{v}^{ij}\rangle + w^{ij}$ is fed back on the right-hand side (rhs) of Eq. (44) of $d\pi^{ij}/dt$.

B. The decoherence-diffusion kernels

Recall that the hybrid dynamics (Sec. II C) assumes a certain metric on the space of canonical coordinates, which is a functional metric on the function space of 3×3 metric tensor fields $g_{ij}(\mathbf{r})$. We restrict ourselves to the metrics

$$(dg)^2 = \int\int \gamma^{ij|kl}(\mathbf{r}|\mathbf{s}) dg_{ij}(\mathbf{r}) dg_{kl}(\mathbf{s}) dV_{\mathbf{r}} dV_{\mathbf{s}}, \tag{46}$$

where the functional metric tensor γ contains explicit coordinate dependence on $(\mathbf{r},\mathbf{s})$ to damp remote correlations; also it may depend on $g_{ij}(\mathbf{r})$ and $g_{kl}(\mathbf{s})$ (meaning nonflat functional geometry). Accordingly, the D&D terms take this form:

$$\mathcal{D}_{\rm Q} = -\frac{1}{8}\int\int \gamma^{-1}_{ij|kl}(\mathbf{r}|\mathbf{s})[\hat{v}^{ij}(\mathbf{r}), [\hat{v}^{kl}(\mathbf{s}), \hat{\rho}]] dV_{\mathbf{r}} dV_{\mathbf{s}}, \tag{47}$$

$$\mathcal{D}_{\rm C} = \frac{1}{2}\int\int \frac{\delta^2(\gamma^{ij|kl}(\mathbf{r}|\mathbf{s})\hat{\rho})}{\delta\pi^{ij}(\mathbf{r})\delta\pi^{kl}(\mathbf{s})} dV_{\mathbf{r}} dV_{\mathbf{s}}, \tag{48}$$

where the covariant and contravariant metrics satisfy the functional relationship

$$\int\int \gamma_{ij|k'l'}(\mathbf{r}|\mathbf{s}')\gamma^{k'l'|kl}(\mathbf{s}'|\mathbf{s}) dV_{\mathbf{s}'} = \delta^{kl}_{ij}\delta(\mathbf{r},\mathbf{s}). \tag{49}$$

It is instructive to consider the simple special case when the kernels are local. Then their structure is perfectly determined by covariance:

$$\begin{aligned}
\gamma_{ij|kl}(\mathbf{r}|\mathbf{s}) &= \frac{\gamma(R)}{N(\mathbf{r})} G^{(\alpha)}_{ij|kl}(\mathbf{r})\delta(\mathbf{r},\mathbf{s}),\\
G^{(\alpha)}_{ij|kl} &= \frac{1}{2}g_{ik}g_{jl} + \frac{1}{2}g_{il}g_{jk} + \alpha g_{ij}g_{kl},
\end{aligned} \tag{50}$$

$$\begin{aligned}
\gamma^{ij|kl}(\mathbf{r}|\mathbf{s}) &= \frac{N(\mathbf{r})}{\gamma(R)} G^{ij|kl}_{(\beta)}(\mathbf{r})\delta(\mathbf{r},\mathbf{s}),\\
G^{ij|kl}_{(\beta)} &= \frac{1}{2}g^{ik}g^{jl} + \frac{1}{2}g^{il}g^{jk} + \beta g^{ij}g^{kl}.
\end{aligned} \tag{51}$$

These kernels are positive if $\alpha,\beta\rangle - 1/3$. If $3\alpha\beta + \alpha + \beta = 0$, then the kernels are each other's inverses as they should be, according to Eq. (49).

With the above kernels, unfortunately, both the D&D terms in Eqs. (47) and (48), respectively, become divergent because of the δ-functions, just as in Sec. IV. However, a rescue procedure seems to be on offer.

We could try the sDM invariant regularization. For instance, we replace the $\delta(\mathbf{r},\mathbf{s})$ in the decoherence kernel (50) by

$$\mathcal{N}_\epsilon(\mathbf{r},\mathbf{s}) \exp\left(-\frac{\ell^2(\mathbf{r},\mathbf{s})}{2\epsilon}\right), \tag{52}$$

where $\mathcal{N}_\epsilon(\mathbf{r},\mathbf{s})$ is for normalization, $\ell(\mathbf{r},\mathbf{s})$ is the geodesic distance between $\mathbf{r}$ and $\mathbf{s}$, and ϵ is the small parameter to go to $+0$. To keep covariance, the index factor, too, should go nonlocal:

$$\begin{aligned}
G^{(\alpha)}_{ij|kl}(\mathbf{r}|\mathbf{s}) = \frac{1}{2}P^{j'}_j\bar{P}^{k'}_k g_{ik'}(\mathbf{r})g_{j'l}(\mathbf{s}) + \frac{1}{2}P^{j'}_j\bar{P}^{l'}_l g_{il'}(\mathbf{r})g_{kj'}(\mathbf{s})\\
+ \alpha g_{ij}(\mathbf{r})g_{kl}(\mathbf{s}).
\end{aligned} \tag{53}$$

Here P^i_j is a geodesic parallel transport of covariant vectors from $\mathbf{s}$ to $\mathbf{r}$ and $\bar{P}^i_j$ is the same from $\mathbf{r}$ to $\mathbf{s}$.

So far so good. The problem is the factor $1/N(\mathbf{r})$, which ensures the tRP invariance. We should keep it but we cannot. It cannot be split for the two locations $\mathbf{r}$ and $\mathbf{s}$. The same problem would come along with the factor $N(\mathbf{r})$ if we regularized the decoherence kernel (51) first.

The lesson goes beyond the example. Any nonlocal generalization of the kernels will necessarily violate the tRP invariance. Local kernels, on the other hand, generate divergences whose removal may or may not be possible. Hence, for the time being, a compromise seems inevitable. We give up tRP invariance and retain sDM invariance that allows regular nonlocal kernels. Just losing tRP invariance means losing relativistic invariance. We are left with NR slow motions in a distinguished frame: sDM is pointless.

Also the spacetime must be nearly flat. That is the Newtonian limit.

VI. NEWTONIAN HYBRID CLASSICAL-QUANTUM GRAVITY

When recapitulating the results of Refs. [11,12], we use a particular approach. These works used the NR HSDE representation of hybrid dynamics. Not for deduction but for comparison, we guide our derivation by the HSDEs (42)–(44) that promised general relativistic postquantum gravity in Sec. V. We present the HSDEs of Newtonian hybrid theory first.

What is the closest NR dynamics to the HSDEs (42)–(45)? The matter Hamiltonian with the hybrid coupling reads

$$\hat{H}_{\mathrm{M}}[\Phi]=\hat{H}_0+\int\hat{\mu}\Phi dV, \tag{54}$$

where Φ is the Newton potential and $\hat{\mu}$ is the NR quantum field of mass density. The quantum monitoring of $\hat{v}^{ij}$ corresponds to the quantum monitoring of $\hat{\mu}(\mathbf{r})$ since the nonrelativistic limit of $\hat{v}^{ij}$ is $\propto\hat{\mu}$. Hence, the NR counterpart of the SDE (42) is

$$\frac{d\hat{P}}{dt}=-\frac{i}{\hbar}[\hat{H}_{\mathrm{M}}[\Phi],\hat{P}]+\mathcal{D}_{\mathrm{Q}}\hat{P}+\frac{1}{\hbar}\mathbb{H}\int(\hat{\mu}-\langle\hat{\mu}\rangle)\hat{P}wdV, \tag{55}$$

with

$$\mathcal{D}_{\mathrm{Q}}\hat{P}=-\frac{1}{8\hbar^2}\int\int\gamma_{\mathbf{rs}}[\hat{\mu}(\mathbf{r}),[\hat{\mu}(\mathbf{s}),\hat{P}]]d\mathbf{r}d\mathbf{s}. \tag{56}$$

The measurement signal is of the standard form

$$\langle\hat{\mu}\rangle+\tilde{w}, \tag{57}$$

where $\tilde{w}(\mathbf{r},t)=\int\gamma^{\mathbf{rs}}w(\mathbf{s},t)d\mathbf{s}$. The covariant and contravariant components $(w,\tilde{w})$ of the same noise satisfy

$$\begin{aligned}\mathbb{M}w(\mathbf{r},t)w(\mathbf{s},\tau)&=\gamma_{\mathbf{rs}}\delta(t-\tau),\\ \mathbb{M}\tilde{w}(\mathbf{r},t)\tilde{w}(\mathbf{s},\tau]&=\hbar^2\gamma_{\mathbf{rs}}\delta(t-\tau),\\ \mathbb{M}w(\mathbf{r},t)\tilde{w}(\mathbf{s},\tau]&=\hbar\delta(\mathbf{r}-\mathbf{s})\delta(t-\tau).\end{aligned} \tag{58}$$

Since gravity has no self-dynamics, $H_{\mathrm{G}}=0$, the backaction (43) and (44) reduces to the Poisson equation sourced by the signal (57), and we can solve it:

$$\begin{aligned}\Phi(\mathbf{r},t)&=\frac{4\pi G}{\nabla^2}(\langle\hat{\mu}(\mathbf{r})\rangle_t+\tilde{w}(\mathbf{r},t))\\ &\equiv\Phi_{\mathrm{mf}}(\mathbf{r},t)+\delta\Phi(\mathbf{r},t).\end{aligned} \tag{59}$$

The deterministic term Φ_{mf} is the mean-field (semiclassical) part, the stochastic term is a white noise of correlation

$$\mathbb{M}\delta\Phi(\mathbf{r},t)\delta\Phi(\mathbf{s},\tau)=\frac{4\pi G}{\nabla_{\mathbf{r}}^2}\frac{4\pi G}{\nabla_{\mathbf{s}}^2}\hbar^2\gamma_{\mathbf{rs}}\delta(t-\tau). \tag{60}$$

When Φ is fed back in Eq. (55), the Hamiltonian $\hat{H}_{\mathrm{M}}[\Phi]$ generates the Newtonian pair potential

$$\hat{V}_{\mathrm{G}}=-\frac{G}{2}\int\int\frac{\hat{\mu}(\mathbf{r})\hat{\mu}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}d\mathbf{r}d\mathbf{s}. \tag{61}$$

Unlike the general relativistic $\hat{H}_{\mathrm{M}}[g]$, where g is a Wiener process, Φ is not, it is the time derivative of a Wiener process. The feedback of the white-noise term in $\hat{H}_{\mathrm{M}}[\Phi]$, proportional to $\delta\Phi$, will contribute to a new decoherence term:

$$\mathcal{D}_{\mathrm{Q}}^{\mathrm{fb}}\hat{P}=-\frac{1}{2\hbar^2}\int\int\left(\frac{4\pi G}{\nabla_{\mathbf{r}}^2}\frac{4\pi G}{\nabla_{\mathbf{s}}^2}\gamma^{\mathbf{rs}}\right)[\hat{\mu}(\mathbf{r}),[\hat{\mu}(\mathbf{s}),\hat{P}]]d\mathbf{r}d\mathbf{s}. \tag{62}$$

This backaction makes a remarkable difference compared to the general relativistic case in Sec. V B. The ambiguity of the D&D kernels can be removed by the principle of least decoherence. Since $\gamma^{\mathbf{rs}}=\gamma_{\mathbf{rs}}^{-1}$, the total decoherence $\mathcal{D}_{\mathrm{Q}}+\mathcal{D}_{\mathrm{Q}}^{\mathrm{fb}}$ possesses a minimum when

$$\begin{aligned}\gamma_{\mathbf{rs}}&=\frac{2\hbar G}{|\mathbf{r}-\mathbf{s}|},\\ \gamma^{\mathbf{rs}}&=-\frac{1}{8\pi\hbar G}\nabla^2\delta(\mathbf{r}-\mathbf{s}).\end{aligned} \tag{63}$$

Accordingly, the least decoherence reads

$$\mathcal{D}_{\mathrm{Q}}^{\mathrm{DP}}=-\frac{G}{2\hbar}\int\int\frac{[\hat{\mu}(\mathbf{r}),[\hat{\mu}(\mathbf{s}),\hat{P}]]d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}, \tag{64}$$

and the correlation of the gravitational fluctuations become

$$\mathbb{M}\delta\Phi(\mathbf{r},t)\delta\Phi(\mathbf{s},\tau)=\frac{\hbar G/2}{|\mathbf{r}-\mathbf{s}|}\delta(t-\tau). \tag{65}$$

We obtain the HSDEs of the Newtonian NR postquantum gravity:

$$\begin{aligned}\frac{d\hat{P}}{dt}=&-\frac{i}{\hbar}[\hat{H}_0+\hat{V}_{\mathrm{G}},\hat{P}]\\ &+\mathcal{D}_{\mathrm{Q}}^{\mathrm{DP}}\hat{P}+\mathbb{H}\frac{1+i}{\hbar}\int(\hat{\mu}-\langle\hat{\mu}\rangle)\hat{P}wdV,\end{aligned} \tag{66}$$

$$\Phi=\frac{4\pi G}{\nabla^2}\langle\hat{\mu}\rangle-\frac{1}{2}w=\Phi_{\mathrm{mf}}-\frac{1}{2}w, \tag{67}$$

where Φ_{mf} is the mean-field (semiclassical) Newton potential, and

$$\mathbb{M}w(\mathbf{r},t)w(\mathbf{s},\tau)=\frac{2\hbar G}{|\mathbf{r}-\mathbf{s}|}\delta(t-\tau). \tag{68}$$

For pointlike particles the theory is divergent and predicts a kinetic energy increase at an infinite rate. Therefore, $\hat{\mu}(\mathbf{r})$ must be smoothened by a short length cutoff parameter, the only free parameter of the theory (see [33] for its experimental limit).

Observe that due to the simple structure of the Newtonian postquantum dynamics the reduced dynamics of the quantized matter is autonomous. Take the stochastic mean of both sides of Eq. (66), and then the following Lindblad master equation is obtained for $\hat{\rho}_{\mathrm{Q}}=\mathbb{M}\hat{P}$:

$$\frac{d\hat{\rho}_{\mathrm{Q}}}{dt}=-\frac{i}{\hbar}[\hat{H}_0+\hat{V}_{\mathrm{G}},\hat{\rho}_{\mathrm{Q}}]-\frac{G}{2\hbar}\int\int[\hat{\mu}(\mathbf{r}),[\hat{\mu}(\mathbf{s}),\hat{\rho}_{\mathrm{Q}}]]\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}. \tag{69}$$

The full HME, equivalent to the HSDE formalism (66)–(68), is derived in Appendix B.

VII. REMARKS AND CONCLUSIONS

The issues of hybrid dynamics relativistic extensions that Secs. IV and V claim are unsolved were carefully discussed by the authors of Refs. [15,16], highlighting some perspectives toward solutions. These are assessed with certain reservations in Ref. [30]. We add that the literature offers no support for hybrid constraints, and little or no support for renormalizability of relativistic effective field theories whether they are classical, quantum, or hybrid. Toward fixing infinities predicted by relativistic Lindblad and Fokker-Planck equations, conclusive research is missing even for the simple special relativistic D&D in Sec. IV.

Some additional details about the nonrelativistic "postquantum" theory (Sec. VI) are to be recalled. It all started in foundations (reviewed in [34,35]), with a gravity-related nonrelativistic model of the quantum-classical transition [6] and a naive formalism of relativistic monitoring-plus-feedback [7]. Recognizing the difficulties of relativistic monitoring, only the Newtonian limit of monitoring-plus-feedback was briefly presented. Much later, the concept of postquantum gravity, called a "conceptually healthier semi-classical theory," was stated literally in [11]: monitoring the quantized energy-momentum tensor $\hat{T}_{ab}$ and its measured value fed back into the Einstein equation of classical general relativity. After two and a half decades, this work and its follow-up [12] must still have adhered to the Newtonian limit. The reason has remained the same: the missing theory of relativistic monitoring. The concrete technical obstacles are the D&D kernels that must be time local for Markovianity. If the suitably covariant kernels exist at all, they generate divergences whose treatment is unknown. Without these difficulties, the monitoring-plus-feedback form (equivalent to the hybrid master equation form) of postquantum general relativity would have been a straightforward step. Vice versa, if the hybrid master equation form of postquantum gravity got rid of its difficulties with the D&D kernels, it would contain a module of relativistic quantum monitoring. This matches with the assessment in Ref. [30].

The pending issues of the recent proposal [15,16] of postquantum gravity are the known old difficulties that have been hindering the relativistic extension of the Newtonian forerunner [6,7,11,12]. The difficulties are rooted in difficulties of Lindblad as well as of Fokker-Planck dynamics of relativistic fields; both dynamics are obligatory parts of postquantum gravity. Although these issues might become fixed later, the contrary is equally likely: relativity and Markovianity of decoherence (or diffusion) may turn out to be just inconsistent [32].

In contrast to the relativistic postquantum gravity, the Newtonian precursor [6,11,12] is a consistent model with a single free parameter. The predicted violation of the superposition principle and the presence of the tiny noise have been looked for by various experiments reviewed, e.g., in [36]. The model, also called the Diósi-Penrose model, is currently neither confirmed nor ruled out. For a conclusive test, the quantum control of the test mass motional states must be improved. Even higher improvement will be requested in the proposed nonrelativistic tests to rule out the classicality of gravity [37–39]. Such tests might or might not rule out unquantized gravity theories. As yet, this is at worst a period of grace for them.

The present author expects that the hybrid of classical gravity and quantized matter is hiding more secrets already in the Newtonian limit, both in theory and in experiments. We should continue to reveal them in the simple nonrelativistic realm before we would cross the bridge toward a certain postquantum general relativity.

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APPENDIX A: DEDUCTION OF HME (4)

We show that our canonical HME (4) with the D&D term (8) is the special case of the general diffusive HME [26–29]:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + 2\mathbb{H}([\bar{G}_{\mathrm{CQ}}]^n_\alpha \hat{L}^\alpha\hat{\rho})_{,n} + \mathcal{D}\hat{\rho}, \tag{A1}$$

$$\mathcal{D} = D^{\mathrm{Q}}_{\beta\alpha}(\hat{L}^\alpha\hat{\rho}\hat{L}^\beta - \mathbb{H}\hat{L}^\beta\hat{L}^\alpha\hat{\rho}) + \frac{1}{2}(D^{nm}_{\mathrm{C}}\hat{\rho})_{nm}, \tag{A2}$$

where, compared to Eq. (36) in [28], we assumed Hermitian Lindblad generators $\hat{L}^\alpha$ and changed the upper/lower greek indices for the lower/upper ones. This HME is valid for any classical subsystem, and the classical coordinates x are not necessarily canonical. When the Lindblad generators $\hat{L}^\alpha(x)$ are independent operators, then minimum noise is achieved if the positive D&D matrices D^{Q} and D_{C}, respectively, are constrained by the matrix of backaction G_{CQ}:

$$G_{\mathrm{CQ}}\frac{1}{D^{\mathrm{Q}}}G^\dagger_{\mathrm{CQ}} = D_{\mathrm{C}}. \tag{A3}$$

Let us first identify the classical variables x^n by our canonical ones. Second, identify the Lindblad generators $\hat{L}^\alpha$ by our velocity operators $\hat{v}^n$, the greek indices will become the latin ones accordingly. Let us equate the backaction terms in (4) and (A1):

$$\mathbb{H}(\hat{v}^n\hat{\rho})_{,n} = -2\mathbb{H}([\bar{G}_{\mathrm{CQ}}]^n_m\hat{v}^n\hat{\rho})_{,n}. \tag{A4}$$

They coincide if $[\bar{G}_{\mathrm{CQ}}]^n_m = -\frac{1}{2}\delta^n_m$. The D&D terms (8) and (A2) coincide if $D^{\mathrm{Q}}_{nm} = \frac{1}{4}\gamma_{nm}$ and $D^{nm}_{\mathrm{C}} = \gamma^{nm}$. The said choices $D_{\mathrm{C}}, D^{\mathrm{Q}}$, and G_{CQ} satisfy the general condition (A3) of minimum noise.

APPENDIX B: DERIVATION OF HME FROM HSDES (66)–(68)

It is incorrect to take the form $\hat{\rho}[\Phi]$ for the hybrid state since Φ is a white noise. The correct form is $\hat{\rho}_t[\chi]$; i.e., the configuration of classical gravity is represented by the Wiener process χ defined by $\Phi = d\chi/dt$. We define the hybrid density as follows:

$$\hat{\rho}_t[\chi] = \mathbb{M}\hat{P}_t\delta[\chi-\chi_t]. \tag{B1}$$

The differentials of both sides read

$$d\hat{\rho}_t[\chi] = \mathbb{M}(d\hat{P}_t\delta[\chi-\chi_t] + \hat{P}_t d\delta[\chi-\chi_t] + d\hat{P}_t d\delta[\chi-\chi_t]), \tag{B2}$$

where the last term on the rhs is the Ito correction to the Leibnitz rule. According to Ito calculus, using the HSDEs (66) and (67) and the white-noise correlation (68) yield

$$\begin{aligned} d\hat{P} = &-\frac{i}{\hbar}[\hat{H}_0+\hat{V}_{\mathrm{G}},\hat{P}]dt + \mathcal{D}^{\mathrm{DP}}_{\mathrm{Q}}\hat{P}dt \\ &+ \mathbb{H}\frac{1+i}{\hbar}\int(\hat{\mu}(\mathbf{r})-\langle\hat{\mu}(\mathbf{r})\rangle)\hat{P}w(\mathbf{r},t)d\mathbf{r}dt, \end{aligned} \tag{B3}$$

$$\begin{aligned} d\delta[\chi-\chi_t] = &-\int\left(\Phi_{\mathrm{mf}}(\mathbf{r})-\frac{1}{2}w(\mathbf{r},t)\right)\frac{\delta}{\delta\chi(\mathbf{r})}\delta[\chi-\chi_t]d\mathbf{r}dt \\ &+\frac{1}{4}\int\int\frac{\hbar G}{|\mathbf{r}-\mathbf{s}|}\frac{\delta^2}{\delta\chi(\mathbf{r})\delta\chi(\mathbf{s})}\delta[\chi-\chi_t]d\mathbf{r}d\mathbf{s}dt, \end{aligned} \tag{B4}$$

$$\begin{aligned} d\hat{P}d\delta[\chi-\chi_t] = &\,\mathbb{H}(1+i)\int\int\frac{G}{|\mathbf{r}-\mathbf{s}|}(\hat{\mu}(\mathbf{s})-\langle\hat{\mu}(\mathbf{s})\rangle)\hat{P} \\ &\times\frac{\delta}{\delta\chi(\mathbf{r})}\delta[\chi-\chi_t]d\mathbf{r}d\mathbf{s}dt. \end{aligned} \tag{B5}$$

Now we insert these three expressions into Eq. (B2), set $w=0$ since $\mathbb{M}w=0$, and use the definition (B1) of $\hat{\rho}[\chi]$, yielding, after dividing both sides by dt,

$$\begin{aligned} \frac{d\hat{\rho}[\chi]}{dt} = &-\frac{i}{\hbar}[\hat{H}_0+\hat{V}_{\mathrm{G}},\hat{\rho}[\chi]] + \mathcal{D}^{\mathrm{DP}}_{\mathrm{Q}}\hat{\rho}[\chi] \\ &-\int\Phi_{\mathrm{mf}}(\mathbf{r})\frac{\delta\hat{\rho}[\chi]}{\delta\chi(\mathbf{r})}d\mathbf{r} \\ &+\frac{1}{4}\int\int\frac{\hbar G}{|\mathbf{r}-\mathbf{s}|}\frac{\delta^2\hat{\rho}[\chi]}{\delta\chi(\mathbf{r})\delta\chi(\mathbf{s})}d\mathbf{r}d\mathbf{s} \\ &+\mathbb{H}(1+i)\int\int\frac{G}{|\mathbf{r}-\mathbf{s}|}(\hat{\mu}(\mathbf{s})-\langle\hat{\mu}(\mathbf{s})\rangle)\frac{\delta\hat{\rho}[\chi]}{\delta\chi(\mathbf{r})}d\mathbf{r}d\mathbf{s}. \end{aligned} \tag{B6}$$

The nonlinear terms on the rhs cancel as they should, and we write the HME in the following form:

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = &-\frac{i}{\hbar}[\hat{H}_0+\hat{V}_{\mathrm{G}},\hat{\rho}] \\ &+G\int\int\left(-\frac{1}{2\hbar}[\hat{\mu}(\mathbf{r}),[\hat{\mu}(\mathbf{s}),\hat{\rho}]] + \mathbb{H}(1+i)\hat{\mu}(\mathbf{r})\frac{\delta\hat{\rho}}{\delta\chi(\mathbf{s})}\right. \\ &\left.+\frac{\hbar}{4}\frac{\delta^2\hat{\rho}}{\delta\chi(\mathbf{r})\delta\chi(\mathbf{s})}\right)\frac{d\mathbf{r}d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}. \end{aligned} \tag{B7}$$

The HME yields the mean-field (semiclassical) gravity:

$$\mathbb{M}\Phi(\mathbf{r}) = \mathrm{tr}\int\frac{d\chi(\mathbf{r})}{dt}\hat{\rho}[\chi]d[\chi] = -G\int\frac{\langle\hat{\mu}(\mathbf{s})\rangle}{|\mathbf{r}-\mathbf{s}|}d\mathbf{s} = \Phi_{\mathrm{mf}}(\mathbf{r}),$$

as well as the "spacetime" diffusion (65) where $\delta\Phi = d\chi/dt - \Phi_{\mathrm{mf}}$.

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