The frictional Schrödinger–Newton equation in models of wave function collapse

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Abstract. Replacing the Newtonian coupling G by $-iG$, the Schrödinger–Newton equation becomes "frictional". Instead of the reversible Schrödinger–Newton equation, we advocate its frictional version to generate the set of pointer states for macroscopic quantum bodies.

1. Introduction

Surely there must be a sort of bottleneck that impedes the unification of both the quantum and the gravitation theories. In the mainstream opinion, the bottleneck is the concept of gravity rather than the concept of quantum, possibly we have to reformulate our concept of spacetime and introduce sophisticated representations of the geometry. In a sidestream opinion (cf., e.g., [1, 2, 3, 4, 5]), the bottleneck is within the concept of quantum theory, maybe in the notorious concept of wave function collapse which blocks our progress to a unified quantum gravity. If we support this sidestream opinion, it deserves a look at Fig. 1 of the Physics Building [3, 4]. It might well be that the path upto a relativistic theory of a quantized Universe goes through the non-relativistic theory of Newtonian Quantum Gravity explaining the quantized motion
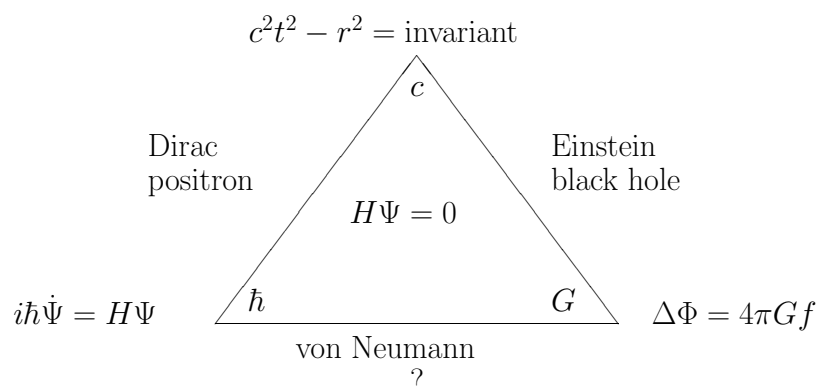

Figure 1. c =velocity of light, G =Newton's gravitational constant, $\hbar$ =Planck constant. The corners of the triangle represent the three fundamental theories, the sides correspond to partially unified theories while the middle symbolizes the fully unified theory.

of common macroscopic objects. If the conflict of wave function collapse with the concept of gravity is indeed the bottleneck then, in the Newtonian limit, the resolution should come in the form of a gravity-related (but non-Hamiltonian) theory of spontaneous (i.e. non-environmental) *decoherence and collapse* of macro-objects' wave function. This task is complex and maybe too elusive. It is, therefore, in tradition that we focus on the quantized motion of a rigid massive ball prepared in a so-called Schrödinger cat state and we suppose a certain Newton-gravity-related decoherence for it. We are going to adopt this focused discussion of the quantized motion of massive objects, which might eventually clear a path to the relativistic unification of the quantum and gravity.

2. The 'rigid ball' Schrödinger cat and the Newtonian $U(\mathbf{x} - \mathbf{x}')$

Our macroscopic system is a rigid homogenous ball. Initially we start from a Schrödinger cat state. It means that the c.o.m. wave function of the ball is a balanced superposition of two distant wave packets. We postulate a quick universal (non-environmental) decoherence-collapse mechanism which destroys the superposition in a short time $\sim t_G$ (decoherence), also drops one of the wave packets and retains the other at random (collapse). This way the initial macroscopic (i.e.: large-distance) superposition has disappeared. The local superpositions (i.e.: internal to
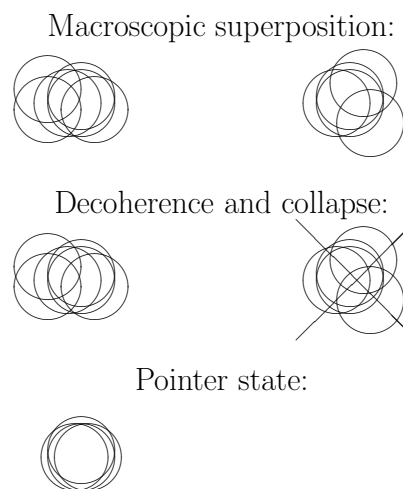
Figure 2. The Schrödinger cat state (upper part) decoheres and collapses to one of the two wave packets (middle part) and a localized pointer state is formed ultimately (lower part).

one wave packet) survive and, according to a further assumption, become uniform later when the wave packet takes a steady shape. We call this ultimate state *pointer state*. This qualitative picture of the process two-packets—one-packet—one-pointer-state, see Fig. 2, if it were real, would relax the controversy between quantum mechanics and macroscopic physics. Anyway, it deserves the concrete study: are there simple equations at all, to model the postulated scenario?

Due to the extreme simplification of the system, we can use transparent equations. All equations will contain the same Newtonian structure. Formally, this structure is just the Newtonian self-interaction potential of two hypothetical interpenetrating copies of our rigid ball centered at $\mathbf{x}$ and $\mathbf{x}'$, respectively:

$$U(\mathbf{x} - \mathbf{x}') = -G \int \frac{f(\mathbf{r}|\mathbf{x}) f(\mathbf{r}'|\mathbf{x}')}{|\mathbf{r}' - \mathbf{r}|} d\mathbf{r} d\mathbf{r}' \,, \tag{1}$$

where G is the Newton constant, $f(\mathbf{r}|\mathbf{x}) = (3M/4\pi R^3)\theta(R - |\mathbf{r} - \mathbf{x}|)$ is the mass density at $\mathbf{r}$ of the homogeneous rigid ball of mass M and radius R, θ stands for the step function. This

potential takes the standard elementary form for large separations and it becomes a quadratic potential for small separations [6]:

$$U(\mathbf{x} - \mathbf{x}') \sim \begin{cases} -GM^2/|\mathbf{x} - \mathbf{x}'| & \text{for } |\mathbf{x} - \mathbf{x}'| \gg R \\ U(0) + \frac{1}{2}M\omega_G^2|\mathbf{x} - \mathbf{x}'|^2 & \text{for } |\mathbf{x} - \mathbf{x}'| \ll R \end{cases} \tag{2}$$

where $\omega_G^2 = GM/R^3$ is the frequency of the corresponding hypothetic oscillator. This $U(\mathbf{x} - \mathbf{x}')$ will be utilized in various equations below, namely, in the equation of the c.o.m. decoherence, in the equation of the pointer states, and in the equation of both.

3. The equation of c.o.m. decoherence time

Before we suggest any differential equation for the process of decoherence, we restrict ourselves to the equation of the decoherence time t_G which is needed to collapse the macroscopic superposition, cf. the step from the upper stage to the middle one on Fig. 2. The suggested general expression reads [7, 8, 9, 10]:

$$t_G =: \frac{\hbar}{U(\mathbf{x} - \mathbf{x}') - U(0)} \ . \tag{3}$$

For distant superposition, like the Schrödinger cat state on Fig. 2, we get:

$$t_G \sim -\hbar/U(0) \sim \hbar R/GM^2 \ . \tag{4}$$

For atomic masses, t_G is extremely long and the corresponding decoherence effect is irrelevant. For nano-objects, t_G is shorter and the postulated universal decoherence may compete with the inevitable environmental decoherence. This would be the regime of experimental interest. For large macro-objects, t_G is unrealistically short, i.e., they would never form a Schrödinger cat state — which is just the consequence we desire.

Hence, the Eq. (3) yields a plausible scale of spontaneous decoherence. There is a serious problem, however. For a point-like massive ball ($R = 0$) as well as for any object containing point-like massive constituents $U(0)$ is $-\infty$ therefore t_G would be zero! This can not be physical, we can not suppose infinite fast decoherence in any case. Therefore, the Eq. (3) can not be universally valid unless, e.g., we introduce a cutoff to resolve the issue that the decoherence speed $1/t_G$ diverges for point-like objects.

Let us anticipate that the simple form (3) of the decoherence-collapse time induces a natural form for the differential equation of the detailed stochastic process. This equation will be discussed in Sec. 6.

4. Pointer states: SNE?

For long enough time we expect that the c.o.m. quantum state of our ball becomes a so-called pointer state which should be localized wave packet of constant shape. Let us consider the so-called Schrödinger—Newton equation (SNE) [6, 11] for the c.o.m. wave function $\psi(\mathbf{x})$:

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} - \frac{i}{\hbar} \int U(\mathbf{x} - \mathbf{x}')|\psi(\mathbf{x}')|^2 d\mathbf{x}' \ \psi(\mathbf{x}) \ . \tag{5}$$

The ground state solution is a standing soliton of width $\Delta\mathbf{x}_G$. Galilean translations and boosts yield an overcomplete set of states and we identify them as the set of pointer states.

For atomic particles, $\Delta\mathbf{x}_G$ is extremely large, the pointer states are practically delocalized. For nano-objects, the localization becomes significant. For a rigid ball of common density the

approximation $\Delta \mathbf{x}_G \ll R$ is valid if $R \gg 10^{-5}$cm and $M \gg 10^{-15}$g. Then $U(\mathbf{x} - \mathbf{x}') \approx U(0) + \frac{1}{2} M \omega_G^2 |\mathbf{x} - \mathbf{x}'|^2$ and the SNE reduces to:

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} - \frac{i}{2\hbar} M \omega_G^2 |\mathbf{x} - \langle \mathbf{x} \rangle|^2 \psi(\mathbf{x}) \ . \tag{6}$$

If there is no external potential, the exact ground state solution is easy because in the special framework $\langle \mathbf{x} \rangle = 0$ the equation describes a harmonic oscillator of mass M and frequency ω_G. We can just read out the ground state wave function of the oscillator from a textbook:

$$\psi(\mathbf{x}) = \mathcal{N} \exp \left(-\frac{\mathbf{x}^2}{4 \Delta x_G^2} \right) \ , \qquad \Delta \mathbf{x}_G = \sqrt{\frac{\hbar}{M \omega_G}} = \left(\frac{\hbar^2}{GM^3} \right)^{1/4} R^{3/4} \ . \tag{7}$$

The SNE (5) is a non-linear and reversible. Contrary to the decoherence-collapse speed $1/t_G$, the SNE remains regular for $R = 0$, i.e., for point-like objects or constituents as well. The SNE is utilized to generate the set of pointer states while it is not clear how the wave function would reduce to these pointer states and whether the SNE has any role in the *process* of reduction.

5. Pointer states: frictional SNE!

The SNE is not the only plausible choice to generate the pointer states. We are going to consider an alternative, the frictional Schrödinger—Newton equation (frSNE):

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} - \frac{1}{\hbar} \int U(\mathbf{x} - \mathbf{x}') |\psi(\mathbf{x}')|^2 d\mathbf{x}' \ \psi(\mathbf{x}) + \frac{1}{\hbar} U_G \psi(\mathbf{x}) \ , \tag{8}$$

where $U_G = \iint U(\mathbf{x}'' - \mathbf{x}') |\psi(\mathbf{x}') \psi(\mathbf{x}'')|^2 d\mathbf{x}' d\mathbf{x}''$. This equation differs from the SNE by the imaginary coupling $-iG$ of the non-linear term instead of the real coupling G. A further technical difference consists of the trivial new term with U_G, that assures the normalization of the state.

We expect that the ground state solution is a standing soliton of width $\Delta \mathbf{x}_G$ of the same order like for the SNE, and Galilean translations-boosts yield the overcomplete set of pointer states.

For atomic particles, similarly to the SNE, $\Delta \mathbf{x}_G$ is extremely large and can not represent any significant localization. For nano-objects the localization becomes significant. For a rigid ball of common density the approximation $\Delta \mathbf{x}_G \ll R$ applies if $R \gg 10^{-5}$cm and $M \gg 10^{-15}$g. Then $U(\mathbf{x} - \mathbf{x}') \approx U(0) + \frac{1}{2} M \omega_G^2 |\mathbf{x} - \mathbf{x}'|^2$ and the frSNE reduces to:

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} - \frac{1}{2\hbar} M \omega_G^2 |\mathbf{x} - \langle \mathbf{x} \rangle|^2 \psi(\mathbf{x}) + \frac{1}{2\hbar} M \omega_G^2 \langle (\Delta \mathbf{x})^2 \rangle \psi(\mathbf{x}) \ , \tag{9}$$

where $\langle (\Delta \mathbf{x})^2 \rangle = \langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2$. If there is no external potential, the exact ground state solution is simple:

$$\psi(\mathbf{x}) = \mathcal{N} \exp \left(-\sqrt{-i} \frac{\mathbf{x}^2}{4 \Delta x_G^2} \right) , \qquad \sqrt{-i} = \frac{1 - i}{\sqrt{2}} \ . \tag{10}$$

As we see, the ground state is identical with the ground state (7) of the SNE provided we replace G by $-iG$.

The non-linear frSNE (5) is, contrary to the SNE, not reversible. The frSNE keeps to be regular for $R = 0$. When we generated the pointer states by the SNE in Sec. 4 it was not clear "how the wave function would reduce to these pointer states and whether the SNE has any role in the *process* of reduction". The frSNE is on the positive side now: its role in the process of reduction is clear. At least there is a natural choice for the stochastic dynamics, see Sec. 6.

6. Matching decoherence-collapse with pointer states

As we mentioned in Sec. 3, there is a straightforward mathematical model [8] of the decoherence-collapse process that emerges directly from the decoherence-collapse speed $1/t_G$ defined by (3). The central structure of the model is the standard von Neumann equation of motion for the c.o.m. density matrix $\rho(\mathbf{x}, \mathbf{x}')$, completed by a non-Hamiltonian decoherence term $-t_G^{-1}\rho(\mathbf{x}, \mathbf{x}')$ [7]:

$$\frac{d\rho(\mathbf{x}, \mathbf{x}')}{dt} = \text{standard q.m. terms} - \frac{1}{\hbar}[U(\mathbf{x} - \mathbf{x}') - U(0)]\rho(\mathbf{x}, \mathbf{x}') \ . \tag{11}$$

Obviously, the new term tends to destroy the interference terms at a time scale t_G — which was the effect to model. This von Neumann—Newton equation (vNNE), see originally in [4], describes decoherence but it does not yet describe collapse and the subsequent reduction to a pointer state either. Nonetheless, there is a natural stochastic modification (also called unraveling) of the vNNE which can do it:

$$\frac{d\rho(\mathbf{x}, \mathbf{x}')}{dt} = \text{standard q.m. terms} \quad - \quad \frac{1}{\hbar}[U(\mathbf{x} - \mathbf{x}') - U(0)]\rho(\mathbf{x}, \mathbf{x}') \tag{12}$$
$$+ \quad \frac{1}{\hbar}[W_t(\mathbf{x}) + W_t(\mathbf{x}') - 2\langle W_t \rangle]\rho(\mathbf{x}, \mathbf{x}') \ ,$$

where $\langle W_t \rangle = \int W_t(\mathbf{x})\rho(\mathbf{x}, \mathbf{x})d\mathbf{x}$ and W_t itself is a white-noise field with spatial correlation: $M[W_t(\mathbf{x})W_{t'}(\mathbf{x}')] = -\hbar U(\mathbf{x} - \mathbf{x}')\delta(t - t')$.

After long time, this stochastic differential equation drives any initial state $\rho(\mathbf{x}, \mathbf{x}')$ into a localized pure state — the pointer state — while the stochastic differential equation reduces to the following stochastic wave equation:

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} \quad - \quad \frac{1}{\hbar}\int U(\mathbf{x} - \mathbf{x}')|\psi(\mathbf{x}')|^2 d\mathbf{x}' \ \psi(\mathbf{x}) + \frac{1}{2\hbar}[U_G + U(0)]\psi(\mathbf{x})$$
$$+ \quad \frac{1}{\hbar}[W_t(\mathbf{x}) - \langle W_t \rangle]\psi(\mathbf{x}) \ . \tag{13}$$

We can recognize the frSNE plus a stochastic term! (The change $U_G \to [U_G + U(0)]/2$ of the trivial renormalizing term compensates the contribution of the singular stochastic term.) We are not able to solve this equation in the general case. We can do it in the limit where the c.o.m. is already well localized with respect to the size R of the ball. In this limit the equation reads:

$$\frac{d\psi(\mathbf{x})}{dt} = \text{standard q.m. terms} - \frac{1}{2\hbar}M\omega_G^2|\mathbf{x} - \langle\mathbf{x}\rangle|^2\psi(\mathbf{x}) + w_t\sqrt{\frac{M}{\hbar}}\omega_G(\mathbf{x} - \langle\mathbf{x}\rangle)\psi(\mathbf{x}) \ , \tag{14}$$

where w_t is standard white-noise of correlation $M[w_t w_{t'}] = \delta(t - t')$. If there is no external potential, the exact steady shape solutions of this equation are the pointer state solutions of the frSNE and the effect of the stochastic term only contributes to the random walk of the steady wave packet.

7. Context

My work advocates the frictional alternative of the SNE instead of the SNE itself. The latter has been studied in a number of works including physics and mathematical ones [12, 13, 14, 15, 16, 17, 18, 19, 20]. The frictional alternative attracted less attention although the special case (9) with the quadratic imaginary potential had been studied [21] and was already utilized to generate pointer states longtime ago [22]. That time, however, it was not quite clear to me why just frictional equations were better pointer state generators than others, and only after many years could Kiefer and I analyze the complexity of the issue [23]. Future investigation of the mathematical properties of the frSNE is of interest.

Finally, I would like to comment on the historic context. Penrose and myself got the same form (3) of the decoherence-collapse time t_G. (Former discrepancies have been clarified in [24].) I derived this equation from the vNNE (11) and for quite a long time I did not see why Penrose did never buy the straightforward construction of the vNNE from t_G. Later I understood his reservation. As mentioned in Sec. 3, too, the form (3) yields a divergent $1/t_G$ for $R \to 0$ and this divergence is inherited by the vNNE. This is indeed a serious problem and we do not exactly know what to do with it, see [4, 24] and references therein. For point-like particles the decoherence speed $1/t_G$ would become infinite and the width of the wave function would tend to zero. But we do not expect the process of reduction to end up with singular wave functions rather we expect it ends up with localized "pointer states" of a certain finite width. Penrose defines the set of pointer states, independently of the equation (3) of t_G, as the ground state solitons of the SNE. So we have decent pointer states even for point-like structures and we may get some hint how to cure the divergence of $1/t_G$. I suggest warily: the hint might come from the stochastic wave equation (13) which unifies the complete scenario of Fig. 2 in one equation. Section 6 invokes Ref. [8] to claim the stochastic wave equation follows naturally from the Eq. (3) of t_G. The deterministic part of (13) is regular for $R = 0$, only its stochastic part is divergent. Shall gravity make a loophole somehow? We don't yet know. It is, nonetheless, remarkable that the deterministic part is the frictional Schrödinger—Newton equation. The frSNE has thus been in business already, and also independent arguments [23] favor this class of *irreversible* modifications of the Schrödinger equation to generate robust pointer states, see also [25, 5].

References

[1] Penrose R 1986 *Quantum Concepts in Space and Time*, eds Penrose R and Isham C J (London: Clarendon)

[2] Gell-Mann M and Hartle J B 1990 *Complexity, Entropy and the Physics of Information*, ed Zurek W H (Reading, MA: Addison-Wesley)

[3] Diósi L 1992 *Quantum Chaos - Quantum Measurement*, eds Cvitanovic P, Percival I and Wirzba A (Dordrecht: Kluwer)

[4] Diósi L 2007 *J. Phys.* A **40** 2989

[5] Adler S L 2004 *Quantum mechanics as an emergent phenomenon* (Cambridge: Cambridge University Press)

[6] Diósi L 1984 *Phys. Lett.* A **105** 199

[7] Diósi L 1987 *Phys. Lett.* A **120** 377

[8] Diósi L 1989 *Phys. Rev.* A **40** 1165

[9] Penrose R 1994 *Shadows of the Mind* (Oxford: Oxford University Press)

[10] Penrose R 1996 *Gen. Rel. Grav.* **28** 581

[11] Penrose R 1998 *Phil. Trans. R. Soc. Lond.* A **356** 1927

[12] Moroz I M, Penrose R and Tod P 1998 *Class. Quant. Grav.* **15** 2733

[13] Bernstein D A, Giladi E and Jones K R W 1998 *Mod. Phys. Lett.* A **13** 2327

[14] Kumar D and Soni V 2000 *Phys. Lett.* A **271** 157

[15] Tod K P 2001 *Phys. Lett.* A **280** 173

[16] Soni V 2002 *Pramana - J. Phys.* **59** 375

[17] Geszti T 2004 *Phys. Rev.* A **69** 032110

[18] Salzman P J and Carlip S 2007 *Preprint* quant-ph/0606120

[19] Greiner D and Wunner G 2006 *Phys. Rev.* A **74** 052106

[20] Adler S L 2007 *J. Phys.* A **40** 755

[21] Hasse R W 1979 *Phys. Lett.* B **85** 197

[22] Diósi L 1987 *Phys. Lett.* A **122** 221

[23] Diósi L and Kiefer C 2000 *Phys. Rev. Lett.* **85** 3552

[24] Diósi L 2005 *Braz. J. Phys.* **35** 260

[25] Gisin N 1984 *Phys. Rev. Lett.* **52** 1657;
Halliwell J and Zoupas A 1995 *Phys. Rev.* D **52** 7294 (1995); *do.* 1997 *Phys. Rev.* D **55** 4697;
Percival I 1998 *Quantum State Diffusion* (Cambridge: Cambridge University Press);
Bassi A and Ghirardi G C 2003 *Phys. Rep.* **379** 257