The gravity-related decoherence master equation from hybrid dynamics

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Abstract. Canonical coupling between classical and quantum systems cannot result in reversible equations, rather it leads to irreversible master equations. Coupling of quantized non-relativistic matter to gravity is illustrated by a simplistic example. The heuristic derivation yields the theory of gravity-related decoherence proposed longtime ago by Penrose and the author.

1. Introduction

There is at least one classical dynamic system whose quantization is still problematic, that is gravity. Experimental evidences up to now have not indicated that gravitation should be quantized. Methods of quantizing the classical equations of gravity have failed to become generally accepted. So, gravitation could happen to be classical. Classical dynamics of gravitation would couple to quantum dynamics of other fields. This mathematical problem is not at all trivial.

Hybrid quantum-classical systems are important not only for gravity but for molecular and nuclear systems as well where we might need to go beyond the mean-field approximation. Hybrid dynamics has obtained certain theoretical importance in foundations, in cosmology, in measurement problem. A very incomplete list of related works [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] shows the diversity of motivations.

The particular goal of the present work is to re-derive a former heuristic gravity-related decoherence model. According to it, the Newton gravitational field ϕ possesses a universal uncertainty $\delta\phi$ which is modelled by a zero-mean white-noise with spatial correlation [13, 14]

$$\langle\delta\phi(r,t')\delta\phi(s,t)\rangle_{noise} = \frac{G\hbar}{|r-s|}\delta(t'-t)\ , \tag{1}$$

where G is the Newton constant. This noise causes a certain universal decoherence between 'macroscopically' different mass distributions. The following quantum master equation can be derived [14, 15] for the density matrix $\hat\rho_Q$ of the quantized system:

$$\frac{d}{dt}\hat\rho_Q = -\frac{i}{\hbar}[\hat H_Q + \hat H_G, \hat\rho_Q] - \frac{1}{2}\int_{r,s}\frac{G/\hbar}{|r-s|}[\hat f(r),[\hat f(s),\hat\rho_Q]\ , \tag{2}$$

where $\hat{H}_Q$ is the Hamiltonian, $\hat{H}_G$ is the standard Newton pair-potential, $\hat{f}(r)$ is the operator of spatial mass density. (We have omitted the integral volume elements dr, ds and we keep this shorthand notation throughout our work.) The decoherence mechanism encoded in Eqs. (1,2) coincides (upto a missing factor 1/2) with Penrose's proposal [16, 17]

$$\frac{1}{4}\int_{r,s}\frac{G/\hbar}{|r-s|}[f(r)-f'(r)][f(s)-f'(s)] \tag{3}$$

for the rate of decoherence between two different mass distributions f and f'.

Sec. 2 introduces the mathematical model of hybrid dynamics which will be applied to the matter-gravity quantum-classical hybrid system in Sec. 3 including a re-derivation of Eqs. (1,2).

2. Hybrid dynamics

We are going to discuss a possible mathematical model for hybrid composite systems consisting of one quantum and one classical subsystem.

2.1. Hybrid state, expectation values

The generic state of a hybrid system is the hybrid density

$$\hat{\rho} \equiv \hat{\rho}(q,p) \tag{4}$$

which is a non-negative matrix in the Hilbert space of the quantum subsystem, depending on the phase space point (q,p) of the classical subsystem. Its quantum marginal

$$\hat{\rho}_Q \equiv \int \hat{\rho}(q,p)dqdp \tag{5}$$

is the usual normalized density matrix of the quantum subsystem, while its classical marginal

$$\rho_C(q,p) \equiv \mathrm{tr}\hat{\rho}(q,p) \tag{6}$$

is the usual normalized Liouville density of the classical subsystem. (We mention the notion of conditional state $\hat{\rho}_Q|_{qp} \equiv \hat{\rho}(q,p)/\rho_C(q,p)$ of the quantum subsystem; conditional state of the classical subsystem wouldn't make sense in general.)

The interpretation of the hybrid state follows from the interpretation of density matrices and Liouville densities, and goes like this. Consider an arbitrary phase-point-dependent Hermitian matrix $\hat{A}(q,p)$, i.e.: a hybrid observable. Its expectation value in the hybrid state $\hat{\rho}$ is defined by

$$\langle\hat{A}(q,p)\rangle_{\hat{\rho}} \equiv \mathrm{tr}\int \hat{A}(q,p)\hat{\rho}(q,p)dqdp\;. \tag{7}$$

The expectation value has the usual statistical interpretation: if we measure the value of the hybrid observable $\hat{A}(q,p)$ repeatedly on a large ensemble of identically prepared hybrid systems characterized by the hybrid state $\hat{\rho}$, the average value of the measured outcomes will tend to the above calculated expectation value.

Once we have specified the general features of hybrid states, we can go on and learn about their dynamics.

2.2. Dirac, Poisson, Aleksandrov brackets

Let us begin with two independent systems, one is quantum and the other is classical. Our quantum system's state is described by the density matrix $\hat{\rho}_Q$. The evolution is governed by the von Neumann equation with Hamiltonian $\hat{H}_Q$:

$$\frac{d}{dt}\hat{\rho}_Q = -\frac{i}{\hbar}[\hat{H}_Q, \hat{\rho}_Q] \equiv -\frac{i}{\hbar}\left(\hat{H}_Q\hat{\rho}_Q - \hat{\rho}_Q\hat{H}_Q\right) . \tag{8}$$

For our classical system, we have chosen the classical canonical Liouville theory because of its best formal match with the quantum theory. The state of the classical system is described by the normalized phase space density $\rho_C = \rho_C(q,p)$. The evolution is governed by the Liouville equation with Hamilton function $H_C = H_C(q,p)$:

$$\frac{d}{dt}\rho_C = \{H_C, \rho_C\}_P \equiv \sum_n \left(\frac{\partial H_C}{\partial q_n}\frac{\partial \rho_C}{\partial p_n} - \frac{\partial \rho_C}{\partial q_n}\frac{\partial H_C}{\partial p_n}\right) , \tag{9}$$

where $q = (q_1, q_2, \dots, q_n)$ and $p = (p_1, p_2, \dots, p_n)$ stand for the canonical pairs of coordinates and momenta. The classical structure $\{\ ,\ \}_P$ is called Poisson bracket, the quantum structure $-(i/\hbar)[\ ,\ ]$ is called Dirac bracket.

Now we can take the above quantum and classical systems to construct a single composite hybrid system. We can easily form the hybrid composite state:

$$\hat{\rho} = \hat{\rho}(q,p) \equiv \hat{\rho}_Q\rho_C(q,p) , \tag{10}$$

as well as the hybrid Hamiltonian:

$$\hat{H} = \hat{H}(q,p) \equiv \hat{H}_Q + H_C(q,p) . \tag{11}$$

We write the hybrid evolution equation in the form

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \{\hat{H}, \hat{\rho}\}_P , \tag{12}$$

which is equivalent with the previous evolution equations (8) and (9) for $\hat{\rho}_Q$ and ρ_C separately.

Of course, we would like to introduce interaction between the quantum and classical subsystems, i.e., we introduce the hybrid interaction Hamiltonian $\hat{H}_{QC} = \hat{H}_{QC}(q,p)$ which is a certain Hermitian matrix that also depends on the canonical coordinates of the classical subsystem. Now our total hybrid Hamiltonian reads

$$\hat{H} = \hat{H}(q,p) \equiv \hat{H}_Q + H_C(q,p) + \hat{H}_{QC}(q,p) , \tag{13}$$

and Aleksandrov [1] proposed the following evolution equation:

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathrm{Herm}\{\hat{H}, \hat{\rho}\}_P . \tag{14}$$

The structure on the r.h.s. is called the Aleksandrov bracket, being the simplest generalization of the naive interaction-free structure (12).

The Aleksandrov equation (14) does not preserve the factorized form (10) of the hybrid state $\hat{\rho}(q,p)$, which is the legitimate consequence of the interaction. But a serious drawback arises. As pointed out by Boucher and Traschen [2], the positivity of $\hat{\rho}(q,p)$ may not be preserved, the Aleksandrov equation is fundamentally defective whereas it can be a useful approximation beyond the mean-field model.

2.3. Blurring Dirac+Poisson

However disappointing it is, the algebraic unification of classical and quantum dynamics runs into difficulties [2, 3, 4, 5]. Perhaps the deepest reason is that the notion of deterministic classical trajectories of one subsystem becomes lost under the influence of the other subsystem which is subject to quantum uncertainties. I proposed a possible remedy long time ago [3], another approach was shown together with Gisin and Strunz [6]; further mathematical structures of hybrid dynamics appear from time to time [7, 8, 9, 10, 11, 12]. A comparative analysis is missing. The present work follows the method of Ref. [3].

In the hybrid Hamiltonian (13), the interaction can always be written in the form

$$\hat{H}_{QC} = \sum_r \hat{J}_Q^r J_C^r \ , \tag{15}$$

where Ref. [3] called $\hat{J}_Q^r$ and J_C^r as 'currents' that build up the hybrid coupling. This time, with one eye on the forthcoming application in Sec. 3, we write the hybrid interaction this way:

$$\hat{H}_{QC} = \sum_r \hat{f}^r \phi^r \ . \tag{16}$$

Let us blur the interacting 'currents' by auxiliary classical noises δf^r and $\delta\phi^r$:

$$\hat{H}_{QC}^{noise} = \sum_r \left(\hat{f}^r + \delta f^r(t)\right)\left(\phi^r + \delta\phi^r(t)\right) \ . \tag{17}$$

Accordingly, we replace the hybrid Hamiltonian (13) by the noisy one:

$$\hat{H}^{noise} = \hat{H}_Q + H_C + \hat{H}_{QC}^{noise} \ , \tag{18}$$

and we replace the Aleksandrov equation (14) by its noise-averaged (blurred, coarse-grained) version:

$$\frac{d}{dt}\hat{\rho} = \left\langle -\frac{i}{\hbar}[\hat{H}^{noise}, \hat{\rho}] + \mathrm{Herm}\{\hat{H}^{noise}, \hat{\rho}\}_P \right\rangle_{noise} \ . \tag{19}$$

Suppose the noises δf and $\delta\phi$ are independent white-noises of zero mean, with the following auto-correlations:

$$\langle \delta f^r(t')\delta f^s(t)\rangle_{noise} \ = \ D_Q^{rs}\delta(t'-t) \ , \tag{20}$$

$$\langle \delta \phi^r(t')\delta \phi^s(t)\rangle_{noise} \ = \ D_C^{rs}\delta(t'-t) \ . \tag{21}$$

The two correlation matrices D_Q^{rs} and D_C^{rs} can be chosen in such a way that the noise-averaged Aleksandrov equation (19) guarantee the positivity of the coarse-grained hybrid density $\hat{\rho}$. The price to pay is the loss of reversibility: the mathematically correct hybrid equation will be a certain irreversible master (kinetic) equation.

2.4. Hybrid master equation

By hand, we added a noisy part (17) to the total hybrid Hamiltonian (13):

$$\hat{H}^{noise} = \hat{H} + \sum_r (\hat{f}^r \delta\phi^r + \phi^r \delta f^r) \ . \tag{22}$$

Compared to Eqs. (17,18), the term $\delta f \delta\phi$ has been omitted because it cancels from both the Dirac and Poisson brackets. In the evolution equation (19), the 1^{st} order contribution of the noisy part is vanishing on average. In 2^{nd} order, the term $\hat{f}\delta\phi$ adds the structure $-[\hat{f}, [\hat{f}, \hat{\rho}]]$ and the term $\phi\delta f$ adds the structure $\{\phi,\{\phi,\hat\rho\}_P\}_P$ to the r.h.s. of the naive Aleksandrov equation (14):

$$\frac{d}{dt}\hat\rho=-\frac{i}{\hbar}[\hat H,\hat\rho]+\mathrm{Herm}\{\hat H,\hat\rho\}_P-\frac{1}{2\hbar^2}\sum_{r,s}D_C^{rs}[\hat f^r,[\hat f^s,\hat\rho]]+\frac{1}{2}\sum_{r,s}D_Q^{rs}\{\phi^r,\{\phi^s,\hat\rho\}_P\}_P\ . \quad (23)$$

This hybrid master equation, i.e.: the noise-averaged Aleksandrov equation (19), preserves the positivity of $\hat\rho$ if the correlation matrices satisfy

$$D_QD_C\geq\frac{\hbar^2}{4}\ . \quad (24)$$

The proof was given in Ref. [3] (it has remained largely unnoticed).

The first irreversible term on the r.h.s. of the hybrid master equation (23) is a typical decoherence term dephasing the superpositions between different configurations of the 'currents' f. The kernel D_C plays the role of decoherence matrix. The second irreversible term concerns the classical subsystem, yielding diffusion in its phase space.

3. Gravity-related decoherence

Hybrid dynamics will be applied to the interaction of quantized massive particles with classical gravitational field.

3.1. Hybrid master equation for matter plus gravity

The Hamiltonian $\hat H_Q$ will stand for the quantized non-relativistic matter. We'll need its mass density field operator $\hat f$, which should be the non-relativistic limit of $\hat T_{00}/c^2$, the 00 component of the Einstein energy-momentum tensor divided by c^2 (c is the speed of light). For point-like particles of mass m and position operator $\hat x$, the mass density would be

$$\hat f(r)=\sum m\delta(r-\hat x)\ , \quad (25)$$

where summation extends for all massive particles. In fact, a certain short-length cutoff is always understood, for details of this issue see Ref. [18, 19, 20] and references therein.

As to the classical gravitational field, we start with a scalar relativistic model and we take the non-relativistic limit $c\to\infty$ afterwards. We define the Newton potential $\phi\equiv\frac{1}{2}c^2(g_{00}-1)$ where g_{00} is the 00 component of the Einstein metric tensor. The canonical coordinates q of Sec. 2 will be played by the Newton field $\phi(r)$, the conjugate canonical momenta p will be denoted by the field $\xi(r)$. The Hamilton function(al) can be chosen as

$$H_C(\phi,\xi)=\int_r\left(2\pi Gc^2\ \xi^2+\frac{|\nabla\phi|^2}{8\pi G}\right)\ . \quad (26)$$

To construct the hybrid interaction, we couple the quantized mass density $\hat f$ to the Newton field:

$$\hat H_{QC}(\phi)=\int_r\hat f(r)\phi(r)\ . \quad (27)$$

Note, in parentheses, that $\hat H_{QC}$ does not depend on the canonical momenta ξ. Looking back to notations in Sec. 2, we recognize the identity of this interaction with (16) provided we replace $\sum_r,\hat f^r,\phi^r$ by $\int_r,\hat f(r),\phi(r)$, respectively. Therefore the equations of Sec. 2 apply here conveniently. Let us form the total hybrid Hamiltonian (13):

$$\hat H(\phi,\xi)=\hat H_Q+H_C(\phi,\xi)+\hat H_{QC}(\phi)\ . \quad (28)$$

Write down the hybrid master equation (23) for the state $\hat\rho(\phi,\xi)$ of our hybrid system:

$$\frac{d}{dt}\hat\rho = -\frac{i}{\hbar}[\hat H,\hat\rho] + \mathrm{Herm}\{\hat H,\hat\rho\}_P \\ -\frac{1}{2\hbar^2}\int_{r,s} D_C(r,s)[\hat f(r),[\hat f(s),\hat\rho]] + \frac{1}{2}\int_{r,s} D_Q(r,s)\{\phi(r),\{\phi(s),\hat\rho\}_P\}_P \ . \tag{29}$$

For preserving the positivity of the state $\hat\rho$, the decoherence and diffusion kernels should satisfy the condition (24). Below, we propose a heuristic resolution of the remaining ambiguity as to the concrete form of D_Q and D_C.

3.2. The gravity-related decoherence matrix

In the Newtonian non-relativistic limit $c\to\infty$, the mean-field Poisson equation is valid. In our notations (7):

$$\Delta\left\langle\phi(r)\right\rangle_{\hat\rho} = 4\pi G\left\langle\hat f(r)\right\rangle_{\hat\rho} \ . \tag{30}$$

Let us recall the definitions (20) and (21) of the correlation matrices (kernels) D_Q and D_C, respectively. If we imposed the Poisson equation $\Delta\delta\phi = 4\pi G\delta f$ on the fluctuations $\delta f, \delta\phi$ in Eqs. (20,21), respectively, then the above two correlation kernels would become related by

$$\Delta\Delta' D_C(r,r') = (4\pi G)^2 D_Q(r,r') \ . \tag{31}$$

(We are aware of a conflict with the derivation of the hybrid master equation where we considered $\delta\phi$ and δf uncorrelated. This is conceptual problem, it remains harmless for our resulting hybrid master equation but it may become relevant for later developments.) From the inequality (24), the minimum blurring corresponds to $D_C D_Q = \hbar^2/4$. Hence the unique translation invariant solution reads

$$D_C(r,s) = \frac{G\hbar}{2}\frac{1}{|r-s|} \ . \tag{32}$$

This is the correlation (1) of universal gravitational fluctuations [13, 14] apart from the numeric factor 1/2.

3.3. Reduced quantum master equation

Let us invoke the expansion (9) of the Poisson bracket and calculate the full expansion of the r.h.s. of the hybrid master equation (29):

$$\frac{d}{dt}\hat\rho = -\frac{i}{\hbar}[\hat H_Q,\hat\rho] - \frac{i}{\hbar}\int_r \phi(r)[\hat f(r),\hat\rho] \\ -\int_r\Big(4\pi Gc^2\xi(r)\frac{\delta\hat\rho}{\delta\phi(r)} + \frac{1}{4\pi G}\Delta\phi(r)\frac{\delta\hat\rho}{\delta\xi(r)}\Big) + \mathrm{Herm}\int_r \hat f(r)\frac{\delta\hat\rho}{\delta\xi(r)} \\ -\frac{1}{2\hbar^2}\int_{r,s} D_C(r,s)[\hat f(r),[\hat f(s),\hat\rho]\ ] + \frac{1}{2}\int_{r,s} D_Q(r,s)\frac{\delta^2\hat\rho}{\delta\xi(r)\delta\xi(s)} \ . \tag{33}$$

We are going to derive the reduced dynamics of the quantized matter. Its density matrix is the marginal (5) of the hybrid state:

$$\hat\rho_Q = \int \hat\rho(\phi,\xi)\mathcal{D}\phi\mathcal{D}\xi \ . \tag{34}$$

Let us try to obtain a closed master equation for $\hat\rho_Q$. Integrate both sides of (33) by $\mathcal{D}\phi$ and $\mathcal{D}\xi$. Most terms on the r.h.s. cancel and we are left with

$$\frac{d}{dt}\hat\rho_Q = -\frac{i}{\hbar}[\hat H_Q,\hat\rho_Q] - \frac{i}{\hbar}\int_r\!\!\phi(r)[\hat f(r),\hat\rho(\phi)]\mathcal{D}\phi - \frac{1}{2\hbar^2}\!\int_{r,s}\!\! D_C(r,s)[\hat f(r),[\hat f(s),\hat\rho]] \ , \tag{35}$$

where $\hat{\rho}(\phi) = \int \hat{\rho}(\phi, \xi)\mathcal{D}\xi$. Suppose the post-mean-field Ansatz:

$$\int \Delta\phi(r)\hat{\rho}(\phi)\mathcal{D}\phi = 4\pi G \;\mathrm{Herm}\hat{f}(r)\hat{\rho}_Q \;, \tag{36}$$

which is stronger than the standard mean-field equation (30). From this Ansatz, we express the functional integral on the r.h.s. of (35):

$$\int \phi(r)\hat{\rho}(\phi)\mathcal{D}\phi = -G\mathrm{Herm}\int_s \frac{\hat{f}(s)}{|r-s|}\hat{\rho}_Q \;. \tag{37}$$

Subsequently, the complete term in question becomes Hamiltonian:

$$-\frac{i}{\hbar}\int_r \phi(r)[\hat{f}(r), \hat{\rho}(\phi)]\mathcal{D}\phi = -\frac{i}{\hbar}[\hat{H}_G, \hat{\rho}_Q] \;, \tag{38}$$

where $\hat{H}_G$ is the well-known Newtonian pair-potential:

$$\hat{H}_G = -\frac{G}{2}\int_{r,s}\frac{\hat{f}(r)\hat{f}(s)}{|r-s|} \;. \tag{39}$$

The Eq. (35) has thus led to the following closed master equation for the evolution of the quantized matter:

$$\frac{d}{dt}\hat{\rho}_Q = -\frac{i}{\hbar}[\hat{H}_Q + \hat{H}_G, \hat{\rho}_Q] - \frac{1}{2\hbar^2}\int_{r,s} D_C(r,s)[\hat{f}(r), [\hat{f}(s), \hat{\rho}_Q] \;, \tag{40}$$

which, invoking D_C from (32), is the master equation (2) [14, 15] apart from an additional numeric factor 1/2 (cf. also Ref. [21]) in front of the decoherence term .

4. Concluding remarks

We have constructed a possible consistent hybrid dynamics coupling classical gravity to quantized non-relativistic matter. We were able to re-derive the former heuristic equations of gravity-related universal decoherence.

We admit a certain elusiveness of our procedure. In order to borrow self-dynamics to the Newtonian field ϕ, we attributed the status of relativistic scalar field to ϕ, with the non-relativistic limit to be taken on the hybrid master equation. On the other hand, the proposed hybrid master equation assumes the Markovian approximation since it contains classical white-noise fields. Coexistence of relativistic and white-noise fields may be an issue in itself. Therefore our hybrid master equation is heuristic rather than exact, and it remains so until, e.g., we construct some explicit solutions at least. The non-relativistic limit $c \to \infty$ was treated formally, without check for its existence. This is why we keep talking about the post-mean-field Ansatz rather than the post-mean-field equation despite the heuristic proof presented in the Appendix.

In any case, the formal re-derivation of the gravity-related decoherence master equation from a certain closed set of equations is reassuring and indicates the possible relevance of hybrid dynamics in foundations [22, 23].

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Appendix

We ought to show that the post-mean-field Ansatz (36) is consistent with the non-relativistic limit $c \to \infty$ of the hybrid master equation (29). We begin with the heuristic derivation of the standard mean-field equation (30).

From the hybrid master equation (33) and the definition of expectation values (7), calculate the time-derivatives of the classical mean fields $\langle\phi\rangle_{\hat\rho}$ and $\langle\xi\rangle_{\hat\rho}$:

$$\frac{d}{dt}\langle\phi(r)\rangle_{\hat\rho} = 4\pi G c^2 \langle\xi(r)\rangle_{\hat\rho} \;, \tag{.1}$$

$$\frac{d}{dt}\langle\xi(r)\rangle_{\hat\rho} = -\frac{1}{4\pi G}\langle\Delta\phi(r) - 4\pi G\hat f(r)\rangle_{\hat\rho} \;. \tag{.2}$$

From the first equation wee see that $\langle\xi\rangle_{\hat\rho} \to 0$ in the non-relativistic limit. It also means that the l.h.s. of the second equation vanishes, hence we have:

$$\langle\Delta\phi(r) - 4\pi G\hat f(r)\rangle_{\hat\rho} = 0 \;. \tag{.3}$$

This is just the standard mean-field equation (30).

Let us upgrade it into the post-mean-field Ansatz (36). To this end, we calculate the time-derivative of the expectation value of the simple hybrid quantity $\hat A\phi$ where $\hat A$ itself is an arbitrary constant Hermitian matrix:

$$\begin{aligned}\frac{d}{dt}\langle\hat A\phi(r)\rangle_{\hat\rho} &= 4\pi G c^2\langle\hat A\xi(r)\rangle_{\hat\rho} + \frac{i}{\hbar}\Big\langle[\hat H_Q + \int_s \phi(s)\hat f(s), \hat A\phi(r)] - \frac{1}{2\hbar^2}\int_{r,s} D_C(u,s)[\hat f(u),[\hat f(s),\hat A\phi(r)]\Big\rangle_{\hat\rho}\\ &= 4\pi G c^2\langle\hat A\xi(r)\rangle_{\hat\rho} + \mathcal{O}(1) \;.\end{aligned} \tag{.4}$$

Here $\mathcal{O}(1)$ stands for terms that does not contain the diverging factor c^2. In the non-relativistic limit, the quantity $\langle\hat A\xi(r)\rangle_{\hat\rho}$ must vanish. Since $\hat A$ can be any Hermitian operator, the 'partial' expectation value, i.e.: the expression (7) without the trace and $\hat A$, does already vanish:

$$\int \xi(r)\hat\rho(\phi,\xi)\mathcal{D}\phi\mathcal{D}\xi = 0 \;. \tag{.5}$$

In general, the hybrid dynamics (33) yields

$$\frac{d}{dt}\langle\hat A\phi(s)F(\xi)\rangle_{\hat\rho} = 4\pi G c^2\langle\hat A\xi(r)F(\xi)\rangle_{\hat\rho} + \mathcal{O}(1) \;, \tag{.6}$$

where $F(\xi)$ can be a generic functional of ξ. Repeating the previous arguments, we conclude that the 'partial' expectation value of $\xi(r)F(\xi)$ must vanish for $c \to \infty$. Since $F[\xi]$ is an arbitrary functional, we come to the conclusion that, in the non-relativistic limit, $\xi(r)$ is identically zero. Now we derive the time-derivative of $\langle\hat A\xi(r)\rangle_{\hat\rho}$ from (33):

$$\begin{aligned}\frac{d}{dt}\langle\hat A\xi(r)\rangle_{\hat\rho} = &-\frac{1}{4\pi G}\langle\hat A\Delta\phi(r) - 4\pi G\,\mathrm{Herm}\hat A\hat f(r)\rangle_{\hat\rho}\\ &+\frac{i}{\hbar}\Big\langle[\hat H_Q + \int_s \phi(s)\hat f(s), \hat A\xi(r)] - \frac{1}{2}\int_{r,s} D_C(u,s)[\hat f(u),[\hat f(s),\hat A\xi(r)]\Big\rangle_{\hat\rho} \;.\end{aligned} \tag{.7}$$

As we said, in the limit $c \to \infty$ we can set ξ to zero identically for all time and position therefore the above equation reduces to:

$$\langle\hat A\Delta\phi(r) - 4\pi G\,\mathrm{Herm}\hat A\hat f(r)\rangle_{\hat\rho} = 0 \;. \tag{.8}$$

The l.h.s. vanishes for all $\hat A$, hence we can remove the trace and $\hat A$ from the hybrid expectation value, yielding:

$$\int \Big(\Delta\phi(r) - 4\pi G\,\mathrm{Herm}\hat f(r)\Big)\,\hat\rho(\phi,\xi)\mathcal{D}\phi\mathcal{D}\xi = 0 \;. \tag{.9}$$

This is exactly the post-mean-field Ansatz (36) if we invoke the definition (5) of $\hat\rho_Q$.

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