Nonlinear Schrödinger equation in foundations: summary of 4 catches

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Abstract. Fundamental modifications of the standard Schrödinger equation by additional nonlinear terms have been considered for various purposes over the recent decades. It came as a surprise when, inverting Abner Shimonyi's observation of "peaceful coexistence" between standard quantum mechanics and relativity, N. Gisin proved in 1990 that any (deterministic) nonlinear Schrödinger equation would allow for superluminal communication. This is by now the most spectacular and best known anomaly. We discuss further anomalies, simple but foundational, less spectacular but not less dramatic.

1. Introduction

We discuss major difficulties encountered by nonlinear modifications of the Schrödinger equation. Before we enter the subject, we make it clear that nonlinear Schrödinger equations and nonlinear effective modifications of quantum mechanics are widely used in physics as approximate means. Most typical is the mean-field approximation. It yields the nonlinear Hartree-Fock equation [1] so basic for many-electron systems, and it yields the semi-classical Einstein equation [2] which is indispensable in quantum cosmology. These are approximate (effective) nonlinear quantum mechanics. One can, on the contrary, consider nonlinear modification of quantum mechanics at the fundamental level. As early as in 1952, Jánossy proposed [3] that a simple state-dependent potential ensure localized stationary wave function of a free quantum particle (of mass M):

$$i\hbar\frac{d\Psi(x)}{dt} = -\frac{\hbar^2}{2M}\Psi''(x) + \frac{1}{2}\alpha^2(x-\langle x\rangle_\Psi)^2\Psi(x), \tag{1}$$

where

$$\langle x\rangle_\Psi = \int x|\Psi(x)|^2 dx \tag{2}$$

and α is a certain parameter. The new contractive 'mean-field' potential counters the unlimited dispersion predicted by the standard Schrödinger equation. The stationary states are localized Gaussians (solitons). Jánossy's equation was and remained forgotten until now [4]. A gravity-related version of his nonlinear term was suggested much later independently by the author and by Penrose [5, 6], both being unaware of Jánossy's equation yet seeking the same effect, namely, to ensure localization of quantum objects – massive ones this time. We choose this equation, called the Schrödinger–Newton equation (SNE), to be our testbed to discuss four typical catches otherwise valid for any nonlinear Schrödinger equations.

2. Peaceful coexistence

It has been known since the beginning of quantum theory that it contains a certain latent non-locality. For instance, the collapse of the single particle wave function, occurring in quantum measurement, is to happen instantaneously over whatever large distances covered by the pre-measurement wave function. For two distant particles, Einstein, Podolski and Rosen (EPR) in 1935 [7] constructed the sharpest example of apparent action-at-a-distance and John Bell in 1963 [8] pointed out a specific non-locality. It was also clear that neither the apparent action-at-a-distance nor the quantum non-locality could result in testable dynamical effects like dynamical action-at-a-distance or superluminal communication. The physicist and philosopher Abner Shimony formulated this paradoxical situation as the peaceful coexistence between quantum theory and special relativity [9].

The reason of why formal violations of locality will all cancel from the measurable outcomes of quantum mechanics lie in the linear structure of the mathematical representation of the state space, dynamics, and measurable outcomes themselves. If one adds non-linear terms to the Schrödinger equation the peaceful coexistence ends and the physical violation of special relativity takes over. Interestingly, Jánossy [3] mentions that his non-linear single particle Schrödinger equation suffers of superluminal physical influence. The most spectacular thesis belongs to Gisin [10]. Any nonlinear modification of the single particle Schrödinger equation,

$$i\hbar\frac{d\Psi(x)}{dt} = -\frac{\hbar^2}{2M}\Psi''(x) + V_\Psi(x)\Psi(x), \tag{3}$$

allows for superluminal communication between the EPR partners for whatever small state-dependent potential $V_\Psi(x)$ (apart from the constant potential, of course).

3. Schrödinger–Newton equation: our testbed

Consider the single-body SNE for the centre-of-mass free motion of a 'large' composite object of mass M:

$$i\hbar\frac{d\Psi(x)}{dt} = -\frac{\hbar^2}{2M}\Psi''(x) + M\Phi_\Psi\Psi(x), \tag{4}$$

where the state-dependent Newton potential reads

$$\Phi_\Psi(x) = -GM\int\frac{|\Psi(r)|^2}{|x-r|}d^3r. \tag{5}$$

Although this equation is the Newtonian limit of the approximate semiclassical Einstein equation, it was suggested that it might be fundamental [5, 6] and it is currently studied as such, both theoretically and experimentally [11].

Our forthcoming analysis needs the following peculiar features of the SNE. It possesses standing (static) soliton solutions $|\bigcirc\rangle$ of diameter

$$D \sim \frac{\hbar^2}{GM^3}, \tag{6}$$

c.f., e.g., [5]. We can construct the Schrödinger cat states formed by two separate static solitons:

$$\Psi_\pm = \frac{1}{\sqrt{2}}\left(|\bigcirc_L\rangle \pm |\bigcirc_R\rangle\right). \tag{7}$$

The left and right solitons $\bigcirc_{L/R}$ are separated by a distance $\ell \gg D$ initially. Because of the mean-field $\Phi_\Psi(x)$ in the SNE, the two solitons in $\Psi_\pm$ attract each other and start to move, like this:

$$\bigcirc\!\!\!\leftarrow\!-\ell-\!\!\rightarrow\!\!\!\bigcirc \quad\Longrightarrow\quad \bigcirc\bigcirc \quad\Longrightarrow\quad \bigcirc\!\!\!\!\bigcirc \quad\Longrightarrow\quad \bigcirc\bigcirc \quad\Longrightarrow\quad \bigcirc\!\!\!\leftarrow\!-\ell-\!\!\rightarrow\!\!\!\bigcirc \quad\Longrightarrow\quad \bigcirc\bigcirc \quad\Longrightarrow\quad \ldots$$

They are oscillating as if they were two (interpenetrating) gravitating bodies of mass $M/2$ each, the periodicity is $T = \pi\sqrt{\ell^3/2GM}$ known from the two-body problem.

The feature we should note is the following. Single soliton states $|\bigcirc_L\rangle$ and $|\bigcirc_R\rangle$ are static. The two-soliton Schrödinger cat states $\Psi_\pm$ oscillate. The initial overlap of $\Psi_\pm$ with $\bigcirc_{L/R}$ is $1/\sqrt{2}$, see (7). In standard linear quantum dynamics the overlap would be constant in time. It is not so here, the overlap will oscillate with amplitude $1/\sqrt{2}$. In fact, it is zero for most of the time. The SNE makes $\Psi_\pm(t)$ orthogonal to $|\bigcirc_{L/R}\rangle$ after a few times

$$\delta t \sim \frac{\hbar\ell}{GM^2} \ll T, \tag{8}$$

when the distance differs from ℓ by (a few times) more than the soliton size D. To confirm the guess, consider the relative acceleration $GM\ell^{-2}$ of the two solitons toward each other and calculate the time lapse until they get closer by a length $\sim D$.

4. Schrödinger–Newton equation: four catches

Based on the above features of Schrödinger cat states under SNE, we are going to present four interrelated foundational issues all originating exclusively from the nonlinearity. Our thought experiments adapt Gisin's two-qubit superluminal telegraph [10]. Suppose Alice and Bob are far away from each other. Alice owns a qubit and Bob owns a large mass M, in the following maximally entangled composite state:

$$|\uparrow_z\rangle \otimes |\bigcirc_L\rangle + |\downarrow_z\rangle \otimes |\bigcirc_R\rangle. \tag{9}$$

Three of the forthcoming testable controversies are based on this composite state, the fourth one uses the single particle dynamics in itself.

4.1. Action-at-a-distance from nothing

Alice measures either $\hat\sigma_z$ (case i) or $\hat\sigma_x$ (case ii), each with random outcomes ±1. In case i, Bob's state collapses into the static single soliton states $|\bigcirc_L\rangle$ or $|\bigcirc_R\rangle$, according to the outcomes ±1. Alternatively, in case ii, Bob's state collapses into the Schrödinger cat states $\Psi_\pm$, respectively. So far the story coincides with the standard EPR scenario of standard quantum mechanics. The salient effect of nonlinearity enters from now on. In case i, Bob's states $|\bigcirc_L\rangle$ or $|\bigcirc_R\rangle$ remain static. In case ii, his states $\Psi_\pm$ heavily oscillate. Using no physical interaction with it, Alice could cause a testable effect to the remote object of Bob.

4.2. Superluminal telegraph

This thought experiment is the continuation of the previous one. Bob tests whether his initial state is preserved (case i) or it is changing (case ii). Bob waits until a few times δt (8) and then he measures the projector

$$|\bigcirc_L\rangle\langle\bigcirc_L| + |\bigcirc_R\rangle\langle\bigcirc_R|. \tag{10}$$

The outcome is 1 in case i and 0 in case ii. This confirms that the action-at-a-distance (Sec. 4.1) is testable. Is it superluminal? The answer is immediate if we substitute eq. (8) into the condition $c\delta t < \ell$ of superluminality. Surprisingly, the distance ℓ cancels and we are left with $M > \sqrt{\hbar c/G}$. Accordingly, the action-at-a-distance (in other words: the communication from Alice to Bob) is always superluminal if the mass M is at least a few times larger than the Planck mass.

4.3. Unsuitability for mixed state

The initial setup is the same as before, Alice and Bob are far away from each other but they are not assumed to cooperate this time. They may even not know about each other yet they are supposed to inherit the entangled states (9). We assume that Alice does not measure her qubit at all or, if she measures anything, Bob cannot learn anything about the measurement and the outcome. Bob's local (reduced) state is anyway a mixed state

$$\rho = \frac{1}{2}\Big(|\bigcirc_L\rangle\langle\bigcirc_L| + |\bigcirc_R\rangle\langle\bigcirc_R|\Big). \tag{11}$$

The SNE does, as a matter of fact, not apply to mixed states but pure ones described by a wave function. It is clear now that Bob cannot calculate the dynamical evolution of his local system. This incapacity is general since under natural conditions local quantum systems are never in pure states since they are always entangled with the rest of the world or with their environment at least.

4.4. Breakdown of statistical interpretation

To understand this catch, we need no EPR situation but the single-body SNE in itself. Consider that the SNE evolves a given initial pure state density matrix ρ^i into a given final pure state density matrix ρ^f and note that the map

$$\rho^f = \mathcal{M}[\rho^i] \tag{12}$$

is nonlinear.

We are going to show that any nonlinear map makes the statistical interpretation of quantum theory impossible. The proof is elementary and quick [12]. Consider the weighted statistical mixing of two states ρ_1, ρ_2 with weights $\lambda_1 + \lambda_2 = 1$, the resulting state reads

$$\rho = \lambda_1\rho_1 + \lambda_2\rho_2. \tag{13}$$

In von Neumann standard theory, the order of mixing and dynamical evolution are interchangeable:

$$\mathcal{M}[\lambda_1\rho_1 + \lambda_2\rho_2] = \lambda_1\mathcal{M}[\rho_1] + \lambda_2\mathcal{M}[\rho_2]. \tag{14}$$

Now we recognize that this is the mathematical condition of $\mathcal{M}$'s linearity! From the structure of the proof we see that the interchangeability of mixing and dynamical map excludes any nonlinear Schrödinger equation not just the SNE. Without such interchangeability the statistical interpretation of quantum states totally collapses. Moreover, the principle of interchangeability of mixing and dynamics is not necessarily quantum, it is carved in marble in classical statistical physics.

5. Summary: catches and loopholes

We listed four foundational issues encountered by nonlinear modifications of the Schrödinger equation. Our example was the so-called Schrödinger–Newton equation but all four anomalies are equally valid for any deterministic nonlinear modification of the dynamics of any simple or composite quantum system. The best known and most spectacular anomaly is superluminality, a clear violation of special relativity. The derivation of superluminality exploits nonlinearity together with standard statistical interpretation of the wave function. Less known is that already the statistical interpretation is in ultimate conflict with any (deterministic) non-linearity of quantum dynamics, cf., e.g., [13, 14]. This anomaly is admittedly less spectacular than superluminality. It is, nonetheless, the deepest anomaly of nonlinear quantum mechanics.

Non-linear quantum mechanics, the SNE first of all, attracts growing attention in foundations. Awareness of the also foundational catches has recently motivated a stochastically re-linearized model [15] and theory [16]. The surviving and even growing interest in SNE as it is can be explained and perhaps justified despite the catches if we point out certain loopholes. In summary, nonlinear Schrödinger equations

- allow for
  - fake action-at-a-distance but it may be extremely weak to be detected
  - superluminal communication but it may be too hard to be realized
- do not allow for
  - local dynamics unless a local pure state is prepared
  - standard statistical interpretation hence a new interpretation is needed

The last catch is the major one. *Any* nonlinear quantum theory needs a radical new interpretation of the wave function because *any* (deterministic) nonlinear dynamics makes the Born interpretation contradict the principles of statistics.

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