

Planck length challenges non-relativistic quantum mechanics of large masses

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Abstract. With the simplest proof ever, we justify the significance of quantum-gravity in non-relativistic quantum mechanics together with the related theories and experiments. Since the de Broglie wave length is inverse proportional to the mass, it would descend towards and below the Planck scale 10^{-33} cm for large masses even at slow non-relativistic motion. The tricky relationship between gravity and quantum mechanics —well-known in the relativistic case— shows up in non-relativistic motion of massive objects. Hence the gravity-related modification of their Schrödinger equation is *mandatory*. We also recall the option of an autonomous Newtonian quantum-gravity, a theory parametrized by \hbar and G . On cancellation of c from the Newtonian limit of Planck scale metric fluctuations is given a new hint.

1. Introduction

Quantum-gravitational limitations of our very notion of space-time were conjectured by Bronstein in 1936 [1], the same issue was famously characterized by Wheeler’s foamy structure of space-time [2] In the back-ground of quantized matter, the classical notion of space-time continuum can not be maintained at short distances. In the vicinity of the Planck length

$$\ell_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-33} \text{ cm}, \quad (1)$$

the quantization of space-time would become unavoidable according to mainstream views. Efforts to construct a satisfactory theory of quantum-gravity have reached partial successes only. There is no consensual model of distances about ℓ_{Pl} [3, 4].

Direct observation of distances like 10^{-33} cm would require incredible high precision — beyond imaginations. Their indirect test would be possible around the extreme high Planck energy $E_{\text{Pl}} = 1.2 \times 10^{19}$ GeV per particle, which existed right after the Big Bang only. At low energies, Planck scale effects are too small to be testable. That’s the mainstream wisdom at least.

“... it certainly is not a viable option to simply wait until the Planck scale becomes accessible experimentally in order to have an indication on how to proceed theoretically, as this will require a long time” — says the recent review Ref. [5] arguing for low-energy gravity-related decoherence theories and experiments. There are many discussions of quantum-gravity related effects that might occur at non-relativistically low energies, cf., e.g. [4, 5, 6, 7, 8].

But, what is the reason that the extreme-relativistic Planck scale effects, suppressed by a factor 10^{-28} at low energies (a number used in [6]), may amplify (or accumulate) to the level of laboratory testability? Here we are going to show the trivial answer lies in non-relativistic



quantized motion of massive objects, and the argument is no doubt simpler than any earlier ones.

2. Non-relativistic quantized motion reaches Planck length

Let us consider an object of mass m and compare its de Broglie wave length λ_{dB} with the Planck length ℓ_{Pl} :

$$\underbrace{\lambda_{\text{deBroglie}}}_{\frac{2\pi\hbar}{mv}\sqrt{1-v^2/c^2}} \sim \underbrace{\ell_{\text{Planck}}}_{\sqrt{\frac{\hbar G}{c^3}}}$$

where v is the object's velocity. Now, λ_{dB} can sink to ℓ_{Pl} in two ways. First: velocity v approaches c . Obviously, the required energy is E_{Pl} , too high to occur apart from the Big Bang. Second: mass m grows macroscopic keeping $v \ll c$. That's the way we go.

2.1. Periods of the non-relativistic plane wave

Construct the non-relativistic time-dependent plane wave $\Psi(x, t)$, at velocity $v \ll c$, along the coordinate x :

$$\begin{aligned} \Psi(x, t) &= \exp(-iEt/\hbar + ipx/\hbar) \\ &= \exp\left(-i\frac{mv^2}{2\hbar}t + i\frac{mv}{\hbar}x\right) \\ &= \exp\left(-2\pi i\frac{t}{\tau} + 2\pi i\frac{x}{\lambda}\right), \end{aligned} \quad (2)$$

where E, p are the energy and momentum, respectively. The plane wave is periodic both in time and space, with the corresponding periods τ and $\lambda = \lambda_{\text{dB}}$:

$$\tau = (4\pi\hbar/mv^2), \quad (3a)$$

$$\lambda_{\text{dB}} = (2\pi\hbar/mv). \quad (3b)$$

These periods should be compared to the Planck time t_{Pl} and length ℓ_{Pl} :

$$t_{\text{Pl}} = \sqrt{\hbar G/c^5} \sim 10^{-43} \text{ s}, \quad (4a)$$

$$\ell_{\text{Pl}} = \sqrt{\hbar G/c^3} \sim 10^{-33} \text{ cm}. \quad (4b)$$

$\Psi(x, t)$ is legitimate non-relativistic wave function as long as its periods are much longer than the Planck time and length, respectively: $\tau \gg t_{\text{Pl}}$ and $\lambda_{\text{dB}} \gg \ell_{\text{Pl}}$, discussed similarly in [9]. That's the case for atomic masses m , where $\tau/t_{\text{Pl}} \sim 10^{18}$ and $\lambda_{\text{dB}}/\ell_{\text{Pl}} \sim 10^{18}$. But much larger masses will push the structure of $\Psi(x, t)$ towards the Planck scales, such role of the mass is what our work aims to emphasize.

2.2. How large should the mass be?

Masses M , requested for a Planck scale structure of $\Psi(x, t)$ non-relativistically, are much greater than the Planck mass $m_{\text{Pl}} = 2.18 \times 10^{-5} \text{ g}$. The time-period τ would sink to the Planck time t_{Pl} if the mass is

$$M|_{\tau=t_{\text{Pl}}} = \frac{c^2}{v^2} m_{\text{Pl}} \sim \frac{c^2}{v^2} 10^{-5} \text{ g}, \quad (5)$$

whereas the spatial period λ_{dB} drops down to ℓ_{Pl} provided

$$M|_{\lambda_{\text{dB}}=\ell_{\text{Pl}}} = \frac{c}{v} m_{\text{Pl}} \sim \frac{c}{v} 10^{-5} \text{ g}. \quad (6)$$

Remember that we keep $v \ll c$. Interestingly, the condition $\tau = t_{\text{Pl}}$ is equivalent with $Mv^2/2 = E_{\text{Pl}}/2$, i.e., the center-of-mass wave function temporal structure approaches the Planck

scale ℓ_{P1} provided the non-relativistic kinetic energy approaches, due to the large M , the Planck energy E_{P1} (see related observation in [9]). Btw, the incoherent preparation of macroscopic objects, unlike of elementary particles, of Planckian kinetic energy E_{P1} would not be an issue.

But, for the spatial structure, the bell rings earlier at much smaller mass M and kinetic energy $Mv^2/2 = (v/c)E_{\text{P1}}$. Through examples, we illustrate that the system can be perfectly non-relativistic, though apparently not available by state-of-the-art coherent preparation of massive objects. Our first example is mass $M = 10$ g at velocity $v = 10$ km/s, yielding $\lambda_{\text{dB}} = 4.2 \times 10^{-33}$ cm which is of the order of ℓ_{P1} ! Of course, the single plane wave is not necessarily illegitimate in itself since its Planck scale structure may be the artifact of the chosen frame of reference. So, we consider the superposition of two opposite plane waves of the mass $M = 10$ g:

$$|+10\text{km/s}\rangle + |-10\text{km/s}\rangle, \quad (7)$$

which is a Schrödinger cat in momentum. Its wave function has a Planck scale spatial structure in any reference frame hence it is not legitimate.

Our second example is a solid mass $M = 10$ kg, with an elastic vibrational mode of frequency $\omega = 100$ kHz. If the amplitude is $a = 0.01$ cm, then λ_{dB} oscillates between infinity and $\lambda_{\text{dB}} = (2\pi\hbar/m\omega) = 4.2 \times 10^{-33}$ cm. Again, standard Schrödinger equation becomes illegitimate for this massive (macroscopic) non-relativistic oscillator.

2.3. How can tiny Planckian effects accumulate for massive objects?

As we mentioned, in non-relativistic atomic quantum systems the Planck scale effects are suppressed by an incredible small factor guessed deliberately by 10^{-28} which is the ratio of typical atomic energies (eV's) to $E_{\text{P1}} \sim 10^{19}$ GeV [6]. We can illustrate how such small effects can accumulate in macroscopic massive systems. Our example is a minimum toy model of the uncertainty u of the classical spatial continuum at the Planckian distances. We suppose the replacement

$$x \Rightarrow x + u \quad (8)$$

where u is a global uncertainty of the coordinate x . It can be quantum or just classically random, we choose the latter option for its simplicity. Let u be a Gaussian random number of squared spread $\mathbb{E}u^2 = \ell_{\text{P1}}^2$. Consider the many-body wave function of a macroscopic object. Due to the random uncertainty u , it should be replaced randomly as

$$\Psi(x_1, x_2, \dots, x_{10^{23}}) \Rightarrow \Psi(x_1 + u, x_2 + u, \dots, x_{10^{23}} + u). \quad (9)$$

The shifts $u \sim \ell_{\text{P1}}$ are irrelevant for the non-relativistic individual constituents, but their effects accumulate for the center-of-mass. The mechanism of accumulation is best seen in momentum representation of Eq. (9):

$$\tilde{\Psi}(p_1, p_2, \dots, p_{10^{23}}) \Rightarrow \exp\left(\frac{i}{\hbar}u \underbrace{(p_1 + p_2 + \dots + p_{10^{23}})}_P\right) \tilde{\Psi}(p_1, p_2, \dots, p_{10^{23}}). \quad (10)$$

Observe the appearance of the phase u/\hbar times the total momentum P of the center-of-mass motion of the macroscopic object. P is the “large” number to compensate the the smallness of u . We can derive closed expression for the influence of the Planckian coordinate uncertainty u on the center-of-mass density matrix if we take the average over u :

$$\rho(P, P') = \mathbb{E} \exp\left(\frac{i}{\hbar}u(P - P')\right) \rho(P, P') = \exp\left(-\frac{\ell_{\text{P1}}^2}{2\hbar^2}(P - P')^2\right) \rho(P, P'). \quad (11)$$

In our toy model, the Planck scale coordinate uncertainty decoheres the center-of-mass momentum superpositions of massive objects. As to our Schrödinger cat (7), the interference term $\rho(mv, -mv)$ is suppressed by the factor

$$\exp\left(-\frac{\ell_{\text{Pl}}^2}{2\hbar^2}(2mv)^2\right) \approx e^{-1.2}. \quad (12)$$

The cat (7) becomes partially decohered. The atom-wise ignorable Planck scale uncertainty grew testable indeed non-relativistically in the massive degree of freedom. Yet a consistent theory (advancing our minimum toy model) of Planck scale uncertainties should either eliminate the illegitimate cat states or make them legitimate.

To improve the phenomenology of our minimum toy model, the unnatural globality and constancy of u should be replaced by some correlated structure of local space-time randomnesses. They result in much stronger decoherence effects whose significance extends for masses M much below m_{Pl} towards mesoscopic masses. Cat states (7) become eliminated at fast decoherence rates. Numerous different choices of space-time fluctuations have been summarized recently in Ref. [10], we discuss the particular DP-model below.

3. Newtonian quantum gravity: $\hbar + \mathbf{G}$, no \mathbf{c}

So far we argued how quantum-gravity, relativistic by its definition, should influence non-relativistic quantum mechanics. The extreme-relativistic quantum-gravity mechanism must have its footprints on non-relativistic quantum mechanics of massive objects. Now, this opens a remarkable perspective: what if velocity of light c cancels from the said footprints? Then there would be an autonomous theory of Newtonian quantum-gravity, coined so by [11]. And, apparently, this shall be the case. A first indication was the non-relativistic limit (cf. footprint) of quantum-gravity in the semi-classical approximation which led to the Newton-Schrödinger equation, containing G and \hbar only, and predicting a regime of significance for quantum dynamics of meso- and macroscopic masses [12, 13]. The other instance is the DP-model [13, 14] of gravity-related decoherence and wave function collapse in massive degrees of freedom, parametrized again by G and \hbar only. It is based on a conjectured ultimate uncertainty of space-time but neither the relationship to the Planck length ℓ_{Pl} (sketched in [15]) nor the cancellation of c are explained in concrete forms.

The question itself was explicitly asked in Ref. [18]: whether the space-time Newtonian unpredictabilities/fluctuations *à la* DP [14] are the non-relativistic limit of the Planckian's? A possible, affirmative, answer may come from Unruh's intuitive quantization of space-time [16], as mentioned in [14, 17, 18]. Below we show a direct affirmative derivation, still starting from a phenomenology of Planckian fluctuations. Let us represent the uncertainty around the Minkowski metric by introducing a perturbative conformal factor $1 + h$ (at $|h| \ll 1$) where the random fluctuations h are proportional to ℓ_{Pl} . We choose the formal expression of relativistic covariance:

$$\mathbb{E}h(x)h(y) = \text{const.} \times \frac{\ell_{\text{Pl}}^2}{(2\pi)^4} \int e^{-ik(x-y)} \frac{\theta(-k^2)}{-k^2} dk, \quad (13)$$

ignoring the issues of regularizing $\theta(-k^2)/k^2$. Now we write h into the form $h = 2\Phi/c^2$ anticipating that Φ plays the role of a (random) Newton potential for non-relativistic matter. In the limit $c \rightarrow \infty$, the good thing about the formal relativistic correlator (13) is that it leads to a convergent correlator for Φ :

$$\mathbb{E}\Phi(t, \mathbf{x})\Phi(s, \mathbf{y}) = \text{const.} \times \frac{\hbar G}{|\mathbf{x} - \mathbf{y}|} \delta(t - s). \quad (14)$$

The conform uncertainties, proportional to ℓ_{Pl} , have become equivalent non-relativistically with the (Newtonian) space-time uncertainties proposed for the DP-model [14]. The way c has gone is being demonstrated directly, tentatively though, for the first time.

4. Summary

We wished relaxing the belief, based mainly on quantum-field-theory considerations, that the significance of quantum-gravity is restricted for the regime of extreme high energies. We simply argue that the Planck length ℓ_{Pl} invokes a *mandatory* —gravity-related, yet to be specified— modification of non-relativistic quantum mechanics of massive (macroscopic) degrees of freedom because their de Broglie wave lengths can descend into the vicinity of ℓ_{Pl} non-relativistically. The modification depends on the yet also unknown behavior of space-time at the Planckian scale.

No matter whether space-time is classical or quantized, some ultimate statistical or quantum uncertainties (fluctuations) are attributed to the background space-time, to result in some particular decoherence of massive degrees of freedom. We explained the accumulation of the tiny extreme-relativistic effects in massive degrees of freedom. Such non-relativistic footprints of the Planck scale fluctuations do, in general, depend on G , \hbar , and c . However, the footprints according to the DP-model are fully non-relativistic, the light velocity c cancels from them. A mechanism of this cancellation has been outlined.

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