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GENERALIZED PATH INTEGRALS

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ABSTRACT

We have investigated the spectrum of the sum of two self-adjoint operators of continuous spectra. Having applied the Trotter theorem path integral expression has been obtained for the resolvent kernel function. The integrand functional can be associated with the action functional of a mechanical system. Applying our formulae for the Hamiltonian of quantum mechanics we recover the canonical path integral expression of Garrod. As a new result we show the possibility of simultaneous application of the Schrödinger equation and the path integrals in the two-potential scattering problem or in the perturbation problem of the Bloch-states for instance.

АННОТАЦИЯ

Изучается спектр суммы двух самосопряженных операторов с непрерывным спектром. Применив теорему Троттера мы получаем континуально-интегральное выражение ядерной функции резолвента. Интегрируемый функционал может быть связан с функционалом действия некоторой механической системы. Применяя наши формулы к Гамильтониану квантовой механики получается канонический интеграл по пути полученный уже Гарродом. Как новый результат демонстрируется возможность одновременного применения уравнения Шредингера и метода интегралов по пути для проблемы рассеяния в поле двух потенциалов или, например, в случае возмущения состояний Блоха.

KIVONAT

Két folytonos spektrumu önadjungált operátor összegének spektrumát vizsgáljuk. A Trotter tételt alkalmazva pályaintegrálokkal fejezzük ki a rezolvens magfüggvényt. Az integrálandó funkcionál kapcsolatba hozható egy mechanikai rendszer hatásfunkcionáljával. Módszereinket alkalmazva a kvantummechanika Hamilton-operátorára megkapjuk a Garrod-féle kanonikus pályaintegrálokat. Uj eredményként megmutatjuk a Schrödinger egyenlet és a pályaintegrálok együttes alkalmazhatóságát a két-potenciál szórási feladatban vagy például a Bloch-állapotok perturbációs problémájában.

I. INTRODUCTION

The practical problem of the quantum mechanics is the diagonalisation of the self-adjoint operator

$$\hat{H} = \frac{\hat{p}^2}{2} + V(\hat{x}) \quad . \qquad /I.1/$$

Here $\hat{p}$ and $\hat{x}$ are the self-adjoint operators of the canonical variables obeying the commutator relation

$$[\hat{x}, \hat{p}] = i \quad . \qquad /I.2/$$

The spectral expansion of the operator $\hat{H}$ can be explicitly obtained by means of the so-called canonical path integrals [1]. The approximate evaluation of these path integrals leads to the semiclassical approximation of the quantum mechanics through the canonical equations of motion of the classical counterpart of the system [2].

In this paper we shall investigate the spectrum of the operator sum $\hat{H} = \hat{A} + \hat{B}$ of more general form than /I.1/. However the operators $\hat{A}$ and $\hat{B}$ still have to be self-adjoint operators with continuous spectra.

We shall prove that the spectrum of $\hat{H}$ also can be analysed by generalized path integrals. The surprising result is that this operator $\hat{H}$ is also associated with a classical mechanical system.

In Sec. II we introduce the necessary notations. In Sec. III we derive the general path integral formulae for some basic kernel functions connected with the operator H. Sec. IV deals with the associated mechanical equations.

In Sec. V some examples will be presented from the quantum mechanics. We shall recover the result of Garrod [1]. The possibility of the simultaneous application of Schrödinger's wave function method and the path integral method will be shown.

Sec. VI contains some conclusions.

II. SOME NOTATIONS

Let $\hat{A}$ be a self-adjoint operator with continuous spectrum. Let us suppose that the eigenstates of $\hat{A}$ can be labelled by the points $a = (a_1, a_2, \dots, a_{n_A})$ of a real vector space of dimension n_A :

$$\hat{A}|a\rangle = f(a)|a\rangle \,, \tag{II.1}$$

where $f(a)$ is the real eigenvalue belonging to the eigenstate $|a\rangle$. The set of the eigenstates is complete:

$$\int |a\rangle\langle a|\, da = \hat{1} \,, \tag{II.2}$$

and orthonormal:

$$\langle a'|a\rangle = \delta(a'-a) \,, \tag{II.3}$$

where da and $\delta(a)$ stand for the n_A-dimensional volume element and Dirac delta function respectively.

Analogously the self-adjoint operator $\hat{B}$ has continuous spectrum with eigenstates $|b\rangle$ and real eigenvalues $g(b)$, where the label b is a point of a given n_B-dimensional real vector space:

$$\hat{B}|b\rangle = g(b)|b\rangle \,. \tag{II.4}$$

The eigenstates $|b\rangle$ also form a complete and orthonormal set:

$$\int |b\rangle\langle b|\, db = \hat{1} \,, \tag{II.5}$$

$$\langle b'|b\rangle = \delta(b'-b) \,. \tag{II.6}$$

From Eqs. /II.1-6/ the spectral expansion of $\hat{A}$ and $\hat{B}$ can be deduced [3]:

$$\hat{A} = \int f(a)|a\rangle\langle a|\, da \,, \tag{II.7}$$

$$\hat{B} = \int g(b)|b\rangle\langle b|\, db \,. \tag{II.8}$$

Generalizing the above formulae for a function ϕ /real or complex/ of the operator $\hat{A}$ and similarly for the function of $\hat{B}$, we can write:

$$\phi(\hat{A}) = \int \phi(f(a))|a\rangle\langle a|\, da \,, \tag{II.9}$$

$$\phi(\hat{B}) = \int \phi(g(b))|b\rangle\langle b|\, db \,. \tag{II.10}$$

Let us introduce the following self-adjoint operator $\hat{H}$:

$$\hat{H} = \hat{A} + \hat{B} \quad , \tag{II.11}$$

and also the real function

$$H(a,b) = f(a) + g(b) \ . \tag{II.12}$$

The relation between the operators $\hat{A}$ and $\hat{B}$ can be characterized by a unitary kernel U even when a simple commutation rule like Eq. /I.2/ can not be found for $\hat{A}$ and $\hat{B}$:

$$U(a,b) = \langle b | a \rangle \ . \tag{II.13}$$

For our further purposes it will be necessary to have U as an exponential expression. Therefore we introduce the phase function σ by the following definition:

$$U(a,b) = e^{i\sigma(a,b)} \ . \tag{II.14}$$

Finally, we have to define the weight function ρ :

$$\rho(a,b) = |U(a,b)|^2 = e^{-2\cdot \mathrm{Im}\,\sigma(a,b)} \ . \tag{II.15}$$

III. THE SPECTRUM OF $\hat{H} = \hat{A} + \hat{B}$, PATH INTEGRALS

Now we are going to investigate the usual kernel functions characterizing the spectrum of the operator $\hat{H}$.

Let us start with the determination of the so-called propagator function of the operator $\hat{H}$. We choose the following heterogeneous representation:

$$K(a,b,t) \equiv \langle a | e^{-it\hat{H}} | b \rangle \ . \tag{III.1}$$

Consider the following variant of the Trotter theorem [4], being valid for self-adjoint $\hat{A}$, $\hat{B}$ and $\hat{A}+\hat{B}$: +

$$e^{-it(\hat{A}+\hat{B})} = \lim_{n\to\infty} \left(e^{-\frac{it}{n}\hat{A}} \cdot e^{-\frac{it}{n}\hat{B}} \right)^n \ . \tag{III.2}$$

+ Using Trotter theorem Feynman's path integral formula was first derived by Nelson [5].

Applying the spectral representations formulated in Eqs. /II.9-10/ we have:

$$e^{-\frac{it}{n}\hat{A}} = \int e^{-\frac{it}{n}f(a)}|a\rangle\langle a|\,da\ , \qquad /III.3/$$

$$e^{-\frac{it}{n}\hat{B}} = \int e^{-\frac{it}{n}g(b)}|b\rangle\langle b|\,db\ . \qquad /III.4/$$

Consider the product of the right hand side terms of these two equations. Using Eqs. /II.13-14/ and the definition /II.12/ of the function H the following expression can be obtained:

$$e^{-\frac{it}{n}\hat{A}}\cdot e^{-\frac{it}{n}\hat{B}} = \int e^{-\frac{it}{n}H(a,b)-i\sigma^*(a,b)}|a\rangle\langle b|\,da\,db\ . \qquad /III.5/$$

The n'th power of the integral on the right hand side can be written as an n-fold integral, the scalar products like $\langle b|a\rangle$ and $\langle a|b\rangle$ can be put into an exponential form by means of Eqs. /II.13-14/. We substitute the result into the right hand side of the Eq. /III.2/:

$$e^{-it\hat{H}} = \lim_{n\to\infty}\int |a_n\rangle e^{-\frac{it}{n}\sum_{r=1}^{n}H(a_r,b_r)-i\sum_{r=1}^{n}\sigma^*(a_r,b_r)+i\sum_{r=1}^{n-1}\sigma(a_r,b_{r+1})}\langle b_1|\cdot\prod_{r=1}^{n}da_r\,db_r\ . \qquad /III.6/$$

Using Eqs. /II.14-15/ it can be proved that the following transformation is allowed:

$$e^{-it\hat{H}} = \lim_{n\to\infty}\int |a_n\rangle e^{-\frac{it}{n}\sum_{r=1}^{n}H(a_r,b_r)+i\sum_{r=1}^{n-1}\left(\sigma(a_r,b_{r+1})-\sigma(a_r,b_r)\right)-i\sigma(a_n,b_n)}\langle b_1|\cdot\prod_{r=1}^{n}\rho(a_r,b_r)\,da_r\,db_r\ . \qquad /III.7/$$

Owing to the Eqs. /III.1,7/ the propagator function can be expressed by the following integral formula:

$$K(a,b,t) = \lim_{n\to\infty}\int e^{-\frac{it}{n}\sum_{r=1}^{n}H(a_r,b_r)+i\sum_{r=1}^{n-1}\left(\sigma(a_r,b_{r+1})-\sigma(a_r,b_r)\right)-i\sigma(a_n,b_n)}\ \times$$
$$\times\,\delta(a-a_n)\,\delta(b-b_1)\prod_{r=1}^{n}\rho(a_r,b_r)\,da_r\,db_r\ . \qquad /III.8/$$

A simple symbolic notation can be obtained introducing a continuous parameter τ running from 0 to t, instead of the original discrete parameter r :

$$K(a,b,t) = \int e^{i\int_{\tau=0}^{t}\left(\frac{\partial\sigma(a(\tau),b(\tau))}{\partial b}\,db(\tau)-H(a(\tau),b(\tau))d\tau\right)-i\sigma(a(t),b(t))}\ \times$$
$$\times\,\delta(a-a(t))\,\delta(b-b(0))\cdot\prod_{\tau\in[0,t]}\rho(a(\tau),b(\tau))\,da(\tau)\,db(\tau)\ . \qquad /III.9/$$

Here we have introduced the symbol of the functional integration as follows:

$$\prod_{\tau\in[0,t]} \rho(a(\tau),b(\tau))\,da(\tau)\,db(\tau) = \lim_{n\to\infty} \prod_{r=0}^{\infty} \Big(\rho(a(\tau),b(\tau))da(\tau)db(\tau)\Big)_{\tau=\frac{r}{n}\cdot t} \cdot \qquad /III.10/$$

Similarly the homogeneous representation /for example the "b" one/ of the propagator function

$$K(b'',b',t) \equiv \langle b''|e^{-it\hat{H}}|b'\rangle = \int e^{i\sigma(a,b'')} K(a,b')\,da \qquad /III.11/$$

also can be expressed by continual integrals. From Eqs. /III.9,11/ we get:

$$K(b'',b',t) = \int e^{i\int\limits_{\tau=0}^{t}\left(\frac{\partial\sigma(a(\tau),b(\tau))}{\partial b}\,db(\tau) - H(a(\tau),b(\tau))d\tau\right)} \times$$

$$\times\,\delta(b''-b(t))\,\delta(b'-b(0))\cdot\prod_{\tau\in[0,t]} \rho(a(\tau),b(\tau))\,da(\tau)\,db(\tau)\ . \qquad /III.12/$$

Let us try to find Green's function /resolvent matrix element/ of the operator $\hat{H}$:

$$G(b'',b',E) \equiv \langle b''|\frac{1}{E-\hat{H}}|b'\rangle = \frac{1}{i}\int_0^{\infty} K(b'',b',t)\,e^{itE}\,dt\ , \qquad \mathrm{Im}E>0\ . \qquad /III.13/$$

Following Ref. [1] we introduce the average of the function $H(a,b)$ along a given path $\{(a(\tau),b(\tau))\,;\ \tau\in[0,t]\}$:

$$\bar{E} = \frac{1}{t}\int_0^t H(a(\tau),b(\tau))\ . \qquad /III.14/$$

Thus Green's function can be written in the following form:

$$G(b'',b',E) = \int \frac{e^{i\int_{b'}^{b''}\frac{\partial\sigma(a,b)}{\partial b}\,db}}{E-\bar{E}} \cdot \delta(b''-b(1))\,\delta(b'-b(0))\prod_{\tau\in[0,1]} \rho(a(\tau),b(\tau))\,da(\tau)\,db(\tau)\ . \qquad /III.15/$$

Note that having performed the transformation /III.11/ we may substitute 1 instead of t and so rescale the parameter τ.

Finally, we consider the projection operator into the invariant subspace of the operator $\hat{H}$ belonging to a given eigenvalue E :

$$P(b'',b',E) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} K(b'',b',t)\,e^{iEt}\,dt = \frac{i}{2\pi}\,\underset{E}{\mathrm{Disc}}\,G(b'',b',E)\ . \qquad /III.16/$$

Taking the discontinuity of the expression /III.15/ on the real E axis we get the following result:

$$P(b'',b',E) = \int \delta(\bar{E}-E)\,e^{i\int_{b'}^{b''}\frac{\partial\sigma(a,b)}{\partial b}\,db}\cdot\delta(b''-b(1))\,\delta(b'-b(0))\prod_{\tau\in[0,1]}\rho(a,b)\,da\,db\ . \qquad /III.17/$$

IV. EXTREMAL PATHS, ASSOCIATED MECHANICS

Let us define the functional Σ on the space of paths $\{(a(\tau),b(\tau));\tau\in[0,t]\}$:

$$\Sigma = \int_0^t \left(\frac{\partial\sigma(a,b)}{\partial b}\dot{b} - H(a,b)\right)d\tau \qquad \text{where } \dot{b} \equiv \frac{db(\tau)}{d\tau} . \qquad /IV.1/$$

The formula /III.12/, expressing the propagator function, can be written as follows:

$$K(b'',b',t) = \int e^{i\Sigma}\,\delta(b''-b(t))\,\delta(b'-b(0)) \prod_{\tau\in[0,t]} \rho(a,b)\,da\,db . \qquad /IV.2/$$

Let us calculate the variation of the functional Σ under the conditions $\delta b(0) = \delta b(t) = 0$:

$$\delta\Sigma = \int_0^t \left[\delta a\left(-\frac{\partial H}{\partial a} + \frac{\partial^2\sigma}{\partial a\partial b}\dot{b}\right) - \delta b\left(\frac{\partial H}{\partial b} + \frac{\partial^2\sigma}{\partial b\partial a}\dot{a}\right)\right]d\tau . \qquad /IV.3/$$

Those paths for which $\delta\Sigma = 0$ will be called classical paths referring to the action principle of the mechanics. Let us denote the points of a given classical path by $(\tilde{a},\tilde{b})$, from Eq. /IV.3/ we arrive at the following equations of motion:

$$M(\tilde{a},\tilde{b})\dot{\tilde{a}} = -\frac{\partial H(\tilde{a},\tilde{b})}{\partial b} ,$$
$$M^T(\tilde{a},\tilde{b})\dot{\tilde{b}} = \frac{\partial H(\tilde{a},\tilde{b})}{\partial a} , \qquad /IV.4/$$

where the matrix M has been introduced by the definition

$$M(a,b) = \frac{\partial^2\sigma(a,b)}{\partial b\,\partial a} . \qquad /IV.5/$$

The function H is an integral of the motion as in the canonical mechanics Hamilton's function is. However our equations formulated in /IV.4-5/ are not canonical ones, moreover they are complex in general because of the matrix M.

Nevertheless the canonical structure of the Eqs. /IV.4/ can be discovered in many cases by proper transformation of variables.

Let us consider the case when the number of components of the vector a is equal to that of the vector b, i. e. $n_A=n_B$. Then the matrix M will be quadratic and we suppose that its determinant does not vanish anywhere. Therefore we can introduce the new variables a^{can} instead of a as follows:

$$a^{can} = \frac{\partial\sigma(a,b)}{\partial b} . \qquad /IV.6/$$

New Hamiltonian has to be introduced as well:

$$\overline{H}(a^{can}, b) \equiv H(a,b) \quad . \qquad /IV.7/$$

We obtain new equations of motion instead of Eqs. /IV.4/ :

$$\dot{\tilde{a}}^{can} = -\frac{\partial \overline{H}(\tilde{a}^{can}, \tilde{b})}{\partial b} \quad ,$$

$$\dot{\tilde{b}} = \frac{\partial \overline{H}(\tilde{a}^{can}, \tilde{b})}{\partial a^{can}} \quad . \qquad /IV.8/$$

These equations are canonical but the solutions are still complex paths because the Hamiltonian $\overline{H}$ is complex function in general.

Knowing the classical pathes we can try to apply the semiclassical method of the quantum mechanics for investigating the spectrum of the operator $\hat{H}$. We can start from the canonical equations of motion /IV.8/ and then we can make use of the E.B.K. axioms of semiclassical quantization [6].⁺

On the other hand one can follow Ref. [2] , summing up only the contributions of the stable classical paths in the path integrals.

V. ILLUSTRATIVE EXAMPLES FROM QUANTUM MECHANICS ⁺⁺

In this section we shall choose $\hat{A}$ and $\hat{B}$ so, that $\hat{H}$ be the Hamiltonian of a material point.

The usual case of the canonical variables is as follows:

$$\hat{A} = \frac{\hat{\underline{p}}^2}{2} = \int \frac{\underline{p}^2}{2} |\underline{p}\rangle\langle\underline{p}| \, d^3p \quad , \qquad /V.1/$$

$$\hat{B} = V(\hat{\underline{x}}) = \int V(\underline{x}) |\underline{x}\rangle\langle\underline{x}| \, d^3x \quad , \qquad /V.2/$$

$$\hat{H} = \hat{A} + \hat{B} = \frac{\hat{\underline{p}}^2}{2} + V(\hat{\underline{x}}) \quad , \qquad /V.3/$$

where $\hat{\underline{p}}$ and $\hat{\underline{x}}$ stand for the operators of the three-dimensional momentum and coordinates respectively. The operators satisfy the canonical commutation relation /I.2/. As is well-known, the phases of the eigenstate $|\underline{p}\rangle$ and $|\underline{x}\rangle$ can be chosen so that

$$U(\underline{p}, \underline{x}) \equiv \langle\underline{x}|\underline{p}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\underline{p}\underline{x}} \quad . \qquad /V.4/$$

⁺ Complex paths are often occuring even in the usual semiclassicism. Recently in Ref. [7] the relevance of the complex closed orbits has been argued in the frame of the E.B.K. quantization.

⁺⁺ In this section $\hbar = m = 1$.

Hence we obtain:

$$\sigma(p, x) = p x + \text{const} \quad , \qquad /V.5/$$

$$\rho(p, x) = \frac{1}{(2\pi)^3} \quad . \qquad /V.6/$$

Finally, we can see that the function $H(p, x)$ is just the Hamiltonian of the corresponding classical system; see Eqs. /II.1,4,12/ :

$$H(p, x) = \frac{p^2}{2} + V(x) \quad . \qquad /V.7/$$

Now we are going to apply Eq. /III.12/ for the propagator:

$$K(x'', x', t) = \int e^{i\int_{\tau=0}^{t}\left[p(\tau)dx(\tau) - \left(\frac{p^2(\tau)}{2} + V(x(\tau))\right)d\tau\right]} \cdot \delta^{(3)}(x'' - x(t))\,\delta^{(3)}(x' - x(0)) \cdot \prod_{\tau\in[0,t]} \frac{d^3p(\tau)\,d^3x(\tau)}{(2\pi)^3} \quad . \qquad /V.8/$$

It is just the canonical path integral formula of Garrod [1].

Now we choose noncanonical variables /the asymptotic momenta/:

$$\hat{A} = \frac{\hat{p}^2}{2} + V_{sc}(\hat{x}) = \frac{(\hat{p}^{in})^2}{2} = \int \frac{(p^{in})^2}{2} |p^{in}\rangle\langle p^{in}| d^3p^{in} \quad , \qquad /V.9/$$

$$\hat{B} = V(\hat{x}) \quad ,$$

$$\hat{H} = \hat{A} + \hat{B} = \frac{\hat{p}^2}{2} + V_{sc}(\hat{x}) + V(\hat{x}) = \frac{(\hat{p}^{in})^2}{2} + V(\hat{x}) \quad , \qquad /V.10/$$

where V_{sc} is a scattering potential, $\hat{p}^{in}$ is the operator of the asymptotic momentum of the in-going particle, $|p^{in}\rangle$ denotes the eigenstate of the operator $\hat{p}^{in}$, that is the so-called in-state. $\hat{p}^{in}$ and $\hat{x}$ do not satisfy the canonical commutator relations.

Solving the Schrödinger equation with the potential V_{sc} we obtain the coordinate representation of the in-states i.e. the usual stationary wave functions $\psi^{(+)}$; see Ref. [8] .

$$U(p^{in}, x) \equiv \langle x | p^{in}\rangle = \psi^{(+)}_{p^{in}}(x) \quad . \qquad /V.11/$$

Therefore we get:

$$\sigma(p^{in}, x) = \frac{1}{i} \ln \psi^{(+)}_{p^{in}}(x) \quad , \qquad /V.12/$$

$$\rho(p^{in}, x) = |\psi^{(+)}_{p^{in}}(x)|^2 \quad , \qquad /V.13/$$

$$H(p^{in}, x) = \frac{(p^{in})^2}{2} + V(x) \quad . \qquad /V.14/$$

We must note that $H(p^{in}, x)$ is still the classical energy function but p^{in} is not the canonically conjugated momentum.

Now we apply the formulae /III.14-15/ for the resolvent of $\hat{H}$:

$$G(\underline{x}'',\underline{x}',E) = \int \frac{e^{\int_{\underline{x}'}^{\underline{x}''} d\underline{x}(\tau)\, \nabla \ln \psi^{(+)}_{p^{in}(\tau)}(\underline{x}(\tau))}}{E-\bar{E}} \cdot \delta^{(3)}(\underline{x}''-\underline{x}(1))\, \delta^{(3)}(\underline{x}'-\underline{x}(0)) \cdot \prod_{\tau\in[0,1]} \left|\psi^{(+)}_{p^{in}(\tau)}(\underline{x}(\tau))\right|^2 d^3p^{in}(\tau)\, d^3x(\tau), \qquad /V.15/$$

$$\bar{E} = \int_0^1 \left(\frac{(p^{in}(\tau))^2}{2} + V(\underline{x}(\tau))\right) d\tau \ . \qquad /V.16/$$

It can be seen that while the effect of the scattering potential is built into $\psi^{(+)}$ via the solution of the Schrödinger equation, the effect of the second potential term V is taken into account by the above integral over paths.

Let us take an other choice for the <u>noncanonical variables</u> / the <u>quasimomentum</u> will be introduced /. The case is similar to the previous one but we replace the scattering potential V_{sc} by a periodic potential V_{per} of a given lattice:

$$\hat{A} = \frac{\hat{p}^2}{2} + V_{per}(\hat{\underline{x}}) = E(\hat{p}^Q) = \int E(p^Q)\, |p^Q\rangle\langle p^Q|\, d^3p^Q \ , \qquad /V.17/$$

$$\hat{B} = V(\hat{\underline{x}}) \ ,$$

$$\hat{H} = \hat{A} + \hat{B} = \frac{\hat{p}^2}{2} + V_{per}(\hat{\underline{x}}) + V(\underline{x}) = E(\hat{p}^Q) + V(\hat{\underline{x}}) \ , \qquad /V.18/$$

where p^Q is the quasimomentum of a given Bloch-electron, $|p^Q\rangle$ is the Bloch-state, $E(p^Q)$ is the energy of the electron, $\hat{p}^Q$ is the operator of the quasi-momentum. $\hat{H}$ is the Hamiltonian of the Bloch-electron perturbated by the potential V.+

It is known that the wave function of a Bloch-electron can be factorized as follows [9] :

$$U(p^Q,\underline{x}) \equiv \langle \underline{x}|p^Q\rangle = \frac{1}{(2\pi)^{3/2}}\, e^{ip^Q\underline{x}} \cdot \phi_{p^Q}(\underline{x}) \ , \qquad /V.19/$$

where the functions ϕ_{p^Q} have the same periodicity as $V_{per}(\underline{x})$ has.

Let us write the equations of motion of the associated classical system. Performing the actual substitutions into the Eqs. /IV.4-5/ we obtain:

$$\underline{\underline{M}}(p^Q,\underline{x}) \cdot \dot{p}^Q = -\frac{\partial V(\underline{x})}{\partial \underline{x}} \ , \qquad /V.20/$$

$$\underline{\underline{M}}^T(p^Q,\underline{x}) \cdot \dot{\underline{x}} = \frac{\partial E(p^Q)}{\partial p^Q} \ , \qquad /V.21/$$

$$\underline{\underline{M}}(p^Q,\underline{x}) = \mathbb{1} + \frac{1}{i}\frac{\overrightarrow{\partial}}{\partial \underline{x}} \ln \phi_{p^Q}(\underline{x}) \frac{\overleftarrow{\partial}}{\partial p^Q} \ . \qquad /V.22/$$

+ For the sake of the simplicity we suppose that there is only one energy band.

VI. CONCLUSIONS

We have investigated the spectrum of the sum of two self-adjoint operators with continuous spectra but of rather general type. Having applied the Trotter theorem path integral expression can be obtained for the resolvent kernel function of the above mentioned operator sum.

The functional which should be integrated over all paths can be taken as an exponential function of the action of an associated /and generalized in some sense/ mechanical system. The equations of motion of this mechanics have been deduced.

The approximate evaluation of the general path integral expressions seems to be possible, but it would require a certain generalization of the semiclassical methods of the quantum mechanics.

Some illustrative examples from the field of the quantum mechanics have been presented. We recover Garrod's canonical path integral formula as a special application. Our methods make possible to combine the Heisenberg and Feynman-Garrod quantization methods, in the two potential scattering problem or in the perturbation of the Bloch electron states for instance.

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