Shortnote on local hidden Grassmann variables vs. quantum correlations

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Grassmannian local hidden variables are shown to generate all possible quantum correlations in a bipartite quantum system. Grassmann representation of fermions, common in field theory, opens a related perspective. Although Grassmann hidden variables can not challange Bell's locality theorem, they can become an interesting mathematical tool to investigate entanglement.

Given two uncorrelated quantum systems A and B with density matrices $\hat\rho_A$ and $\hat\rho_B$, respectively, the state of the bipartite composite system is the tensor product $\hat\rho_A\hat\rho_B$. Let the states of both A and B depend on a certain variable λ so that the composite state were $\hat\rho_B(\lambda)\hat\rho_B(\lambda)$ had we known the value of λ. However, we suppose that λ is *hidden* variable in the sense that we only know the statistics of it. Therefore, the emerging composite state is the statistical mean value of $\hat\rho_B(\lambda)\hat\rho_B(\lambda)$:

$$\hat\rho_{AB} = \mathrm{M}[\hat\rho_A(\lambda)\hat\rho_A(\lambda)] \tag{1}$$

defined through the normalized probability p of the hidden variable:

$$\mathrm{M}[\dots] = \int \dots p(\lambda)\mathrm{d}\lambda \ . \tag{2}$$

After Werner [1], we consider eq. (1) the *separability* condition for the state $\hat\rho_{AB}$. If A and B were classical systems their composite states would always be separable, i.e., each composite classical density is a weighted mixture of uncorrelated densities. We say classical correlations emerge from ignorance regarding some hidden variables. This is not so in quantum theory. The separable quantum states which are mixtures of uncorrelated states will be called classically correlated quantum states. The non-separable quantum states $\hat\rho_{AB}$, for which the expansions (1) do not exist, are called quantum correlated or, equivalently, entangled. The existence of non-classical correlations is a principal difference of quantum theory from the classical one.

The lack of separability (1) shows up for two Pauli spins $\hat{\boldsymbol\sigma}_A$ and $\hat{\boldsymbol\sigma}_B$ already. The most general forms of the two spin states, respectively, read:

$$\hat\rho_A(\mathbf{a}) = \frac{1}{2}(\hat1_A + \mathbf{a}\hat{\boldsymbol\sigma}_A) \ , \quad \hat\rho_A(\mathbf{b}) = \frac{1}{2}(\hat1_A + \mathbf{b}\hat{\boldsymbol\sigma}_B) \ , \tag{3}$$

where $\mathbf{a}=(a_1,a_2,a_3)$ and $\mathbf{b}=(b_1,b_2,b_3)$ are real spatial polarization vectors satisfying $\mathbf{a}^2,\mathbf{b}^2\leq 1$. Without restricting generality, the hidden variable is the pair of the polarization vectors, $\lambda=(\mathbf{a},\mathbf{b})$, with the probability distribution $p(\mathbf{a},\mathbf{b})$. The composite state $\hat\rho_{AB}$ is separable if the probability distribution $p(\mathbf{a},\mathbf{b})$ exists such that

$$\hat\rho_{AB} = \mathrm{M}[\hat\rho_A(\mathbf{a})\hat\rho_B(\mathbf{b})] \ . \tag{4}$$

Let us apply this condition to the rotational invariant special case where $\mathrm{M}[\mathbf{a}]=\mathrm{M}[\mathbf{b}]=0$ and, in particular,

$$\mathrm{M}[a_ib_j] = \eta\delta_{ij} \tag{5}$$

for $i,j=1,2,3$. Most importantly, the correlation η is constrained by $|\eta|\leq 1/3$ since the values of $\mathbf{a}$ and $\mathbf{b}$ were constrained by $\mathbf{a}^2,\mathbf{b}^2\leq 1$. If we substitute (3) and (5) into the separability condition (4), we get the following form:

$$\hat\rho_{AB} = \frac{1}{4}(\hat1_A\hat1_B + \eta\hat{\boldsymbol\sigma}_A\hat{\boldsymbol\sigma}_B) \ , \tag{6}$$

which is thus separable if $|\eta|\leq 1/3$ and non-separable otherwise [1]. The matrix $\hat\rho_{AB}$ is non-negative for $\eta\in[-1,1/3]$ hence the states are indeed non-separable (quantum correlated, entangled) for $\eta\in[-1,-1/3)$. The constraints $\mathbf{a}^2,\mathbf{b}^2\leq 1$ have forbidden the existence of a hidden variable probability distribution $p(\mathbf{a},\mathbf{b})$ that could provide the (anti-)correlation stronger than $\eta=-1/3$. Quantum mechanics can achieve $\eta=-1$ as well. To generate stronger than $\eta=-1/3$ anti-correlations via hidden variables, we could adopt the blunt compromise as to allow $p(\mathbf{a},\mathbf{b})\ngeq 0$ which means that we would give up the statistical interpretation of the hidden variables.

We take an even more radical step instead: in the separability condition (1) we assume formally that the hidden classical variable λ is Grassmann variable. As for eq. (2), the theory of Grassmann variables contains the notion of integral [2] and of the corresponding measure $p(\lambda)$.

In particualar, we are going to discuss the case of the two correlated Pauli spins. Suppose $\mathbf{a},\mathbf{b}$ are Grassmann variables:

$$a_ia_j + a_ja_i = b_ib_j + b_jb_i = a_ib_j + b_ja_i = 0 \ , \tag{7}$$

for all $i,j=1,2,3$. Let us choose the following normalized rotation invariant Gaussian distribution [2] of the Grasmannian hidden variables:

$$p(\mathbf{a},\mathbf{b})\mathrm{d}\mathbf{b}\mathrm{d}\mathbf{a} = \eta^3\exp\left(\eta^{-1}\mathbf{ab}\right)\mathrm{d}\mathbf{b}\mathrm{d}\mathbf{a} \ , \tag{8}$$

satisfying $\mathrm{M}[\mathbf{a}]=\mathrm{M}[\mathbf{b}]=0$ and the correlation equation (5). Since η is not constrained at all, we conclude that the Grassmann hidden variables $\mathbf{a},\mathbf{b}$ can generate all possible correlations that two Pauli spins may have in quantum mechanics.

It is plausible to conjecture that quantum correlations can universally be reproduced by Grassmann hidden variables. Namely, a multipartite composite state can be expressed in the form

$$\hat{\rho}_{AB\dots K\dots} = \mathrm{M}[\hat{\rho}_A(\lambda)\hat{\rho}_B(\lambda)\dots\hat{\rho}_K(\lambda)\dots] \tag{9}$$

where λ is the hidden variable. If the state is separable then λ can be chosen real valued; if the state is entangled then λ must be a combination of real and Grassmann numbers. Perhaps the real numbers generate the classical while the Grassmann ones generate the quantum correlations, respectively, although the existence of such separation is an open issue itself. The combination of real number and Grassmann algebras might give some new insight into the generic entanglement structure.

I got the hint of Grassmann hidden variables from quantum field theory where the fermionic quantum fields $\hat{\psi}(t,\mathbf{r})$ can equivalently be represented by the corresponding local Grassmann fields $\psi(t,\mathbf{r})$ that satisfy a certain normalized distribution [2]. This means, at least for equilibrium states, that all fermion-mediated quantum correlations (entanglements) emerge from the local Grassmann variables $\psi(t,\mathbf{r})$ which play the role of hidden variables.

Another motivation comes from the recent work [4] by Christian, who suggested Clifford algebra valued hidden variables to violate Bell inequalities. In both his and my proposal the non-commutative hidden variables are able to generate entanglement of the composite state. Grassmann might have some theoretical advantage over Clifford numbers because of the mentioned universal Grassmann-fermion correspondence. Therefore I see a certain mathematical perspective to treat entanglement in the language of Grassmann hidden variables. The present work has no intention to challange the Bell theorem [3] since Grassmann (or Clifford) numbers can not parametrize individual measurement results [5], only real valued hidden variables can, cf. [6]. Yet, the proposed hidden Grassmann variables λ are local in the spirit close to Bell's: they are by construction independent of the occasional local experimental settings at sides A or B.

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[2] F. A. Berezin: *The Method of Second Quantization*, (Academic Press, 1966).
[3] J. S. Bell, Physics 1, 195 (1964).
[4] J. Christian, arXiv:quant-ph/0703244.
[5] E.g., to the measurement of $\hat{\sigma}_{A3}$ the eq. (3) would assign the value $\mathrm{tr}(\hat{\sigma}_{A3}\hat{\rho}_A)$ which is the Grassmannian a_3.
[6] Ph. Grangier, arXiv:0707.2223v1 [quant-ph].