Quantum information loss by frame averaging

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Contents

Von Neumann S vs thermodynamic S^{th} entropies

Homogeneous equilibrium reservoir at temperature $k_B T = 1/\beta$ and volume V, with Hamiltonian H:

$$\rho_\beta = Z_\beta^{-1} \mathrm{e}^{-\beta H} \ .$$

Von Neumann (microscopic) entropy:

$$S(\rho_\beta) =: -\mathrm{tr}(\rho_\beta \log \rho_\beta)$$

coincides with the thermodynamic (macroscopic) entropy S^{th} in the thermodynamic limit $V \to \infty$.
For non-equilibrium: general proof is missing. Let's enforces the coincidence of von Neumann and thermodynamic entropy productions.
Issue: ΔS is zero as long as $\rho_\beta \to U \rho_\beta U^\dagger$, while $\Delta S^{\mathrm{th}} > 0$.
Solution: a 'graceful' irreversible map

$$\rho \to \mathcal{M}\rho$$

constrained by

$$\Delta S =: S(\mathcal{M} U \rho_\beta U^\dagger) - S(\rho_\beta) = \Delta S^{\mathrm{th}}.$$

Key quantity will be the *relative q-entropy*:

$$S(\sigma|\rho) =: \mathrm{tr}[\sigma(\log \sigma - \log \rho)] \ .$$

S and S^{th} in non-equilibrium

Apply an external field, limited in space and time:

$$\rho_\beta \to \rho'_\beta = U\rho_\beta U^\dagger \ .$$

To engineer von Neumann entropy production, we assume an irreversible map $\mathcal{M}$ to be specified later:

$$\Delta S =: S(\mathcal{M}\rho'_\beta) - S(\rho_\beta) \ > 0 \ .$$

To make it equal with ΔS^{th}, we need ΔS^{th}'s microscopic expression! The field performs work:

$$W =: \mathrm{tr}(H\rho'_\beta) - \mathrm{tr}(H\rho_\beta) = \mathrm{tr}[(\rho'_\beta - \rho_\beta)H] \ .$$

From ρ_β, express $H = -\beta^{-1}\log(Z_\beta\rho_\beta)$, and consider $\rho'_\beta = U\rho_\beta U^\dagger$:

$$W = -\beta^{-1}\mathrm{tr}[(\rho'_\beta - \rho_\beta)\log\rho_\beta] = \beta^{-1}S(\rho'_\beta|\rho_\beta) \ .$$

Suppose W is *completely* dissipated, i.e.: $\Delta S^{\mathrm{th}} = W/k_B T = \beta W$, hence:

$$\Delta S^{\mathrm{th}} = S(\rho'_\beta|\rho_\beta) \ > 0 \ .$$

We'll find $\mathcal{M}$ such that $\Delta S = \Delta S^{\mathrm{th}}$ for $V \to \infty$.

A graceful irreverzible map $\mathcal{M}$

$$\lim_{V\to\infty}\overbrace{\left[S(\mathcal{M}\rho'_\beta)-S(\rho_\beta)\right]}^{\Delta S}=\lim_{V\to\infty}\overbrace{S(\rho'_\beta|\rho_\beta)}^{\Delta S^{\mathrm{th}}}\ .$$

$\mathcal{M}$ is 'graceful' if it preserves the free dynamics of the reservoir:

$$\mathcal{M}\left[\mathrm{e}^{-itH}\rho\mathrm{e}^{itH}\right]\equiv\mathcal{M}\rho\quad\text{for all }\rho\ .$$

Hint from Maxwell gas (D. 2002), spin chain (D.,Feldmann,Kosloff 2006): $\mathcal{M}$ is complete permutation of molecules/spins.
This time we consider a correlated many-body system in box V with periodic boundary conditions. Let $U(x)$ *translate the frame* by the spatial vector x. (Don't confuse $U(x)$ with the local perturbation U.)
If the Hamiltonian is translation invariant, so is the equilibrium state:

$$U(x)HU(-x)\equiv H\quad\Longrightarrow\quad U(x)\rho_\beta U(-x)\equiv\rho_\beta\ .$$

The non-equilibrium state $\rho'_\beta=U\rho_\beta U^\dagger$ is not. For it, consider the following irreversible map:

$$\mathcal{M}\rho'_\beta=\frac{1}{V}\int_{x\epsilon V}U(x)\rho'_\beta U(-x)\mathrm{d}x\ .$$

This map is 'graceful' and *makes S increase by* ΔS^{th}.

Proof, 1st part

$$\lim_{V\to\infty}\left[S(\mathcal{M}\rho'_\beta) - S(\rho_\beta) - S(\rho'_\beta|\rho_\beta)\right] = 0 \ .$$

Extension of the rigorous method (of Csiszár,Hia,Petz 2007). Inspect the identity (from translation inv.):

$$S(\mathcal{M}\rho'_\beta|\rho_\beta) = -S(\mathcal{M}\rho'_\beta) + S(\rho'_\beta) + S(\rho'_\beta|\rho_\beta) \ .$$

Hence the eq. to be proven becomes:

$$\lim_{V\to\infty} S(\mathcal{M}\rho'_\beta|\rho_\beta) = 0 \ .$$

The Hiai-Petz (1991) lemma:

$$S(\sigma|\rho) \leq S_{BS}(\sigma|\rho) \ ,$$

where $S_{BS}(\sigma|\rho) = \mathrm{tr}[\sigma \log(\sigma^{1/2}\rho^{-1}\sigma^{1/2})]$ is the Belavkin-Staszewski relative entropy which one re-writes in terms of the function $\eta(s) = -s\log s$: $S_{BS}(\sigma|\rho) = -\mathrm{tr}[\rho\eta(\rho^{-1/2}\sigma\rho^{-1/2})] \geq 0$. Let us chain the Klein and the Hiai-Petz inequalities for $\sigma = \mathcal{M}\rho'_\beta$ and $\rho = \rho_\beta$:

$$0 \leq S(\mathcal{M}\rho'_\beta|\rho_\beta) \leq S_{BS}(\mathcal{M}\rho'_\beta|\rho_\beta) = -\mathrm{tr}[\rho\eta(\mathcal{M}E_\beta)] \ ,$$

where $E_\beta = \rho_\beta^{-1/2}\rho'_\beta\rho_\beta^{-1/2}$ and $\mathcal{M}E_\beta = \frac{1}{V}\int U(x)E_\beta U(-x)\mathrm{d}x$. If we prove $\mathcal{M}E_\beta = \mathrm{I}$ for $V\to\infty$, it means $\eta(\mathcal{M}E_\beta) = 0$. Then the above inequalities yield $S(\mathcal{M}\rho'_\beta|\rho_\beta) = 0$ for $V\to\infty$, which will complete the proof.

Proof, 2nd part

For $\mathcal{M}E_\beta = \mathrm{I}$, we use heuristic arguments. We consider second quantized formalism where all quantized fields satisfy $A(x,t) = \exp(itH)A(x)\exp(-itH)$. Assume pair-potential that vanishes at $> \ell$. It is plausible to assume that perturbations have a maximum speed v of propagation. Hence, at any given time t after the unitary perturbation $\rho'_\beta = U\rho_\beta U^\dagger$ e.g. around the origin, there exists a finite volume of radius r such that

$$[U, A(x,t)] = 0 \quad \text{for all } |x| > r$$

and for all local quantum fields $A(x,t)$. Let us write E_β in the form $E_\beta = \rho_\beta^{-1/2} U\rho_\beta U^\dagger \rho_\beta^{-1/2} = u_\beta u_\beta^\dagger$ with

$$u_\beta = \rho_\beta^{-1/2} U \rho_\beta^{1/2} = \mathrm{e}^{\beta H/2} U \mathrm{e}^{-\beta H/2} \ .$$

u_β is the (non-unitary) equivalent of U, transformed by the operator $\mathrm{e}^{\beta H/2}$. By analytic continuation $\beta \Rightarrow i\beta$ and because of finite speed of perturbations, the operator u_β and thus E_β, too, will commute with all remote fields: $[u_\beta, A(x,t)] = [E_\beta, A(x,t)] = 0$ provided $|x| \gg r + v\beta$. Take the infinite volume limit $V \to \infty$! Since the sub-volume where $A(x,t)$ do *not* commute with E_β is finite and since E_β is a bounded operator, the averaged operator $\mathcal{M}E_\beta$ will commute *with all fields* $A(x,t)$ for *all* coordinates x! Hence $\mathcal{M}E_\beta = \lambda\mathrm{I}$ and the identity $\mathrm{tr}(\rho_\beta \mathcal{M}E_\beta) = \mathrm{tr}(\rho_\beta E_\beta) = 1$ yields $\lambda = 1$.

Realistic versions of $\mathcal{M}$

Graceful irreversible map $\mathcal{M}$ at less artificial conditions: many-body system in infinite V.

$$\mathcal{M}\rho'_\beta = \lim_{R\to\infty} \frac{1}{8\pi R^3} \int \mathrm{e}^{-|x|/R} U(x)\rho'_\beta U(-x)\mathrm{d}x \ .$$

It's plausible that $\mathcal{M}$ makes the reservoir *forget* the information about the *location* of perturbation, that amounts exactly to the thermodynamic entropy production.

A real quantum reservoir would gracefully forget the *location* of perturbation. It does not need to forget it immediately; it may do it at any later time. It does not need to forget it completely; it may do it on a certain finite scale R of *spatial frame coarse-graining*. In concrete cases, the information loss can be well saturated at some finite scale $R \gg r + v\beta$. Instead of the spatial frame, the temporal one can be made forgotten:

$$\mathcal{M}\rho'_\beta = \lim_{\tau\to\infty} \frac{1}{\tau} \int_{-\infty}^{0} \mathrm{e}^{t/\tau} U(-t)\rho'_\beta U(t)\mathrm{d}t \ ,$$

where $U(t) = \exp(-iHt)$. This state is definitely different from the result of spatial averaging. Conjecture: for $\tau, V \to \infty$ it gains the same entropy.

Summary

$$\lim_{V\to\infty}[S(\mathcal{M}\rho') - S(\rho)] = \lim_{V\to\infty} S(\rho'|\rho)\ ,$$

where ρ is translation invariant and $\rho' = U\rho U^\dagger$ is not, and

$$\mathcal{M}\rho' = \frac{1}{V}\int_{x\epsilon V} U(x)\rho' U(-x)\mathrm{d}x\ .$$

This is a novel mathematical theorem for the entropy gain of complete frame averaging.
We came to this conjecture by postulating a calculable model of both thermodynamic and von Neumann entropy gains.
While the merit of the new theorem is independent of the validity of our model for real physics, our model and results shed more light on the the physical mechanism of microscopic irreversibility, i.e., on Nature's graceful way to *forget* microscopic data.
On the other hand: the theorem provides a new interpretation of the q-relative entropy. And it has direct application in q-info: "Limit relation for quantum entropy and channel capacity per unit cost" - Csiszár,Hiai,Petz 2007.

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