Theorem on von Neumann entropy conjectured from physics of irreversibility

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Abstract

The talk is about the physicist's story of a theorem on relative von Neumann entropies:

$$S(\mathcal{T}(\sigma \otimes \rho^{\otimes N})) - S(\sigma \otimes \rho^{\otimes N}) \underset{N\to\infty}{\longrightarrow} S(\sigma|\rho)$$

where $\mathcal{T}$ (notation after average-over-group *twirlings* in quantum information) is symmetrization for permutations of the N+1 subsystems. (Mathematician Imre Csiszár told me: I cannot see how you came to this conjecture. I answered: Oh, and I cannot follow how you proved it.) Emergence of macroscopic irreversibility from the underlying microscopic dynamics is a fundamental puzzle in theoretical physics. In a Boltzmann gas under local dynamic perturbation we pointed out that the restoration of indistiguishability of the molecules (i.e.: a non-equilibrium generalization of Gibbs famous factor 1/N!) generates entropy in exact coincidence with the thermodynamic entropy production of the local perturbation, provided the Shannon relative entropies satisfy a certain asymptotic relationship. Assuming the similar mechanism in an abstract quantum gas led the speaker, Feldmann and Kosloff (2006) to the conjecture proved by Csiszár, Hiai and Petz (2007).

A new entropy theorem

Product state $\rho = \underset{1}{\sigma'} \otimes \underset{2}{\sigma} \otimes \underset{3}{\sigma} \otimes \underset{\dots}{\dots} \otimes \underset{N}{\sigma} = \sigma' \otimes \sigma^{\otimes(N-1)}$

von Neumann entropy: $S[\sigma' \otimes \sigma^{\otimes(N-1)}] = S[\sigma'] + (N-1)S[\sigma]$

Irreversible operation *twirling* $\mathcal{T}$:

$$\mathcal{T}\left(\sigma' \otimes \sigma^{\otimes(N-1)}\right) = \frac{\sigma' \otimes \sigma^{\otimes(N-1)} + \sigma \otimes \sigma' \otimes \sigma^{\otimes(N-2)} + \ldots + \sigma^{\otimes(N-1)} \otimes \sigma'}{N}$$

Limit theorem for entropy production:

$$\lim_{N=\infty} \left(S[\mathcal{T}(\sigma' \otimes \sigma^{\otimes(N-1)})] - S[\sigma' \otimes \sigma^{\otimes(N-1)}] \right) = S[\sigma' | \sigma].$$

Csiszár-Hiai-Petz: We don't see how you got the conjecture.
D.-Feldmann-Kosloff: We don't see how you prove it.

Microscopic irreversibility?

MIcroscopic dynamics – reversibility in closed systems:

$$\rho \to U\rho U^\dagger, \qquad S[U\rho U^\dagger] = S[\rho]$$

Macrosopic dynamics – irreversibility, entropy production:

$$\Delta S^{\mathrm{thermo}} \geq 0$$

An irreversible mechanism $\mathcal{T}$ must settle on unitary evolution.

$$\rho \to \mathcal{T}(U\rho U^\dagger), \qquad S[\mathcal{T}(U\rho U^\dagger)] > S[\rho]$$

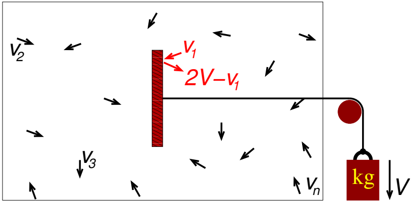
What can "coarse-graining" $\mathcal{T}$ be?
Find a system such that both(!)

- microscopic dynamics U is tractable analytically,
- thermodynamic entropy production $\Delta S^{\mathrm{thermo}}$ is calculable.

Assume $\Delta S \equiv S[\mathcal{T}(U\rho U^\dagger)] - S[U\rho U^\dagger] = \Delta S^{\mathrm{thermo}} \qquad (k_B = 1)$.
Then you guess concrete form of $\mathcal{T}$.

Mechanical friction in Maxwell gas

Constant force pulling a disk at small velocity V through gas.



Friction force: $2\nu m V$. (molecular mass m, collision rate ν)
Thermodynamic entropy-production rate at temperature $1/\beta$:

$$dS^{\text{thermo}}/dt = 2\beta\nu m V^2.$$

Microsopic dynamics must generate Boltzmann entropy

$$\Delta S/\text{collision} = 2\beta m V^2.$$

What information becomes erased?

Initial microscopic state:

$$\rho(v_1, ..., v_k, ..., v_N) = \mathcal{N} \prod_k \exp\left(-\frac{\beta m}{2} v_k^2\right) .$$

State after collision of the kth molecule:

$$\rho_k(v_1, ..., v_k, ..., v_N) = \rho(v_1, ..., 2V - v_k, ..., v_N).$$

So far no entropy production: $S[\rho_k] = S[\rho]$.
Now, restore Gibbs' molecular indistigushability:

$$\mathcal{T}\rho_k = (\rho_1 + \rho_2 + ... + \rho_N)/N.$$

And voila, for small V, in thermodynamic limit $N \to \infty$:

$$S[\mathcal{T}\rho_k] - S[\rho] = 2\beta m V^2.$$

Erasure of information by twirling $\mathcal{T}$ makes

$$\Delta S = \Delta S^{\mathrm{thermo}}.$$

D. 2002

MIcroscopic dissipation in abstract quantum gas

Initial Gibbs equilibrium state: $\rho = \left(\mathcal{N} e^{-\beta H}\right)^{\otimes N} \equiv \sigma^{\otimes N}$.
State after 'collision' of the kth 'molecule':

$$\rho_k = \sigma^{\otimes(k-1)} \otimes \sigma' \otimes \sigma^{\otimes(N-k)}$$

where $\sigma' = U\sigma U^\dagger$. The energy change is always non-negative:

$$\Delta E = \mathrm{tr}(H\sigma') - \mathrm{tr}(H\sigma) = S[\sigma'|\sigma]/\beta$$

Suppose ΔE is dissipated: $\Delta S^{\mathrm{thermo}} = S[\sigma'|\sigma]$ (β gone).

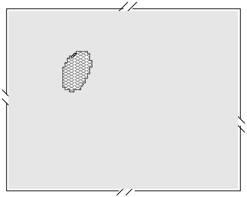
So far no von Neumann entropy gain: $S[\rho_k] = S[\rho]$.

Conjecture: $\Delta S = S[\mathcal{T}\rho_k] - S[\rho_k] = \Delta S^{\mathrm{thermo}}$ if $N \to \infty$.

$$\lim_{N=\infty}\left(S[\mathcal{T}(\sigma' \otimes \sigma^{\otimes(N-1)})] - S[\sigma' \otimes \sigma^{\otimes(N-1)}]\right) = S[\sigma'|\sigma].$$

D.-Feldmann-Kosloff 2006. Proof Csiszár-Hiai-Petz 2007.

Quantum many-body (gas, liquid,etc.)



Initial equilibrium Gibbs state $\rho = \mathcal{N} \exp(-\beta H)$
Locally perturbed by local external field $\rho' = U \rho U^\dagger$
Assume the work is dissipated, then $\Delta S^{\mathrm{thermo}} = S[\rho'|\rho]$
So far no von Neumann entropy gain: $S[\rho'] = S[\rho]$.
Make ρ' forget the original location of perturbation:

$$\mathcal{T}\rho' = \lim_{V=\infty} \frac{1}{V} \int_V U(x) \rho' U^\dagger(x) dx$$

von Neumann entropy production coincides with $\Delta S^{\mathrm{thermo}}$ if

$$\lim_{V\to\infty} \left(S[\mathcal{T}\rho'] - S[\rho'] \right) = \lim_{V\to\infty} S[\rho'|\rho]$$

Proof: D. 2012

Twirling $\mathcal{T}$

is a non-unitary (irreversible) map (Bennett et al. 1996):

$$\rho \to \mathcal{T}\rho \equiv \int U(g)\rho U^{\dagger}(g)dg$$

$U(g)$: unitary representation of a group; dg: Haar measure

- Friction in ideal gas: permutation group of molecules
- Dissipation of work local work: translational group

$U(g)HU^{\dagger}(g)=H$: no harm to dynamics, 'graceful' entropy production
Postulating $\Delta S = \Delta S^{\mathrm{thermo}}$ yields non-trivial entropic conjectures.
Phenomenology led us to exact math.

Outlook: Nature's forgetfulness

What is the mechanism of microscopic irreversibility?

- Is there a 'graceful' way Nature is forgetting microscopic data?
- Is there a universal non-unitary mechanism?
- Can it produce as much entropy as expected thermodynamically?

Answer: 3xYES
If Nature is twirling (shaking?) reference frames, She is producing just the observed thermodynamic entropy. Whether this is the real and ultimate way for Nature to 'forget' some particular microscopic data remains an open question.

Summary

- Notorious tension: reversible micro vs. irrev. macro
- Case study: mechanical friction in Maxwell gas (D. 2002)
- Quantitative entropic constraint on microscopic mechanism (D. et al. 2006)
- Nature may use *twirl* to erase information
- Bye-product: new quantum informatic theorem (Csiszár et al. 2007)
- Reality: twirling local perturbation of Gibbs state (D. 2012)

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